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Exponential Dichotomy and Eberlein-Weak Almost Periodic Solutions*

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ABSTRACT

We give sufficient conditions ensuring the existence and uniqueness of an Eberlein-weakly almost periodic solution to the following linear equation

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t)=A(t)x(t)+f(t),$$

in a Banach space X, where $(A(t))_{t\in\mathbb{R}}$ is a family of infinitesimal generators such that for all $t\in\mathbb{R}$, $A(t+T)=A(t)$ for some $T>0$, for which the homogeneous linear equation $\frac{\mathrm{d}x}{\mathrm{d}t}(t)=A(t)x(t)$ is well posed, stable and has an exponential dichotomy, and $f:\mathbb{R}\to X$ is Eberlein-weakly amost periodic.

Keywords: Bounded Solutions; Almost Periodic and Eberlein Weak Almost Periodic Functions; Exponential Dichotomy; Linear Differential Equations

1. Introduction

The aim of this work is to investigate the existence and uniqueness of a weakly almost periodic solution in the sense of Eberlein for the following linear equation :

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t)=A(t)x(t)+f(t),\text{ for all } t\in\mathbb{R} \tag{1}$$

for $x\in X$, where X is a complex Banach space, $A(t)$ is (unbounded) linear operator acting on X for every fixed $t\in\mathbb{R}$ such that for all $t\in\mathbb{R}$, $A(t+T)=A(t)$ for some $T>0$, and the input function $f:\mathbb{R}\to X$ is weakly almost periodic in the sense of Eberlein (Eberlein-weakly almost periodic). In the sequel, we essentially assume that:

(H_1) $(A(t))_{t\in\mathbb{R}}$ is a family of infinitesimal generators for which the corresponding homogeneous equation of (1) is well posed and stable in the following sense: there exists a T-periodic strongly continuous evolutionary process $(U(t,s))_{t\geq s}$, which is uniformly bounded and strongly continuous such in particular that:

for all

$$t\geq s,\ U(t,t)=I_X,\ U(t+T,s+T)=U(t,s)$$

and

$$U(t,s)U(s,\tau)=U(t,\tau),\text{ for all } t\geq s\geq\tau.$$

Further, if $x\in D(A(s))$ is given and $t\geq s$, then $U(t,s)x\in D(A(t))$, and

$$\begin{cases}\dfrac{\partial}{\partial t}U(t,s)x=A(t)U(t,s)x,\\[2mm] \dfrac{\partial}{\partial s}U(t,s)x=-U(t,s)A(s)x.\end{cases}$$

We also assume

(H_2) The corresponding homogeneous equation of (1) has an exponential dichotomy, *i.e.*, there exist a family of projections $Q(t)$, $t\in\mathbb{R}$ and positive constants M, α such that the following conditions are satisfied :

1) For every fixed $x\in X$ the map $t\mapsto Q(t)x$, is continuous and T-periodic,

2) $Q(t)U(t,s)=U(t,s)Q(s)$,

$\forall(t,s)\in\Delta:=\{(t,s)\in\mathbb{R}^2 : t\geq s\}$,

3) $\|U(t,s)P(s)\|\leq Me^{-\alpha(t-s)},\forall(t,s)\in\Delta,$ where for all

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$t \in \mathbb{R}$ and $x \in X$, $P(t)x := x - Q(t)x$,

4) $\|U(t,s)Q(s)\| \le Me^{\alpha(t-s)}, \forall (s,t) \in \Delta$,

5) $U(t,s)\big|_{\operatorname{Im} Q(s)}$ is an isomorphism from $\operatorname{Im} Q(s)$ onto $\operatorname{Im} Q(t)$, $\forall (s,t) \in \Delta$.

The problem of the existence of almost periodic solutions has been extensively studies in the literature [1-6]. Eberlein-weak almost periodic functions are more general than almost periodic functions and they were introduced by Eberlein [7], for more details about this topics we refer to [8-11] where the authors gave an important overview about the theory of Eberlein weak almost periodic functions and their applications to differential equations. In the literature, many works are devoted to the existence of almost periodic and pseudo almost periodic solutions for differential equations (a pseudo almost periodic function is the sum of an almost periodic function and of an ergodic perturbation), but results about Eberlein weak almost periodic solutions are rare [7,12-16].

