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A Modified Precondition in the Gauss-Seidel Method

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ABSTRACT

In recent years, a number of preconditioners have been applied to solve the linear systems with Gauss-Seidel method (see [1-13]). In this paper we use S_l instead of $(S + S_m)$ and compare with M. Morimoto's precondition [3] and H. Niki's precondition [5] to obtain better convergence rate. A numerical example is given which shows the preference of our method.

Keywords: Preconditioning; Gauss-Seidel Method; Regular Splitting; *Z*-Matrix; Nonnegative Matrix

1. Introduction

Consider the linear system

$$Ax = b, \tag{1}$$

where $A = (a_{ij}) \in R^{n\times n}$ is a known nonsingular matrix and $x, b \in R^n$ are vectors. For any splitting $A = M - N$ with a nonsingular matrix M, the basic splitting iterative method can be expressed as

$$x^{(k+1)} = M^{-1}Nx^{(k)} + M^{-1}b. \qquad k = 0,1,2,\cdots \tag{2}$$

Assume that

$$a_{ii} \neq 0,\ i = 1,2,\cdots,n,$$

without loss of generality we can write

$$A = I - L - U, \tag{3}$$

where I is the identity matrix, $-L$ and $-U$ are strictly lower triangular and strictly upper triangular parts of A, respectively. In order to accelerate the convergence of the iterative method for solving the linear system (1), the original linear system (1) is transformed into the following preconditioned linear system

$$PAx = Pb, \tag{4}$$

where P, called a preconditioner, is a nonsingular matrix.

In 1991, Gunawardena *et al*. [2] considered the modified Gauss-Seidel method with $P = (I + S)$, where

$$S = (s_{ij}) = \begin{cases} -a_{ii+1} & \text{for } i = 1,2,\cdots,n-1,\ j = i+1 \\ 0 & \text{otherwise.} \end{cases}$$

Then, the preconditioned matrix $A_S = (I + S)A$ can be written as

$$\begin{aligned} A_S &= I - L - SL - U + S - SU \\ &= (I - D) - (L + E) - (U - S + SU), \end{aligned}$$

where D and E are the diagonal and strictly lower triangular parts of SL, respectively. If $a_{ii+1}a_{i+1i} \neq 1 (1 \le i \le n-1)$, then $\{(I - D) - (L + E)\}^{-1}$ exists. Therefore, the preconditioned Gauss-Seidel iterative matrix T_S for A_S becomes

$$T_S = \{(I - D) - (L + E)\}^{-1} (U - S + SU),$$

which is referred to as the modified Gauss-Seidel iterative matrix. Gunawardena *et al*. proved the following inequality:

$$\rho(T_S) \le \rho(T) < 1,$$

where $\rho(T)$ denotes the spectral radius of the Gauss-Seidel iterative matrix T. Morimoto *et al*. [3] have proposed the following preconditioner,

$$P_{s+m} = \left(a_{ij}^{(s+m)}\right) = I + S + S_m.$$

In this preconditioner, S_m is defined by

$$S_m = \left(s_{ij}^{(m)}\right) = \begin{cases} -a_{il_i} & 1 \le i < n-1, i+1 < j \le n, \\ 0 & \text{otherwise,} \end{cases}$$

where $l_i = \min I_i$ and $I_i = \left\{ j : |a_{ij}| \text{ is maximal for } i+1 < j \le n \right\}$, for $1 \le i < n-1$. The preconditioned matrix $A_{s+m} = (I + S + S_m)A$ can then be written as

$$\begin{aligned} A_{s+m} &= I - L - SL - U + S \\ &\quad - SU + S_m - S_mL - S_mU \\ &= \{(I - D_{s+m}) - (L + E_{s+m})\} \\ &\quad - (U - S - S_m + SU + S_mU + F_{s+m}), \end{aligned}$$

where D_{s+m}, E_{s+m} and F_{s+m} are the diagonal, strictly lower and strictly upper triangular parts of $(S+S_m)L$ respectively. Assume that the following inequalities are satisfied:

$$\begin{cases} 0 < a_{ii+1}a_{i+1i} + a_{il_i}a_{l_i i} < 1 & 1 \le i < n-1 \\ 0 < a_{ii+1}a_{i+1i} < 1 & i = n-1. \end{cases}$$

