*Theoretical Economics Letters*, **2012, 2, 300-306**
http://dx.doi.org/10.4236/tel.2012.23056 Published Online August 2012 (http://www.SciRP.org/journal/tel)

# Stackelberg-Cournot and Cournot Equilibria in a Mixed Markets Exchange Economy

**Ludovic A. Julien**[1,2,3]
[1]LEG, University of Burgundy, Dijon, France
[2]EconomiX, University of Paris West-Nanterre, Paris, France
[3]IRES, Catholic University of Louvain, Louvain, Belgium
Email: ludovic.julien@u-paris10.fr

Received April 6, 2012; revised May 8, 2012; accepted June 9, 2012

## ABSTRACT

In this note, we compare two strategic general equilibrium concepts: the Stackelberg-Cournot equilibrium and the Cournot equilibrium. We thus consider a market exchange economy including atoms and a continuum of traders, who behave strategically. We show that, when the preferences of the *small* traders are represented by Cobb-Douglas utility functions and the atoms have the same utility functions and endowments, the Stackelberg-Cournot and the Cournot equilibrium equilibria coincide *if and only if* the followers' best responses functions have a zero slope at the SCE.

**Keywords:** Stackelberg-Cournot Equilibrium; Conjectural Variations; Preferences

## 1. Introduction

Oligopolistic competition in general equilibrium has been developed in two main directions. The first is the Cournot-Walras equilibrium approach, which is modeled by Gabszewicz and Vial [1] in an economy with production, and in exchange economies by Codognato and Gabszewicz [2,3], Gabszewicz and Michel [4], and Busetto, Codognato and Ghosal [5,6]. This class of models includes agents who behave strategically (the atoms), while other agents behave competitively (the atomless continuum of traders). The second is the Cournot equilibrium (CE) based on strategic market games as notably modeled by Shapley and Shubik [7], Dubey and Shubik [8], Sahi and Yao [9], and Amir, Sahi, Shubik and Yao [10]. In this approach, all traders always behave strategically and can send quantity signals indicating how much of any commodity they are willing to buy and/or sell. Some contributions aim at comparing the CE with other strategic equilibria. Codognato [11] studies the equivalence between the Cournot-Walras equilibrium and the CE, while Codognato [12] compares two Cournot-Nash equilibrium models. In this note, we compare the CE and the Stackelberg-Cournot equili- brium (SCE) defined for finite economies in Julien and Tricou [13,14]. From the benchmark of strategic market games, the SCE concept inserts Stackelberg competition into interrelated markets. We determine the conditions under which the CE and the SCE are equivalent.

The equivalence is studied in an economy embodying atoms and a continuum of traders. We thus consider a mixed exchange economy *a la* Shitovitz [15] and Codognato [11], in which the traders who are endowed with a corner endowments are atoms, while the traders endowed with all other commodities are represented by an atomless continuum. Markets are complete and prices are consistent. We assume the individual positions and the timing of moves as given. In addition, existence and uniqueness of oligopoly equilibrium are deleted. We rather focus on the case for which both sets of strategic equilibria can have a nonempty intersection. Indeed, when the preferences of the small traders are represented by Cobb-Douglas utility functions, and when the atoms have the same endowments and utility functions, the SCE and the CE coincide if and only if the followers' best responses functions have a zero slope at the SCE. We so spread the result obtained by Codognato [11] for Cournotian economies to a class of exchange economies in which the strategic interactions recover from sequential decisions. We also provide a generalization of Julien [16] because henceforth all the traders behave strategically.

The paper is organized as follows. Section 2 specifies the mixed markets exchange economy. Section 3 provides a characterization and a definition of the SCE. Section 4 is devoted to the statement and the proof of the proposition. In Section 5, an example is given. In Section 6, we conclude.