In ([17], Chap. 3) the authors investigate the existence and uniqueness of an almost periodic solution for equation (1) when the corresponding homogeneous equation of (1) has an exponential dichotomy and the function *f* is almost periodic. In ([17], Chap. 3) the authors showed that, if the corresponding homogeneous equation of (1) has an exponential dichotomy and the function *f* is almost periodic, the equation (1) has a unique bounded integral solution on $\mathbb{R}$ which is also almost periodic. Here we propose to extend the result in [17] to the Eberlein-weakly almost periodic case.

2. Eberlein-Weak Almost Periodic Functions

In the sequel, we give some properties about weak almost periodic functions in the sense of Eberlein (Eberlein-weak almost periodic functions).

Let X and Y be two Banach spaces. Denote by $C(X,Y)$ the space of all continuous functions from X to Y. Let $BC(\mathbb{R},X)$ be the space of all bounded and continuous functions from $\mathbb{R}$ to X, equipped with the norm of uniform topology.

Definition 2.1 *A bounded continuous function* $x:\mathbb{R} \to X$ *is said to be almost periodic, if the orbit of x, the set of translates of x:*

$$O(x) := \{x_\tau := \{t \to x(t+\tau)\} : \tau \in \mathbb{R}\}$$

is a relatively compact set in $BC(\mathbb{R},X)$ with respect to the supremum norm.

We denote these functions by

$$AP(\mathbb{R},X) := \{x \in BC(\mathbb{R},X) : x \text{ is almost periodic}\}$$

Definition 2.2 *A function* $x \in BC(\boldsymbol{J},X)$, *for* $\boldsymbol{J} \in \{\mathbb{R},\mathbb{R}^+\}$ *is said to be weakly almost periodic in the sense of Eberlein* (*Eberlein-weakly almost periodic*) *if the orbit of x with respect to* $\boldsymbol{J}$:

$$O_J(x) := \{x_\tau := \{t \to x(t+\tau)\} : \tau \in \boldsymbol{J}\}$$

is relatively compact with respect to the weak topology of the sup-normed Banach space $BC(\boldsymbol{J},X)$.

For the sequel, $W(\boldsymbol{J},X)$ will denote the set of Eberlein-weakly almost periodic X valued functions.

Theorem 2.3 *Equipped with the norm*

$$\|x\|_\infty := \sup_{t\in\mathbb{R}} \|x(t)\|,$$

the vector space $W(\boldsymbol{J},X)$ is a Banach space.

In [18,19] Deleeuw and Glicksberg proved that if we consider the subspace of those Eberlein weakly almost periodic functions, which contain zero in the weak closure of the orbit (weak topology of $(BC(\boldsymbol{J},X),\|\cdot\|_\infty)$), *i.e.*;

$$W_0(\boldsymbol{J},X) := \{x \in W(\boldsymbol{J},X) : \text{for a sequence } (s_n)_{n\in\mathbb{R}} \subseteq \boldsymbol{J}, x_{s_n} \rightharpoonup 0\},$$

the following decomposition

$$W(\boldsymbol{J},X) = AP(\mathbb{R},X)\big|_J + W_0(\boldsymbol{J},X)$$

holds. Moreover, if $x \in W(\boldsymbol{J},X)$, $x^a \in AP(\boldsymbol{J},X)$, and $x_0 \in W_0(\boldsymbol{J},X)$ with $x = x^a\big|_J + x_0$, then

$$\|\cdot\| - \lim_{T\to\infty} \frac{1}{T}\int_0^T (x_t(s) - x_0(s))\,\mathrm{d}s = 0,$$

uniformly in $t \in \boldsymbol{J}$.

For a more detailed information about the decomposition and the ergodic result we refer to the book of Krengel [20,21].