Then $\{(I-D_{s+m})-(L+E_{s+m})\}$ is nonsingular. The preconditioned Gauss-Seidel iterative matrix T_{s+m} for A_{s+m} is then defined by

$$T_{s+m} = \{(I-D_{s+m})-(L+E_{s+m})\}^{-1} \cdot (U-S-S_m+SU+S_mU+F_{s+m}).$$

Morimoto *et al*. [3] proved that $\rho(T_{s+m}) \le \rho(T_s)$. To extend the preconditioning effect to the last row, Morimoto *et al*. [7] proposed the preconditioner

$$P_R = I + R,$$

where R is defined by

$$R = (r_{nj}) = \begin{cases} -a_{nj} & 1 \le j \le n-1 \\ 0 & \text{otherwise} \end{cases}$$

The elements a_{nj}^R of A_R are given by

$$A_R = (I+R)A = (a_{ij}^R),$$

$$a_{ij}^R = \begin{cases} a_{ij} & 1 \le i < n, 1 \le j \le n \\ a_{nj} - \sum_{k=1}^{n-1} a_{nk}a_{kj} & 1 \le j \le n. \end{cases}$$

And Morimoto *et al*. proved that $\rho(T_R) \le \rho(T)$ holds, where T_R is the iterative matrix for A_R. They also presented combined preconditioners, which are given by combinations of R with any upper preconditioner, and showed that the convergence rate of the combined methods are better than those of the Gauss-Seidel method applied with other upper preconditioners [7]. In [14], Niki *et al*. considered the preconditioner $P_{SR} = (I+S+R)$. De- note $A_{SR} = M_{SR} - N_{SR}$. In [5], Niki *et al*. proved that if the following inequality is satisfied,

$$a_{nj} \le a_{nj}^R = a_{nj} - \sum_{k=1,\, k\ne j}^{n-1} a_{nk}a_{kj} \quad (j = 1,2,\cdots,n-1), \quad (5)$$

then $\rho(T_{SR}) \le \rho(T_S)$ holds, where T_{SR} is the iterative matrix for A_{SR}. For matrices that do not satisfy Equation (5), by putting $R = (r_{nj}) = -\sum_{k=1,\, k\ne j}^{n-1} a_{nk}a_{kj} - a_{nj}, 1 \le j < n,$

Equation (5) is satisfied. Therefore, Niki *et al*. [5] proposed a new preconditioner $P_G = I + \gamma G(\gamma \ge 1)$, where

$$G = (g_{nj}) = \begin{cases} \sum_{k=1,\, k\ne j}^{n-1} a_{nk}a_{kj} - a_{nj}, & 1 \le j \le n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Put $A_G = P_G A = (a_{ij}^G) = M_G - N_G$, and $T_G = M_G^{-1}N_G$.

Replacing P_G by $P_G = I + S + S_m + \gamma G$, and setting $\gamma = 1$, the Gauss-Seidel splitting of A_G can be written as

$$A_G = \{(I-D_{s+m}) - (L+E_{s+m}) - (G(L+U)-G)\} - (U-S-S_m+SU+S_mU+F_{s+m}),$$

where $G(L+U)-G$ is constructed by the elements $a_{nj}^G = g_{nj}a_{jn}$. Thus, if the preconditioner P_G is used, then all of the rows of A are subject to preconditioning. Niki *et al*. [5] proved that under the condition $\gamma_u > \gamma$, $\rho(T_R) \ge \rho(T_G)$, where γ_u is the upper bound of those values of γ for which $\rho(T_G) < 1$. By setting $a_{nj}^G = 0$, they obtained

$$\gamma = -a_{nj} \Big/ \left(g_{nj} + \sum_{k=1, k\ne j}^{n-1} g_{nk}a_{kj} \right).$$

Niki *et al*. [5] proved that the preconditioner P_G satisfies the Equation (5) unconditionally. Moreover, they reported that the convergence rate of the Gauss-Seidel method using preconditioner P_G is better than that of the SOR method using the optimum ω found by numerical computation. They also reported that there is an optimum $\gamma(\gamma_{opt})$ in the range $\gamma_u > \gamma_{opt} > \gamma_m$, which produces an extremely small $\rho(T_{\gamma opt})$, where γ_m is the upper bound of the values of γ for which $a_{nj}^G \ge 0$, for all j.