 

## 2. A Mixed Markets Exchange Economy

### 2.1. The Framework

The space of commodites is $\mathbb{R}_+^\ell$. The economy thus includes a finite set $\mathcal{L}$ of divisible consumption goods, indexed by $h = 1, 2, \cdots, \ell$. Let $(T, \mathcal{T}, \mu)$ be a complete measure space of agents, where $T$ denotes the set of traders, $\mathcal{T}$ a $\sigma$-algebra of all measurable subsets of $T$, and $\mu$ a real valued (with $\mu\{\varnothing\} = 0$), non negative and additive measure defined on $\mathcal{T}$. The space of agents embodies large traders, represented by atoms, and small traders, represented by an atomless continuum. So, let $T = T_0 \cup T_1$, where $T_0$ is the set of atoms, while $T_1$ is the set of small traders. The set of atoms embodies two subsets: the subset of leaders $T_0^1$ and the subset of followers $T_0^2$, so $T_0 = T_0^1 \cup T_0^2$. The measure space $\left(T_0, \mathcal{T}_{T_0}, \mu_0\right)$ is purely atomic, while the measure space $\left(T_1, \mathcal{T}_{T_1}, \mu_1\right)$ is atomless. Therefore, $\mu$ is the counting measure on $\mathcal{T}$, when restricted to $\mathcal{T}_{T_0} = \left\{E \cap T_0 : E \in \mathcal{T}\right\}$, and the Lebesgue measure, when restricted to $\mathcal{T}_{T_1} = \left\{E \cap T_1 : E \in \mathcal{T}\right\}$.

### 2.2. Assumptions

Any trader is represented by his initial endowments $\omega(t)$, his utility function $U_t(\mathbf{x})$ which represents his preferences among the commodity bundles $\mathbf{x}$, and his strategy set (see thereafter). A commodity bundle is a point in $X_t$, where $X_t \subset \mathbb{R}_+^\ell$ (a closed convex set). An assignment (of commodity bundles to traders) is an integrable function $\mathbf{x}(t)$ from $T$ to $\mathcal{R}_+^\ell$. All integrals are with respect to $t$. We consider the following set of assumptions regarding utility and endowments.

**Assumption 1.** For all $t \in T$, $U_t : X_t \subseteq \mathbb{R}_+^\ell \to \mathbb{R}$, $\mathrm{x} \mapsto U_t(\mathbf{x})$ is continuous, strictly monotone in $\mathbb{R}_{++}^\ell$ and concave for $t \in T_0$, and strictly quasi-concave on $X_t$ for $t \in T_1$. In addition, $U_t$ is measurable.

**Assumption 2.** The distribution of initial endowments among traders satisfies:

$$\boldsymbol{\omega}(t) = \left(\omega_1(t), 0, \cdots, 0\right), t \in T_0 \tag{1}$$
$$\boldsymbol{\omega}(t) = \left(0, \omega_2(t), \cdots, \omega_\ell(t)\right), t \in T_1,$$
$$\text{with} \quad \boldsymbol{\omega} \equiv \int_T \boldsymbol{\omega}(t)\mathrm{d}\mu(t) \gg 0.$$

Traders will exchange some amounts of their endowments in order to reach their final allocations. A feasible allocation is an assignment $\mathbf{x}$ for which $\int_T \mathbf{x}(t)\mathrm{d}\mu(t) = \int_T \boldsymbol{\omega}(t)\mathrm{d}\mu(t)$. The price vector is given by $\mathbf{p} \in \mathbb{R}_+^\ell$.

### 2.3. Strategy Sets

Each trader uses fractions of his initial endowment to trade them for the $\ell$ commodities. The strategic behavior then involves all the amounts of the owned good(s) that are engaged in exchange of all commodities. A strategy for a trader $t$, $t \in T$, may be represented by an $\ell \times \ell$ matrix $B = \left(b_{kl}\right)$, where $b_{kl}$ represents the amount of commodity $k$ any trader $t$ offers in exchange for commodity $l$. A strategy set for any trader $t \in T$ may be written:

$$S_t = \left\{b_{kl} \ge 0, k, l = 1, \cdots, \ell \text{ and } \forall l \sum_k b_{lk} \le \omega_l(t)\right\} \quad t \in T \tag{2}$$