In order to prove the weak compactness of the translates, Ruess and Summers extended the double limits criterion of Grothendieck [22] to the following:

Proposition 2.4 *A subset* $H \subset BC(\boldsymbol{J},X)$, *is relatively weakly compact if and only if*,

1) H is bounded in $BC(\boldsymbol{J},X)$, and

2) for all $(h_m)_{m\in\mathbb{N}} \subseteq H$, $(t_n)_{n\in\mathbb{N}} \subseteq \boldsymbol{J}$, and $(x_n^*)_{n\in\mathbb{N}} \subseteq \boldsymbol{B}_{X^*}$, the following double limits condition holds:

$$\lim_{m\to\infty}\lim_{n\to\infty}\langle h_m(t_n), x_n^*\rangle = \lim_{n\to\infty}\lim_{m\to\infty}\langle h_m(t_n), x_n^*\rangle,$$

whenever the iterated limits exist.

This result will be the main tool in verifying weak almost periodicity. For the other task we will use.

Proposition 2.5 *For every Eberlein weakly almost periodic function f there exists a sequence* $(t_n)_{n\in\mathbb{N}}$ *such that if g is the almost periodic part* $f_{t_n} \rightharpoonup g$.

3. Statement of the Main Result

In this section, we state a result of the existence and uniqueness of an Eberlein-weakly almost periodic solution of the Periodic Inhomogeneous Linear Equation (1). The existence and uniqueness of an almost periodic and bounded solution has been studied by M. N'Guérékata ([17]). More precisely, the author proved the following result.

Theorem 3.1 ([17]) *Assume that* (H_1) *and* (H_2) *hold. If the function f is continuous and bounded on* $\mathbb{R}$, *then Equation* (1) *has a unique bounded mild solution* x_f *on* $\mathbb{R}$. *Moreover*, *if f is almost periodic*, *then* x_f *is almost periodic.*

We propose to extend the above theorem to the case where *f* is Eberlein-weakly almost periodic.

Theorem 3.2 *Assume that* (H_1) *and* (H_2) *hold. If the function f is Eberlein-weakly almost periodic with a relatively compact range*, *then Equation* (1) *has a unique bounded mild solution* x_f *on* $\mathbb{R}$ *which is Eberlein-weakly almost periodic.*

For the proof of theorem (3.2), we use the following lemmas.

Lemma 3.3 *Let* $f:\boldsymbol{J}\to X$ *be a bounded uniformly continuous function with relatively compact range*, $(t_m)_{m\in\mathbb{N}}\subseteq\boldsymbol{J}$, $(w_n)_{n\in\mathbb{N}}\subseteq\boldsymbol{J}$, *and* $(x_m^*)_{m\in\mathbb{N}}\subseteq\boldsymbol{B}_{X^*}$. *If* $(t_m)_{m\in\mathbb{N}}$, *or* $(w_n)_{n\in\mathbb{N}}$ *is bounded*, *then one has*

$$\lim_{m\to\infty}\lim_{n\to\infty}\left\langle f(t_m+w_n),x_m^*\right\rangle=\lim_{n\to\infty}\lim_{m\to\infty}\left\langle f(t_m+w_n),x_m^*\right\rangle,$$

whenever the iterated limits exist.

Proof. Noting that only the equality of the iterated limits has to be proved, we may pass to subsequences. Therefore we assume that the following limits exists

1) $\omega-\lim_{n\to\infty}\omega-\lim_{m\to\infty}f(t_m+w_n):=a,$

2) $\omega-\lim_{m\to\infty}\omega-\lim_{n\to\infty}f(t_m+w_n):=b,$

3) $\omega-\lim_{m\to\infty}x_m^*=x^*$ and

4) $\lim_{m\to\infty}t_m=t,$ or $\lim_{n\to\infty}w_n=w.$

here 1) and 2) can be obtained by a diagonalization argument. Since $f(\boldsymbol{J})$ is separable, we may assume that 3) holds.