In this paper we use different preconditions for solving (1) by Gauss-Siedel method, that assuming none of the components of the matrix A to be zero. If the largest component of the column j is not $a_{i,i+1}$, then the value of $\rho(T)$ will be improved.

2. Main Result

In this section we replace S_l by $(S+S_m)$ of Morimoto such that $S_l = S_n + S_m$ and define S_n by

$$S_n = (s_{ij}) = \begin{cases} -a_{k_i j} & i = 1,2,\ldots,n-2,\, j = i+1, \\ -a_{ij} & i = n-1, \\ 0 & \text{otherwise,} \end{cases}$$

where $-a_{k_i j} = \max|a_{kj}|$ s.t $k = i, i+2, i+3, \cdots, n$, and S_m has the same form as the S_m proposed by Morimoto *et al*. [3].

The precondition Matrix $A_{S_l} = (I+S_l)A$ can then be written as

$$A_{S_l} = I - L - U + S_l - S_lL - S_lU = (I-D_{S_l}) - (L+E_{S_l}) - (U - S_l + S_lU + F_{S_l}),$$

where D_{S_l}, E_{S_l}, and F_{S_l} are the diagonal, strictly lower and strictly upper triangular parts of S_lL, respectively. Assume that the following inequalities are satisfied:

$$\begin{cases} -a_{ii+1} + a_{k_i i+1} + a_{il_i} a_{l_i i+1} \ge 0 & i = 1, 2, \cdots, n-2 \\ 0 < a_{k_i i+1} a_{i+1i} + a_{il_i} a_{l_i i} < 1 & i = 1, 2, \cdots, n-2 \\ 0 < a_{ii+1} a_{i+1i} < 1 & i = n-1, \end{cases} \quad (6)$$

Therefore $M_{S_l}^{-1}$ exists and the preconditioned Gauss-Seidel iterative matrix T_{S_l} for A_{S_l} is defined by

$$\begin{aligned} T_{S_l} &= M_{S_l}^{-1} N_{S_l} \\ &= \left[\left(I - D_{S_l} \right) - \left(L + E_{S_l} \right) \right]^{-1} \left(U - S_l + S_l U + F_{S_l} \right). \end{aligned}$$

For $A = (a_{ij})$ and $B = (b_{ij}) \in R^{n \times n}$, we write $A \ge B$ whenever $a_{ij} \ge b_{ij}$ holds for all $i, j = 1, 2, \cdots, n.$ A is non-negative if $A \ge 0, (a_{ij} \ge 0; i, j = 1, \cdots, n)$, and $A \ge B$ if and only if $A - B \ge 0$.

Definition 2.1 (Young, [15]). A real $n \times n$ matrix $A = (a_{ij})$ with $a_{ij} \le 0$ for all $i \ne j$ is called a *Z*-matrix.

Definition 2.2 (Varga, [16]). A matrix *A* is irreducible if the directed graph associated to *A* is strongly connected.

Lemma 2.3. *If A is an irreducible diagonally dominant Z-matrix with unit diagonal, and if the assumption* (6) *holds, then the preconditioned matrix* A_S *is a diagonally dominant Z-matrix.*

Proof. The elements $a_{ij}^{S_l}$ of A_{S_l} are given by

$$a_{ij}^{S_l} = \begin{cases} a_{ij} - a_{k_i,i+1} a_{i+1,j} - a_{i,l_i} a_{l_i,j} & 1 \le i < n-1, \\ a_{ij} - a_{i,i+1} a_{i+1,j} & i = n-1, \\ a_{ij} & i = n. \end{cases} \quad (7)$$

Since *A* is a diagonally dominant *Z*-matrix, so we have

$$\begin{aligned} &0 \le a_{k_i,i+1} a_{i+1,j} \le 1 && \text{for } j \ne i+1, \\ &0 \le a_{i,l_i} a_{l_i,j} \le 1 && \text{for } j \ne l_i, \quad (8) \\ &-1 \le a_{k_i,i+1} a_{i+1,i+1} \le 0. \end{aligned}$$