The strategy set of any trader $t \in T$ is the set of all matrices $B$ satisfying $S_t$. A strategy selection for $t \in T$ is a function $B(t) = \left(\mathbf{b}_{kl}(t)\right)$, defined on $T$ such that $B(t) \in S_t$ for all $t \in T$, and such that $\mathbf{b}_{kl}(t)$, $k, l = 1, \cdots, \ell$ are real valued integrable functions on $T$. Therefore, from (1) a strategy selection for $t \in T_0^i$ is a function $B^i(t) = \left(\mathbf{b}_{kl}^i(t)\right)$ defined on $T_0^i$ such that $B^i(t) \in S_t^i$, with $S_t^i \subset S_t$ for all $t \in T_0^i$, $i = 1, 2$, and such that $\mathbf{b}_{kl}^i(t)$, $k, l = 1, \cdots, \ell$ are real valued integrable functions on $T_0^i$, $i = 1, 2$. Similarly, a strategy selection for $t \in T_1$ is a function $B^3(t) = \left(\mathbf{b}_{kl}^3(t)\right)$ defined on $T_1$ such that $B^3(t) \in S_t^3$, with $S_t^3 \subseteq S_t$ for all $t \in T_1$, and such that $\mathbf{b}_{kl}^3(t)$, $k, l = 1, \cdots, \ell$ are real valued integrable functions on $T_1$. Given $B^i(t) \in S_t^i$ (resp. $B^3(t) \in S_t^3$) for all $t \in T_0^i$ (resp. $t \in T_1$), $i = 1, 2$, one can define a strategy profile $\mathbf{B}^i$ as the aggregate matrix $\mathbf{B}^i = \left(\int_{T_0^i} \mathbf{b}_{kl}^i(t) d\mu(t)\right)$ (resp. $\mathbf{B}^3 = \left(\int_{T_1} \mathbf{b}_{kl}^3(t) d\mu(t)\right)$), $i = 1, 2$. In addition, we define $\mathbf{B}$ as the aggregate matrix $\mathbf{B} = \left(\int_T \mathbf{b}_{kl}(t) d\mu(t)\right)$. We also denote by $\mathbf{B}^i \setminus B^i(t)$ a strategy profile obtained by replacing $\mathbf{B}^i(t)$ in $\mathbf{B}^i$ by $B^i(t) \in S_t^i$, $i = 1, 2, 3$. The definition of a CE is given in Codognato [7] for mixed exchange economies. We now characterize and define the SCE.

## 3. The Stackelberg-Cournot Equilibrium

### 3.1. The SCE: Characterization

A SCE can be modeled as a sequential game in two steps, which is solved by backward induction. The characterization of the SCE relies on the strategic market game mechanism provided by Sahi and Yao [9], since it generates consistent relative prices. Thus, given a strategy profile $\mathbf{B}$, $\mathbf{p}(\mathbf{B})$, with $\mathbf{p} \in \mathbb{R}_{++}^\ell$, is the solution to:

$$\begin{aligned} &\sum_{k=1}^\ell p_k \left(\int_T \mathbf{b}_{kl}(t)\mathrm{d}\mu(t)\right) \\ &= p_l \sum_{k=1}^\ell \left(\int_T \mathbf{b}_{lk}(t)\mathrm{d}\mu(t)\right), l = 1, \cdots, \ell. \end{aligned} \tag{3}$$

These conditions stipulate that the aggregate value of all goods supplied to buy any commodity $l$ must be equal to the aggregate value of this good $l$ supplied to buy any other commodity. From Sahi and Yao [9], we know that when the matrix **B** is irreducible, the market price $\mathbf{p}(\mathbf{B})$ exists and is unique.

The strategic plan of follower $t$, $t \in T \setminus T_0^1$ is determined by two elements: he manipulates the $(\ell-1)$ consistent relative prices, and he takes as given the matrices of bids of all leaders and all other followers. We thus denote by $\mathbf{B}^2 \setminus B^2(\tau)$ (resp. $\mathbf{B}^3 \setminus B^3(\tau)$) a strategy profile which coincides with $\mathbf{B}^2$ (resp. $\mathbf{B}^3$) for all $t \in T_0^2$ (resp. $t \in T_1$) except for $t = \tau$ with $B^2(\tau) \in S_\tau^2$ (resp. $B^3(\tau) \in S_\tau^3$), $B^2(\tau) \neq \mathbf{B}^2(\tau)$ (resp. $B^3(\tau) \neq \mathbf{B}^3(\tau)$). The strategic plan of follower $\tau \in T_0^2$ (resp. $\tau \in T_1$) may be written:

$$\underset{b^2(\tau)\in S_\tau^2}{Max U_\tau}\left(\omega_1(\tau) - \sum_h \mathbf{b}_{1h}^2(\tau), \mathbf{b}_{12}^2(\tau)\frac{p_1(\mathbf{B})}{p_2(\mathbf{B})}, \cdots, \mathbf{b}_{1\ell}^2(\tau)\frac{p_1(\mathbf{B})}{p_\ell(\mathbf{B})}\right) \tag{4}$$