Let $\lim_{m\to\infty}t_m=t;$ then by the uniform continuity of *f*, we find

$$\lim_{m\to\infty}\left\|f_{t_m}-f_t\right\|_\infty=0$$

and

$$\lim_{m\to\infty}\lim_{n\to\infty}\left\langle f(t_m+w_n),x_m^*\right\rangle=\lim_{n\to\infty}\lim_{m\to\infty}\left\langle f(t_m+w_n),x_m^*\right\rangle.$$

Again by uniform continuity of *f*, and by the choice of subsequences we find for the interchanged limits.

Lemma 3.4 *Let* $(f_n)_{n\in\mathbb{N}}\subset BC(\boldsymbol{J},X)$ *such that for a subcompact set* $\boldsymbol{K}\subset X$

$$\bigcup_{t\in\boldsymbol{J},n\in\mathbb{N}}\{f_n(t)\}\subset\boldsymbol{K},$$

and

$$f_n\rightharpoonup 0 \text{ in } BC(\boldsymbol{J},X).$$

Then $\|f_n\|\rightharpoonup 0$ in $BC(\boldsymbol{J})$.

Proof. We first prove that the set $\left(\|f_n(.)\|\right)_{n\in\mathbb{N}}$ is weakly relatively compact in $BC(\boldsymbol{J})$. Thus, for given sequences $(n_j,t_m)_{j,m\in\mathbb{N}}\subseteq\mathbb{N}\times\boldsymbol{J}$, we have to verify the following identity :

$$\lim_{m\to\infty}\lim_{j\to\infty}\left\|f_{n_j}(t_m)\right\|=\lim_{j\to\infty}\lim_{m\to\infty}\left\|f_{n_j}(t_m)\right\|,$$

whenever the iterated limits exist. Since $x_{j,m}:=f_{n_j}(t_m)\in\boldsymbol{K}$ for all $j,m\in\mathbb{N}$, as a consequence of the metric weak compactness of $\boldsymbol{K}$, we may pass to subsequences of $(n_j)_{j\in\mathbb{N}}$ and $(t_m)_{m\in\mathbb{N}}$ such that the iterated limits of $(x_{j,m})_{j,m\in\mathbb{N}}$ exist in *X*, without loss of generality the sequences are chosen in this way. The characterization of weak compactness gives,

$$\lim_{m\to\infty}\lim_{j\to\infty}x_{j,m}=\lim_{j\to\infty}\lim_{m\to\infty}x_{j,m}.$$

Since $(\boldsymbol{K},\text{week})=(\boldsymbol{K},\|.\|)$ (the convergence holds in norm), hence one will obtain that $\left(\|f_n(.)\|\right)_{n\in\mathbb{N}}$ is weakly relatively compact in $BC(\boldsymbol{J})$. Using the fact that :

$$\lim_{n\to\infty}\|f_n(x)\|=0,$$

a standard trick of topology gives $\|f_n(.)\|\rightharpoonup$ in $BC(\boldsymbol{J})$.

Lemma 3.5 *Let*, *for a Banach space* $X, Y=L(X)$, (*the space of all bounded linear operators acting on X*) *and* $A(.)\in BC(\mathbb{R},Y)\,T-$*periodic. Then*, *for any given* $g:\mathbb{R}\to X$ *Eberlein-weakly almost periodic with a relatively compact range*,

$$\{t\to A(t)g(t)\}\in W(\mathbb{R},X).$$

Proof. In Order to prove that $\{t\to A(t)g(t)\}$ is Eberlein-weakly almost periodic, by W. M. Ruess and W. H. Summers's criterion (2.4), we have to verify that for given sequences $(t_m)_{m\in\mathbb{N}}\subseteq\mathbb{R}$, $(w_n)_{n\in\mathbb{N}}\subseteq\mathbb{R}$, and $(x_n^*)_{n\in\mathbb{N}}\subseteq\boldsymbol{B}_{X^*}$

$$\lim_{m\to\infty}\lim_{n\to\infty}\left\langle A(t_m+w_n)g(t_m+w_n),x_n^*\right\rangle$$
$$=\lim_{n\to\infty}\lim_{m\to\infty}\left\langle A(t_m+w_n)g(t_m+w_n),x_n^*\right\rangle,$$