Therefore, the following inequalities hold:

$$p_i = a_{k_i,i+1} a_{i+1,i} \ge 0,$$

$$q_i = a_{i,l_i} a_{l_i,i} \ge 0,$$

$$r_i = a_{k_i,i+1} \sum_{j=1}^{i-1} a_{i+1,j} \ge 0,$$

$$s_i = a_{i,l_i} \sum_{j=1}^{i-1} a_{l_i,j} \ge 0,$$

$$t_i = a_{k_i,i+1} \sum_{j=i+1}^{n} a_{i+1,j} \le 0,$$

$$u_i = a_{i,l_i} \sum_{j=i+1}^{n} a_{l_i,j} \le 0.$$

We denote that $p_n = q_n = r_n = s_n = t_n = u_n = 0$. Then the following inequality holds:

$$\begin{aligned} &p_i + q_i + r_i + s_i + t_i + u_i \\ &= a_{k_i,i+1} \sum_{j=1}^{n} a_{i+1,j} + a_{i,l_i} \sum_{j=1}^{n} a_{l_i,j} \le 0 \quad 1 \le i < n. \end{aligned}$$

Furthermore, if $a_{i,i+1} \ne 0$, and $\sum_{j=1}^{n} a_{i+1,j} < 0$, for some $i < n$, then we have

$$p_i + q_i + r_i + s_i + t_i + u_i < 0 \quad \text{for some } i < n. \quad (9)$$

Let $d(S_l)_i$, $l(S_l)_i$ and $u(S_l)_i$ be the sums of the elements in row *i* of D_{S_l}, L_{S_l}, and U_{S_l}, respectively. The following equations hold:

$$\begin{aligned} d(S_l) &= a_{ij}^{S_l} = 1 - p_i - q_i, && 1 \le i \le n, \\ l(S_l) &= -\sum_{j=1}^{i-1} a_{ij}^{S_l} = l_i + r_i + s_i && 1 \le i \le n, \quad (10) \\ u(S_l) &= -\sum_{j=i+1}^{n} a_{ij}^{S_l} = u_i + t_i + u_i && 1 \le i \le n, \end{aligned}$$

where l_i and u_i are the sums of the elements in row *i* of *L* and *U* for $A = I - L - U$, respectively. Since *A* is a diagonally dominant *Z*-matrix, by (8) and by the condition (6) the following relations hold:

$$1 - a_{k_i,i+1} a_{i+1,j} - a_{i,l_i} a_{l_i,j} > 0, \quad \text{for } j = i$$

$$a_{ij} - a_{k_i,i+1} \sum_{j=i+1}^{n} a_{i+1,j} - a_{i,l_i} \sum_{j=i+1}^{n} a_{l_i,j} \le 0, \quad \text{for } i > j$$

$$a_{ij} - a_{k_i,i+1} - a_{i,l_i} a_{l_i,i+1} - a_{k_i,i+1} \sum_{j=i+2}^{n} a_{i+1,j} - a_{i,l_i} \sum_{j=i+2}^{n} a_{l_i,j} \le 0$$

for $i < j$.

Therefore, $l(S_l) \ge 0$, $u(S_l) \ge 0$, and A_{S_l} is a *Z*-matrix. Moreover, by (9) and by the assumption, we can easily obtain

$$\begin{aligned} &d(S_l) - l(S_l) - u(S_l) \\ &= (d_i - l_i - u_i) - (p_i + q_i + r_i + s_i + t_i + u_i) > 0 \quad (11) \end{aligned}$$

for all *i*.

Therefore, A_{S_l} satisfies the condition of diagonal dominance.

Lemma 2.4 [10, Lemma 2]. *An upper bound on the spectral radius* $\rho(T)$ *for the Gauss-Seidel iteration matrix T is given by*

$$\rho(T) \le \max_i \frac{\tilde{u}_i}{1 - \tilde{l}_i},$$

where $\tilde{l}_i$ and $\tilde{u}_i$ are the sums of the moduli of the elements in row *i* of the triangular matrices *L* and *U*, respectively.

Theorem 2.5. *Let A be a nonsingular diagonally dominant Z-matrix with unit diagonal elements and let the condition* (6) *holds, then* $\rho(T_{S_l}) < 1$.