$$\underset{b^3(\tau)\in S_\tau^3}{Max U_\tau}\left(\sum_k \mathbf{b}_{h1}^3(\tau)\frac{p_k(\mathbf{B})}{p_1(\mathbf{B})}, \cdots \omega_l - \sum_k \boldsymbol{b}_{lk}^3(\tau) + \sum_k \mathbf{b}_{kl}^3(\tau)\frac{p_k(\mathbf{B})}{p_l(\mathbf{B})}, \cdots\right) \tag{5}$$

The solution to these programs yields the best response functions $\psi^2\left(\tau, \mathbf{B}^1, \mathbf{B}^2 \setminus B^2(\tau), \mathbf{B}^3\right)$ of follower $\tau \in T_0^2$ and $\psi^3\left(\tau, \mathbf{B}^1, \mathbf{B}^2, \mathbf{B}^3 \setminus B^3(\tau)\right)$ of follower $\tau \in T_1$. Let $\boldsymbol{\psi}^2\left(., \mathbf{B}^1, \mathbf{B}^2 \setminus B^2(\tau), \mathbf{B}^3\right)$ (resp. $\boldsymbol{\psi}^2\left(., \mathbf{B}^1, \mathbf{B}^2 \setminus B^2(\tau), \mathbf{B}^3\right)$) be the real valued integrable function on $T_0^2$ (resp. $T_1$) with values in $\mathbb{R}_+^\ell \times \mathbb{R}_+^\ell$ defined by $\boldsymbol{\psi}^2\left(., \mathbf{B}^1, \mathbf{B}^2 \setminus B^2(t), \mathbf{B}^3\right) = \psi^2\left(\tau, \mathbf{B}^1, \mathbf{B}^2 \setminus B^2(\tau), \mathbf{B}^3\right)$ for all $\tau \in T_0^2$ (resp. $\boldsymbol{\psi}^3\left(., \mathbf{B}^1, \mathbf{B}^2, \mathbf{B}^3 \setminus B^3(\tau)\right) = \psi^3\left(\tau, \mathbf{B}^1, \mathbf{B}^2, \mathbf{B}^3 \setminus B^3(\tau)\right)$ for all $\tau \in T_1$). In the symmetric equilibrium, $\mathbf{B}^2(\tau) = \mathbf{B}^2(t)$ (resp. $\mathbf{B}^3(\tau) = \mathbf{B}^3(t)$) for $\tau \neq t$, $\tau, t \in T_0^2$ (resp. $\tau \neq t$, $\tau, t \in T_1$). The resulting best responses are $\bar{\boldsymbol{\psi}}^3\left(., \mathbf{B}^1, \mathbf{B}^2(\mathbf{B}^1)\right) = \bar{\psi}^3\left(t, \mathbf{B}^1, \mathbf{B}^2(\mathbf{B}^1)\right)$ for all $t \in T_1$, and $\bar{\boldsymbol{\psi}}^2\left(., \mathbf{B}^1, \mathbf{B}^3(\mathbf{B}^1)\right) = \bar{\psi}^2\left(t, \mathbf{B}^1, \mathbf{B}^3(\mathbf{B}^1)\right)$ for all $t \in T_0^2$. Then, the system of aggregate best response functions may be written:

$$\int_{t\in T_0^2} \mathbf{B}^2(t)\mathrm{d}\mu(t) = \int_{t\in T_0^2} \bar{\boldsymbol{\psi}}^2\left(., \mathbf{B}^1, \mathbf{B}^3(\mathbf{B}^1)\right)\mathrm{d}\mu(t) \tag{6}$$

$$\int_{t\in T_1} \mathbf{B}^3(t)\mathrm{d}\mu(t) = \int_{t\in T_1} \bar{\boldsymbol{\psi}}^3\left(t, \mathbf{B}^1, \mathbf{B}^2(\mathbf{B}^1)\right)\mathrm{d}\mu(t).$$