whenever the iterated limits exist. Assuming that the iterated limits exist and by the fact that we only have to prove the equality of them, we may pass to subsequences for the verification. Since g is Eberlein weakly almost periodic with a relatively compact range, by a use of a diagonalization routine, we may assume that

$$\lim_{n\to+\infty}\lim_{m\to+\infty} g\left(t_{k_m}+w_{l_n}\right)=\lim_{m\to+\infty}\lim_{n\to+\infty} g\left(t_{k_m}+w_{l_n}\right)=z,$$

for a suitable choice of subsequences $\left\{t_{k_m}\right\}_{m\in\mathbb{N}}$ and $\left\{t_{l_n}\right\}_{n\in\mathbb{N}}$. We define

$$\begin{cases} b_{n,m}:=\left\langle A\left(t_{k_m}+w_{l_n}\right)g\left(t_{k_m}+w_{l_n}\right),x_m^*\right\rangle,\\ b=\lim\limits_{n\to+\infty}\lim\limits_{m\to+\infty} b_{n,m},\\ a_{n,m}:=\left\langle A\left(t_{k_m}+w_{l_n}\right)z,x_m^*\right\rangle.\end{cases}$$

By hypothesis, we have $t\to A(t)z$ is periodic, thus $\left\{a_{n,m}\right\}_{n,m\in\mathbb{N}}$ satisfies the double limits condition. Let $a\in\mathbb{R}$ be the double limit.

Now,

$$|b-a|\le\left|b-b_{n,m}\right|+\left|a-a_{n,m}\right|+\left|a_{n,m}-b_{n,m}\right|.$$

From the convergence of $\left\{a_{n,m}\right\}_{n,m\in\mathbb{N}}$ and $\left\{b_{n,m}\right\}_{n,m\in\mathbb{N}}$, we derive that for every $\varepsilon>0$ there exists an $n_0\in\mathbb{N}$, such that for $n\ge n_0$, there exist an m_n, such that

$$\begin{cases} \left|b_{n,m}-b\right|\le\dfrac{\varepsilon}{3}\,\forall m\ge m_n,\\ \qquad\text{and}\\ \left|a_{n,m}-a\right|\le\dfrac{\varepsilon}{3}\,\forall m\ge m_n.\end{cases}$$

Using the double limits condition of the sequence $\left\{x_{n,m}\right\}_{n,m\in\mathbb{N}}$, for given $\delta>0$, there exists $n_1\in\mathbb{N}$, such that for all $n\ge n_1$, there is an m_n^1, such that $\left\|x_{n,m}-z\right\|<\delta$, for all $m\ge m_n^1$. Applying the continuity of the map $x\in\overline{g(\mathbb{R})}^{\omega}\mapsto\left\{t\to A(t)x\right\}$, for $\varepsilon>0$, we find a $\delta>0$, and according to the previous observation, there exists an $n_1\in\mathbb{N}$, such that for all $n\ge n_1$, we find an m_n^1, with

$$\sup_{t\in\mathbb{R}}\left\|A(t)x_{n,m}-A(t)z\right\|<\varepsilon,\text{for every } m\ge m_n^1.$$

This yields, by a standard estimate, that $|b-a|<\varepsilon$, and hence $b=a$.

The following example shows that the compactness assumption on the range of g is essential and that the periodicity of $\left\{A(t)\right\}_{t\in\mathbb{R}}$ is not sufficient even if additional algebraic structure is given.

Example 3.6 *We let* $X=\pounds^2$ *and choose*

$$A(t)\left(\left\{x_n\right\}_{n\in\mathbb{N}}\right):=\left\{\exp(\mathrm{int})\left\{x_n\right\}\right\}_{n\in\mathbb{N}},\text{for all } t\in\mathbb{N},$$

$$\text{and } \left\{x_n\right\}_{n\in\mathbb{N}}\in\pounds^2.$$

Further, if χ_A denotes the indicator function for the set A, we choose $g\in BC(\mathbb{R},X)$