Proof. From (11) and $u(S_l) \ge 0$ we have

$$d(S_l) - l(S_l) > u(S_l) \ge 0 \quad \text{for all } i.$$

This implies that

$$\frac{u(S_l)}{d(S_l) - l(S_l)} < 1. \tag{12}$$

Hence, by Lemma (2.4) we have $\rho(T_{S_l}) < 1$.

Definition 2.6. Let A be an $n \times n$ real matrix. Then, $A = M - N$ is referred to as:

1) a regular splitting, if M is nonsingular, $M^{-1} \ge 0$ and $N \ge 0$.

2) a weak regular splitting, if M is nonsingular, $M^{-1} \ge 0$ and $M^{-1}N \ge 0$.

3) a convergent splitting, if $\rho(M^{-1}N) < 1$.

Lemma 2.7 (Varga, [10]). *Let $A \in R^{n \times n}$ be a nonnegative and irreducible $n \times n$ matrix. Then*

1) A has a positive real eigenvalue equal to its spectral radius $\rho(A)$;

2) for $\rho(A)$, there corresponds an eigenvector $x > 0$;

3) $\rho(A)$ is a simple eigenvalue of A;

4) $\rho(A)$ increases whenever any entry of A increases.

Corollary 2.8 [16, Corollary 3.20]. *If $A = (a_{ij})$ is a real, irreducibly diagonally dominant $n \times n$ matrix with $a_{ij} < 0$ for all $i \ne j$, and $a_{ii} > 0$ for all $1 \le i \le n$, then $A^{-1} > O$.*

Theorem 2.9 [16, Theorem 3.29]. *Let $A = MN$ be a regular splitting of the matrix A. Then, A is nonsingular with $A^{-1} > O$ if and only if $\rho(M^{-1}N) < 1$, where*

$$\rho(M^{-1}N) = \frac{\rho(A^{-1}N)}{1 + \rho(M^{-1}N)} < 1$$

Theorem 2.10 (Gunawardena *et al.* [2, Theorem 2.2]). *Let A be a nonnegative matrix. Then*

1) If $\alpha x \le Ax$ for some nonnegative vector x, $x \ne 0$, then $\alpha \le \rho(A)$.

2) If $Ax \le \beta x$ for some positive vector x, then $\rho(A) \le \beta$. Moreover, if A is irreducible and if $0 \ne \alpha x \le Ax \le \beta x$ for some nonnegative vector x, then $\alpha \le \rho(A) \le \beta$ and x is a positive vector.

Let B be a real Banach space, B' its dual and $L(B)$ the space of all bounded linear operator mapping B into itself. We assume that B is generated by a normal cone K [17]. As is defined in [17], the operator $A \in L(B)$ has the property "*d*" if its dual A' possesses a Fro- benius eigenvector in the dual cone K' which is de- fined by

$$K' = \{x' \in B' : \langle x, x' \rangle = x'(x) \ge 0 \quad \text{for all } x \in K\}.$$

As is remarked in [1,17], when $B = R^n$ and $K = R^n_+$, all $n \times n$ real matrices have the property "*d*". Therefore the case are discussing fulfills the property "*d*". For the space of all $n \times n$ matrices, the theorem of Marek and Szyld can be stated as follows:

Theorem 2.11 (Marek and Szyld [17, Theorem 3.15]). *Let $A_1 = M_1 - N_1$ and $A_2 = M_2 - N_2$ be weak regular splitting with $T_1 = M_1^{-1}N_1, T_2 = M_2^{-1}N_2$. Let $x \ge 0, y \ge 0$ be such that $T_1 x = \rho(T_1)x$ and $T_2 y = \rho(T_2)y$. If $M_1^{-1} \ge M_2^{-1}$,* and if either $(A_1 - A_2)x \ge 0$, $A_1 x \ge 0$, or $(A_1 - A_2)y \ge 0, A_1 y \ge 0$, with $y > 0$, then

$$\rho(T_1) \le \rho(T_2).$$

Moreover, if $M_1^{-1} > M_2^{-1}$ and $N_1 \ne N_2$, then

$$\rho(T_1) < \rho(T_2).$$

Now in the following lemma we prove that $A_{S_l} = (I + S_n + S_m)A = M_{S_l} - N_{S_l}$ is Gauss-Seidel convergent regular splitting.