The system of equations given by (6) determines a consistency among the followers' best response functions. We assume that the solution $\left(\mathbf{B}^2(\mathbf{B}^1), \mathbf{B}^3(\mathbf{B}^1)\right)$ exists and is unique. We denote $\mathbf{B}^1 \setminus B^1(\tau)$ a strategy profile which coincides with $\mathbf{B}^1$ for all $t \in T_0^1$ except for $t = \tau$ with $B^1(\tau) \in S_\tau^1$, $B^1(\tau) \neq \mathbf{B}^1(\tau)$. Leader $\tau \in T_0^1$, then solves the following program:

$$\max_{b^1(\tau)\in S_\tau^1} U_t\left(\omega_1(\tau) - \sum_\ell \boldsymbol{b}_{1\ell}^1(\tau), \cdots, x_L(\tau, \boldsymbol{p}(\mathbf{B}^1, \mathbf{B}^2(\mathbf{B}^1), \mathbf{B}^3(\mathbf{B}^1))\right) \tag{7}$$

The solution to this program yields the best response function $\psi^1\left(\tau, \mathbf{B}^1 \setminus B^1(\tau)\right)$ of leader $\tau \in T_0^1$. Let $\boldsymbol{\psi}^1\left(., \mathbf{B}^1 \setminus B^1(\tau)\right)$ be the real valued integrable function on $T_0^1$ with values in $\mathbb{R}_+^L$ defined by $\boldsymbol{\psi}^1\left(., \mathbf{B}^1 \setminus B^1(\tau)\right) = \psi^1\left(\tau, \mathbf{B}^1 \setminus B^1(\tau)\right)$ for all $\tau \in T_0^1$. In the symmetric SCE, $\mathbf{B}^1(\tau) = \mathbf{B}^1(t)$ for $\tau \neq t$, $\tau, t \in T_0^1$, so one gets the strategy profile $\hat{\mathbf{B}}^1$, from which we deduce $\hat{\mathbf{B}}^2$ and $\hat{\mathbf{B}}^3$. The vector of equilibrium relative prices is $\hat{\mathbf{p}}\left(\hat{\mathbf{B}}\right)$, where $\hat{\mathbf{B}} = \left(\hat{\mathbf{B}}^1, \hat{\mathbf{B}}^2, \hat{\mathbf{B}}^3\right)$. The equilibrium allocation for any $t \in T$ corresponds to the assignment:

$$\hat{\mathbf{x}}_h\left(t, \hat{\mathbf{B}}, \mathbf{p}\left(\hat{\mathbf{B}}\right)\right) = \omega_h(t) - \sum_{k=1}^{\ell} \mathbf{b}_{hk}(t) + \sum_{k=1}^{\ell} \mathbf{b}_{kh}(t)\frac{p_k\left(\hat{\mathbf{B}}\right)}{p_h\left(\hat{\mathbf{B}}\right)}, h \in \mathcal{L}. \tag{8}$$

### 3.2. The SCE: Definition

A SCE is a noncooperative equilibrium of a game where the players are the traders, the strategies are their supply decisions and the payoffs are their utility levels.

**Definition.** *(SCE) A Stackelberg-Cournot equilibrium is given by a matrix* $\hat{\mathbf{B}}$, *consistent prices* $\hat{\mathbf{p}}\left(\hat{\mathbf{B}}\right)$ *and an allocation* $\hat{\mathbf{x}}(t) = \left(\hat{\mathbf{x}}_1(t), \cdots, \hat{\mathbf{x}}_\ell(t)\right)$ *such that*:

i. $\hat{\mathbf{x}}(t) = \mathbf{x}\left(t, \hat{\mathbf{B}}(t), \hat{\mathbf{p}}\left(\hat{\mathbf{B}}\right)\right)$ for all $t \in T$

ii. $\int_T \hat{\mathbf{x}}(t)\mathrm{d}\mu(t) = \int_T \boldsymbol{\omega}(t)\mathrm{d}\mu(t)$

iii. $U_t\left(\hat{\mathbf{x}}\left(t, \hat{\mathbf{B}}^3, \hat{\mathbf{p}}\left(\hat{\mathbf{B}}\right)\right)\right) \geq U_t\left(\mathbf{x}\left(t, B^3(t), \hat{\mathbf{p}}\left(\hat{\mathbf{B}} \setminus B^3(t)\right)\right)\right)$, $\forall B^3(t) \in S_t^3$, $t \in T_1$