$$g(t):=\left\{\sin^2(t)\chi_{[n\pi,(n+1)\pi))}\right\}_{n\in\mathbb{N}}.$$

Using Lemma 2. 16 in ([13]), we obtain that g is Eberlein weakly almost periodic. Now, for the sequences

$$s_n:=\left(n+\frac{1}{2}\right)\pi+\frac{1}{n},\ t_n:=\left(n+\frac{1}{2}\right)\pi-\frac{1}{n},$$

some calculations lead to the identity :

$$\lim_{n\to\infty}\left\|A(s_n)g(s_n)-A(t_n)g(t_n)\right\|=\frac{2}{\sin(1)},$$

hence $t\to A(t)g(t)$ is not uniformly continuous, hence not Eberlein weakly almost periodic.

Proof. (of Theorem 3.2) Since f is Eberlein-weakly almost periodic, then f is continuous and bounded on $\mathbb{R}$. The existence and uniqueness of the bounded mild solution x_f on $\mathbb{R}$ result of theorem (3.1).

We claim that

$$\forall t\in\mathbb{R},$$
$$x_f(t)=\int_{-\infty}^{t}U(t,s)P(s)f(s)\mathrm{d}s-\int_{t}^{+\infty}U(t,s)Q(s)f(s)\mathrm{d}s. \tag{2}$$

In fact, for any $t\ge s$ we have

$$\begin{aligned}x_f(t)&=\int_{-\infty}^{t}U(t,\tau)P(\tau)f(\tau)\mathrm{d}\tau\\&\quad-\int_{t}^{+\infty}U(t,\tau)Q(\tau)f(\tau)\mathrm{d}\tau\\&=U(t,s)\Big(\int_{-\infty}^{s}U(s,\tau)P(\tau)f(\tau)\mathrm{d}\tau\\&\quad-\int_{s}^{-\infty}U(s,\tau)Q(\tau)f(\tau)\mathrm{d}\tau\Big)\\&\quad+\int_{s}^{t}U(t,\tau)P(\tau)f(\tau)d\tau+\int_{s}^{t}U(t,\tau)Q(\tau)f(\tau)\mathrm{d}\tau\Big)\\&=U(t,s)x_f(s)+\int_{s}^{t}U(t,\tau)f(\tau)\mathrm{d}\tau.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\left\|x_f(t)\right\|&\le M\int_{-\infty}^{+\infty}\mathrm{e}^{-\alpha|t-s|}\left\|f(s)\right\|\mathrm{d}s\\&\le M\int_{-\infty}^{+\infty}\mathrm{e}^{-\alpha|\zeta|}\left\|f(t-\zeta)\right\|\mathrm{d}\zeta.\end{aligned}$$

which ends the claim.

Now, to complete the proof, it remains for us to prove that x_f is Eberlein-weakly almost periodic. By Ruess and Summers's double limits criterion, we have to verify that for given sequences

$$(w_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}, (t_m)_{m\in\mathbb{N}} \subseteq \mathbb{R} \text{ and } (x_m^*)_{m\in\mathbb{N}} \subseteq \boldsymbol{B}_{X^*},$$

$$\lim_{m\to\infty}\lim_{n\to\infty}\langle x_f(w_n+t_m), x_m^*\rangle = \lim_{n\to\infty}\lim_{m\to\infty}\langle x_f(w_n+t_m), x_m^*\rangle,$$

whenever the iterated limits exist. Assuming that the iterated limits exist and by the fact that we only have to prove the equality of them, we may pass to subsequences.

Since x_f is uniformly continuous, by Lemma (3.3), we may assume that $w_n \to \infty$, and $t_m \to \infty$. Furthermore without loss of generality $w_n = k_n T + r$, otherwise we have $w_n = k_n T + r_n$ and by going over to subsequence $r_{n_l} \to r \in [0,T]$, the uniform continuity gives us that the double limits for these both sequences coincide. Bringing the equality (2) into play we obtain:

$$\begin{aligned}
&\left\|x_f(t_m+k_nT+r) - x_f(t_m+k_lT+r)\right\| \\
&\le \left\|\int_{-\infty}^{t_m+k_nT+r} U(t_m+k_nT+r,s)P(s)f(s)\mathrm{d}s - \int_{-\infty}^{t_m+k_lT+r} U(t_m+k_lT+r,s)P(s)f(s)\mathrm{d}s\right\| \\
&\le \left\|\int_{-\infty}^{t_m+k_nT+r} U(t_m+k_nT+r,s)P(s)f(s)\mathrm{d}s - \int_{-\infty}^{t_m+k_lT+r} U(t_m+k_lT+r,s)P(s)f(s)\mathrm{d}s\right\| \\
&\quad + \left\|\int_{t_m+k_nT+r}^{+\infty} U(t_m+k_nT+r,s)Q(s)f(s)\mathrm{d}s - \int_{t_m+k_lT+r}^{+\infty} U(t_m+k_lT+r,s)Q(s)f(s)\mathrm{d}s\right\| \\
&\quad + \left\|\int_{t_m+k_nT+r}^{+\infty} U(t_m+k_nT+r,s)Q(s)f(s)\mathrm{d}s - \int_{t_m+k_lT+r}^{+\infty} U(t_m+k_lT+r,s)Q(s)f(s)\mathrm{d}s\right\|
\end{aligned}$$

Since $\forall t \ge s,\ U(t+T,s+T) = U(t,s)$, we obtain:

$$\begin{aligned}
&\left\|x_f(t_m+k_nT+r) - x_f(t_m+k_lT+r)\right\| \\
&\le \left\|\int_{-\infty}^{t_m} U(t_m+r,\tau+r)\big(P(\tau+k_nT+r)f(\tau+k_nT+r) - P(\tau+k_lT+r)f(\tau+k_lT+r)\big)\mathrm{d}\tau\right\| \\
&\le \left\|\int_{-\infty}^{t_m} U(t_m+r,\tau+r)\big(P(\tau+k_nT+r)f(\tau+k_nT+r) - P(\tau+k_lT+r)f(\tau+k_lT+r)\big)\mathrm{d}\tau\right\| \\
&\quad + \left\|\int_{t_m}^{+\infty} U(t_m+r,\tau+r)\big(P(\tau+k_nT+r)f(\tau+k_nT+r) - P(\tau+k_nT+r)f(\tau+k_nT+r)\big)\mathrm{d}\tau\right\| \\
&\quad + \left\|\int_{t_m}^{+\infty} U(t_m+r,\tau+r)\big(P(\tau+k_nT+r)f(\tau+k_nT+r) - P(\tau+k_nT+r)f(\tau+k_nT+r)\big)\mathrm{d}\tau\right\|
\end{aligned}$$

thus,

$$\left\|x_f(t_m+k_nT+r) - x_f(t_m+k_lT+r)\right\| \le M(I_1+I_2).$$

where

$$I_1 = \int_{-\infty}^{t_m} e^{-\alpha(t_m-\tau)}\left\|P(\tau+k_nT+r)f(\tau+k_nT+r) - P(\tau+k_lT+r)f(\tau+k_lT+r)\right\|\mathrm{d}\tau$$

and

$$I_2 = \int_{t_m}^{+\infty} e^{\alpha(t_m-\tau)}\left\|Q(\tau+k_nT+r)f(\tau+k_nT+r) - Q(\tau+k_nT+r)f(\tau+k_nT+r)\right\|\mathrm{d}\tau$$

Since by Lemma (3.5)