Theorem 2.12. *Let A be an irreducibly diagonally dominant Z-matrix with unit diagonal, and let the condition (6) holds, then $A_{S_l} = M_{S_l} - N_{S_l}$ is Gauss-Seidel convergent regular splitting. Moreover*

$$\rho(T_{S_l}) \le \rho(T_{s+m}) < 1.$$

Proof. If A is an irreducibly diagonally dominant Z-matrix, then by Lemma (2.3), A_{S_l} is a diagonally dominant Z-matrix. So we have $A_{S_l}^{-1} \ge 0$. By hypothesis we have $(I - D_{S_l})^{-1} \ge I$. Thus the strictly lower triangular matrix $(L + E_{S_l})$ has nonnegative elements. By considering Neumanns series, the following inequality holds:

$$\begin{aligned} M_{S_l}^{-1} = \Big[I &+ (I - D_{S_l})^{-1}(L + E_{S_l}) \\ &+ \left\{(I - D_{S_l})^{-1}(L + E_{S_l})\right\}^2 + \cdots \\ &+ \left\{(I - D_{S_l})^{-1}(L + E_{S_l})\right\}^{n-1} \Big] (I - D_{S_l})^{-1} \ge 0. \end{aligned}$$

Direct calculation shows that $N_{S_l} \ge 0$ holds. Thus, by definition (2.6) $A_{S_l} = M_{S_l} - N_{S_l}$ is the Gauss-Seidel convergent regular splitting. Also in [3] we have $A_{s+m} = M_{s+m} - N_{s+m}$ and

$$\begin{aligned} M_{s+m}^{-1} = [I &+ (I - D_{S+M})^{-1}(L + E_{S+M}) \\ &+ \left\{(I - D_{S+M})^{-1}(L + E_{S+M})\right\}^2 + \cdots, \\ &+ \left\{(I - D_{S+M})^{-1}(L + E_{S+M})\right\}^{n-1} \Big] (I - D_{S+M})^{-1} \ge 0. \end{aligned}$$

Direct comparison of the two matrix elements $(I - D_{S+M})^{-1}$ and $(I - D_{S_l})^{-1}$ also $(L + E_{S+M})$ and $(L + E_{S_l})$ we obtain

$$\left(I - D_{S+M}\right)^{-1} \leq \left(I - D_{S_l}\right)^{-1},$$
$$\left(L + E_{S+M}\right) \leq \left(L + E_{S_l}\right).$$

Thus

$$0 \leq M_{s+m}^{-1} \leq M_{S_l}^{-1}.$$

Furthermore, since $S_n \geq S$, we have $A_{S_l}x - A_{s+m}x = \left(S_n - S\right)Ax \geq 0$. From Lemma (2.7), x is an eigenvector of T_{S_l}, and x is also a Perron vector of T_{S_l}. Therefore, from Theorem (2.11),

$$\rho\left(T_{S_l}\right) \leq \rho\left(T_{s+m}\right)$$

holds.

Denote

$$A_{smr} = \left(I + S + S_m + R\right)A,$$
$$A_{S_l r} = \left(I + S_l + R\right)A,$$
$$A_G = \left(I + S + S_m + \gamma G\right)A,$$
$$A_{GS_l} = \left(I + S_l + \gamma G\right)A,$$

and also let T_{smr}, $T_{S_l r}$, T_G and $T_G S_l$ be the iterative matrix associated to A_{smr}, $A_{S_l r}$, A_G and A_{GS_l} respectively. Then we can prove $\rho\left(T_{S_l r}\right) \leq \rho\left(T_{smr}\right)$ and $\rho\left(T_G S_l\right) \leq \rho\left(T_G\right)$, similarly. In summary, we have the following inequalities:

$$\rho(T) \geq \rho(T_s) \geq \rho(T_{s+m}) \geq \rho\left(T_{S_l}\right) \geq \rho(T_{smr})$$
$$\geq \rho\left(T_{S_l} r\right) \geq \rho(T_G) \geq \rho\left(T_{GS_l}\right).$$

Remark 2.13. W. Li, in [18] used the M-matrix instead of irreducible diagonally dominant *Z*-matrix, therefore we can say that the Lemma 2.3 and the Theorems 2.5 and 2.12 are hold for M-matrices.

3. Numerical Results

In this section, we test a simple example to compare and contrast the characteristics of the different preconditioners. Consider the matrix

$$A = \begin{pmatrix} 1 & -0.2 & -0.6 & -0.2 \\ -0.2 & 1 & -0.3 & -0.1 \\ -0.1 & -0.2 & 1 & -0.3 \\ -0.2 & -0.3 & -0.2 & 1 \end{pmatrix}$$

Applying the Gauss-Seidel method, we have $\rho(T) = 0.5099$. By using preconditioner $P_S = \left(I + S\right)A$, we find that A_S and T_S have the following forms:

$$A_S = \begin{pmatrix} 0.96 & 0 & -0.66 & -0.22 \\ -0.23 & 0.94 & 0 & -0.19 \\ -0.16 & -0.29 & 0.94 & 0 \\ -0.2 & -0.3 & -0.2 & 1 \end{pmatrix}$$

$$T_S = \begin{pmatrix} 0 & 0 & 0.6875 & 0.2292 \\ 0 & 0 & 0.1682 & 0.2582 \\ 0 & 0 & 0.1689 & 0.1187 \\ 0 & 0 & 0.2217 & 0.1470 \end{pmatrix}$$

and $\rho\left(T_S\right) = 0.3205$.

Using the preconditioner P_{s+m}, we obtain

$$A_{s+m} = \begin{pmatrix} 0.90 & -0.12 & -0.06 & -0.4 \\ -0.25 & 0.91 & -0.2 & -0.09 \\ -0.16 & -0.29 & 0.94 & 0 \\ -0.2 & -0.3 & -0.2 & 1 \end{pmatrix}$$

$$T_{s+m} = \begin{pmatrix} 0 & 0.133 & 0.067 & 0.444 \\ 0 & 0.037 & 0.040 & 0.221 \\ 0 & 0.034 & 0.023 & 0.144 \\ 0 & 0.044 & 0.030 & 0.184 \end{pmatrix}$$

and $\rho\left(T_{s+m}\right) = 0.2581$.

For $P_{S_l} = I + S_l$, we have

$$A_{S_l} = \begin{pmatrix} 0.88 & -0.02 & -0.09 & -0.41 \\ -0.25 & 0.91 & -0.2 & -0.09 \\ -0.16 & -0.29 & 0.94 & 0 \\ -0.2 & -0.3 & -0.2 & 1 \end{pmatrix}$$

$$T_{S_l} = \begin{pmatrix} 0 & 0.023 & 0.102 & 0.465 \\ 0 & 0.006 & 0.050 & 0.227 \\ 0 & 0.006 & 0.033 & 0.149 \\ 0 & 0.007 & 0.042 & 0.1911 \end{pmatrix}$$

and $\rho\left(T_{S_l}\right) = 0.2328$.

For $P_{smr} = I + S + S_m + R$, we have

$$A_{smr} = \begin{pmatrix} 0.90 & -0.12 & -0.06 & -0.4 \\ -0.25 & 0.91 & -0.2 & -0.09 \\ -0.16 & -0.29 & 0.94 & 0 \\ -0.08 & -0.08 & -0.21 & 0.87 \end{pmatrix}$$

$$T_{smr} = \begin{pmatrix} 0 & 0.133 & 0.067 & 0.444 \\ 0 & 0.037 & 0.040 & 0.221 \\ 0 & 0.034 & 0.023 & 0.144 \\ 0 & 0.024 & 0.015 & 0.096 \end{pmatrix}$$

and $\rho\left(T_{smr}\right) = 0.1694$.

For $P_{S_l r} = I + S_l + R,$ we have

$$A_{S_l r} = \begin{pmatrix} 0.88 & -0.02 & -0.09 & -0.41 \\ -0.25 & 0.91 & -0.02 & -0.09 \\ -0.16 & -0.29 & 0.94 & 0 \\ -0.08 & -0.08 & -0.21 & 0.87 \end{pmatrix}$$

$$T_{S_l r}=\begin{pmatrix}0 & 0.023 & 0.102 & 0.465\\ 0 & 0.006 & 0.050 & 0.227\\ 0 & 0.006 & 0.033 & 0.149\\ 0 & 0.004 & 0.022 & 0.1\end{pmatrix}$$

and $\rho\left(T_{S_l r}\right)=0.1415$.