$$\{t \to P(t)f(t)\}$$

and

$$\{t \to Q(t)f(t)\}$$

are Eberlein weakly almost periodic, we may assume that

$$P(k_nT+.)f(k_nT+.) \rightharpoonup g_P$$

and

$$Q(k_nT+.)f(k_nT+.) \rightharpoonup g_Q$$

Bringing the last estimate into play we obtain

$$\left\|x_f\left(t_m+k_nT+r\right)-x_f\left(t_m+k_lT+r\right)\right\|$$
$$\le\int_{t_{-\infty}}^{t_m}e^{-\alpha(t_m-\tau)}\left\|P\left(\tau+k_nT+r\right)f\left(\tau+k_nT+r\right)-g_P(\tau)\right\|\mathrm{d}\tau$$
$$\le\int_{t_{-\infty}}^{t_m}e^{-\alpha(t_m-\tau)}\left\|P\left(\tau+k_nT+r\right)f\left(\tau+k_nT+r\right)-g_P(\tau)\right\|\mathrm{d}\tau$$
$$+\int_{t_{-\infty}}^{t_m}e^{-\alpha(t_m-\tau)}\left\|P\left(\tau+k_lT+r\right)f\left(\tau+k_lT+r\right)-g_P(\tau)\right\|\mathrm{d}\tau$$
$$+\int_{t_{-\infty}}^{t_m}e^{-\alpha(t_m-\tau)}\left\|P\left(\tau+k_lT+r\right)f\left(\tau+k_lT+r\right)-g_P(\tau)\right\|\mathrm{d}\tau$$
$$+\int_{t_m}^{+\infty}e^{\alpha(t_m-\tau)}\left\|Q\left(\tau+k_nT+r\right)f\left(\tau+k_nT+r\right)-g_Q(\tau)\right\|\mathrm{d}\tau$$
$$+\int_{t_m}^{+\infty}e^{\alpha(t_m-\tau)}\left\|Q\left(\tau+k_nT+r\right)f\left(\tau+k_nT+r\right)-g_Q(\tau)\right\|\mathrm{d}\tau$$
$$+\int_{t_m}^{+\infty}e^{\alpha(t_m-\tau)}\left\|Q\left(\tau+k_lT+r\right)f\left(\tau+k_lT+r\right)-g_Q(\tau)\right\|\mathrm{d}\tau$$
$$+\int_{t_m}^{+\infty}e^{\alpha(t_m-\tau)}\left\|Q\left(\tau+k_lT+r\right)f\left(\tau+k_lT+r\right)-g_Q(\tau)\right\|\mathrm{d}\tau$$

Thus,

$$\left\|x_f\left(t_m+k_nT+r\right)-x_f\left(t_m+k_lT+r\right)\right\|$$
$$\le A_m^P+B_{m,n}^P+B_{m,k}^P+A_m^Q+B_{m,n}^Q+B_{m,k}^Q.$$

The uniform boundedness of the sequences of linear functional

$$\phi_m:BC(\mathbb{R})\to\mathbb{R}$$
$$h\mapsto\int_{-\infty}^{t_m}e^{-\alpha(t_m-\tau)}h(\tau)\,\mathrm{d}\tau$$

and

$$\times\sigma_m:BC(\mathbb{R})\to\mathbb{R}$$
$$h\mapsto\int_{-\infty}^{t_m}e^{-\alpha(\tau-t_m)}h(\tau)\,\mathrm{d}\tau$$

and the fact that Lemma (3.4) applies to

$$P(k_nT+.)\,f(k_nT+.)-g_P(.)$$
$$\text{and } Q(k_nT+.)\,f(k_nT+.)-g_Q(.)$$

By going to appropriate subsequences, we can assume that the iterated double limits for $\left\{B_{n,m}^P\right\}_{n,m\in\mathbb{N}}$ (resp. for $\left\{B_{n,m}^Q\right\}_{n,m\in\mathbb{N}}$) exist. Since they have to coincide, they have to be zero. By the triangle inequality we find,

$$\left\|x_f\left(t_m+k_nT+r\right)-x_f\left(t_m+k_lT+r\right)\right\|$$
$$\le A_m^P+B_{m,n}^P+B_{m,k}^P+A_m^Q+B_{m,n}^Q+B_{m,k}^Q.$$

Starting with $\lim_{l\to\infty}$, then $\lim_{m\to\infty}$, and at last $\lim_{n\to\infty}$, we obtain

$$\lim_{m\to\infty}\lim_{l\to\infty}\left\langle x_f\left(w_l+t_m\right),x_m^*\right\rangle=\lim_{n\to\infty}\lim_{m\to\infty}\left\langle x_f\left(w_l+t_m\right),x_m^*\right\rangle,$$

which concludes the proof.

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