From the above results, we have $G_{nj}=(0.28,0.38,0.41,0)$. Then $A_G(\gamma=1)$ and T_G have the forms:

$$A_G=\begin{pmatrix}0.90 & -0.12 & -0.06 & -0.4\\ -0.25 & 0.91 & -0.02 & -0.09\\ -0.16 & -0.29 & 0.94 & 0\\ -0.04 & -0.06 & -0.07 & 0.78\end{pmatrix}$$

$$T_G=\begin{pmatrix}0 & 0.133 & 0.067 & 0.444\\ 0 & 0.037 & 0.040 & 0.221\\ 0 & 0.034 & 0.024 & 0.144\\ 0 & 0.012 & 0.008 & 0.050\end{pmatrix}$$

and $\rho(T_G)=0.1218$.

For $P_{GS_l}(\gamma=1)=I+S_l+G$, we have

$$A_{GS_l}=\begin{pmatrix}0.88 & -0.02 & -0.09 & -0.41\\ -0.25 & 0.91 & -0.02 & -0.09\\ -0.16 & -0.29 & 0.94 & 0\\ -0.04 & -0.06 & -0.07 & 0.78\end{pmatrix}$$

$$T_{GS_l}=\begin{pmatrix}0 & 0.023 & 0.102 & 0.465\\ 0 & 0.006 & 0.050 & 0.227\\ 0 & 0.006 & 0.033 & 0.149\\ 0 & 0.002 & 0.011 & 0.052\end{pmatrix}$$

and $\rho\left(T_{GS_l}\right)=0.0940$. Since the preconditioned matrices differ only in the values of their last rows, the related matrices also differ only in these values, as is shown in the above results. Thus the elements of new A_G and A_{GS_l} are similar to elements of A_G and A_{GS_l}, respectively than the elements of last rows. Therefore, we hereafter show only the last row.

By putting $\gamma_1=1.22699$, the matrices $A_{G_{\gamma_1}}$, $T_{G_{\gamma_1}}$, $A_{GS_l\gamma_1}$ and $T_{GS_l\gamma_1}$ have the following forms:

$$A_{G_{\gamma_1}}=(0,-0.003,-0.042,0.733),$$
$$T_{G_{\gamma_1}}=(0,0.0021,0.0015,0.0093),$$

and $\rho\left(T_{G_{\gamma_1}}\right)=0.0779$,

$$A_{GS_l\gamma_1}=(0,-0.003,-0.042,0.733),$$
$$T_{GS_l\gamma_1}=(0,0.00036,0.0021,0.0096),$$

and $\rho\left(T_{GS_l\gamma_1}\right)=0.0509$.

For $\gamma_2=1.23966$ we have:

$$A_{G_{\gamma_2}}=(0.0021,0,-0.0413,0.7310),$$
$$T_{G_{\gamma_2}}=(0,0.0015,0.0011,0.0069),$$

and $\rho\left(T_{G_{\gamma_2}}\right)=0.0751$ and

$$A_{GS_l\gamma_2}=(0.0021,0,-0.0413,0.7310),$$
$$T_{GS_l\gamma_2}=(0,0.00026,0.0016,0.0071),$$

and $\rho\left(T_{GS_l\gamma_2}\right)=0.0483$.

For $\gamma_3(opt)=1.5625$ we have:

$$A_{G_{\gamma_3(opt)}}=(0.0547,0.0781,0,0.6609),$$
$$T_{G_{\gamma_3(opt)}}=(0,-0.0154,-0.0102,-0.0629),$$

and $\rho\left(T_{G_{\gamma_3(opt)}}\right)=0.0227$ and

$$A_{GS_l\gamma_3(opt)}=(0.0547,0.0781,0,0.6609),$$
$$T_{GS_l\gamma_3(opt)}=(0,-0.0026,-0.0144,-0.0654),$$

and $\rho\left(T_{GS_l\gamma_3(opt)}\right)=0.0216$.

From the numerical results, we see that this method with the preconditioner $P_{GS_l}(\gamma_{opt})$ produces a spectral radius smaller than the recent preconditioners that above was introduced.

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