iv. $U_t\left(\hat{\mathbf{x}}\left(t, \hat{\mathbf{B}}^2, \hat{\mathbf{p}}\left(\hat{\mathbf{B}}\right)\right)\right) \geq U_t\left(\mathbf{x}\left(t, B^2(t), \hat{\mathbf{p}}\left(\hat{\mathbf{B}} \setminus B^2(t)\right)\right)\right)$, $\forall B^2(t) \in S_t^2$, $t \in T_0^2$

v. $U_t\left(\hat{\mathbf{x}}\left(t, \hat{\mathbf{B}}^1, \hat{\boldsymbol{p}}\left(\hat{\mathbf{B}}^1, \hat{\mathbf{B}}^2\left(\hat{\mathbf{B}}^1\right), \hat{\mathbf{B}}^3\left(\hat{\mathbf{B}}^1\right)\right)\right)\right) \geq U_t\left(\mathbf{x}(t, B^1(t), \hat{\mathbf{p}}\left(\hat{\mathbf{B}}^1 \setminus B^1(t), \hat{\mathbf{B}}^2\left(\hat{\mathbf{B}}^1 \setminus B^1(t)\right), \hat{\mathbf{B}}^3\left(\hat{\mathbf{B}}^1 \setminus B^1(t)\right)\right)\right)$, $\forall B^1(t) \in S_t^1$, $t \in T_0^1$.

## 4. Equivalence between the SCE and the CE

**Proposition.** *Assume that the preferences of the small* *traders are represented by Cobb-Douglas utility functions, and the atoms have the same endowments and utility functions. Then, the Stackelberg-Cournot and the Cournot equilibria coincide if and only if the followers' best responses functions have a zero slope at the Stackelberg-Cournot equilibrium.*

*Proof.* Consider $n$ atoms, each being indexed by $i$, $i=1,\cdots,n$ ($m$ leaders and $n-m$ followers), and a continuum of traders, each being indexed by $t$, $t\in[0,1]$. To simplify, suppose $\ell=2$. Assume (A1) and (A2):

$$U_{a_i}\left(\mathbf{x}(a_i)\right)=\Gamma\left(x_1(a_i),x_2(a_i)\right),\ \Gamma\subset\mathcal{C}^2,\ i=1,\cdots,n \quad \text{(A1)}$$

$$U_t\left(\mathbf{x}(t)\right)=x_1^{\alpha_1}(t).x_2^{\alpha_2}(t),\ \alpha_h>0,\ h=1,2,t\in[0,1],$$

$$\boldsymbol{\omega}(a_i)=\left(\omega_1,0\right),\ i=1,\cdots,n \quad \text{(A2)}$$

$$\boldsymbol{\omega}(t)=\left(0,\omega_2(t)\right),\ t\in[0,1].$$

The strategy profiles are given by:

$$S_{a_i}=\left\{b_{12}(a_i)\in\mathbb{R}^+:0\le b_{12}(a_i)\le\omega_1\right\},\ i=1,\cdots,n$$

$$S_t=\left\{b_{21}(t)\in\mathbb{R}^+:0\le b_{21}(t)\le\omega_2(t)\right\},\ t\in[0,1].$$

We first determine the SCE. Given strategy profiles $\mathbf{b}_{12}\in\prod_{i=1}^n S_{a_i}$ and $\mathbf{b}_{21}\in S_t^{[0,1]}$ the market clearing condi- tion given by (3) leads to:

$$p_2\left(\int_0^1 b_{21}(t)\mathrm{d}\mu(t)\right)=p_1\left(\sum_{i=1}^n b_{12}(a_i)\right).$$

The first strategic step consists in determining the best-response functions of follower $\{a_i\}$, $i=m+1,\cdots,n$, and follower $t$, $t\in[0,1]$, which are the solutions to:

$$b_{12}\left(a_i\right)\in\operatorname{argmax}\Gamma\left(\omega_1-b_{12}(a_i),\frac{\int_0^1 b_{21}(t)\mathrm{d}\mu(t)}{\sum_{i=1}^n b_{12}(a_i)}b_{12}(a_i)\right)$$

$$b_{21}(t)\in\operatorname{argmax}\left(\frac{\sum_{i=1}^n b_{12}(a_i)}{\int_0^1 b_{21}(\tau)d\mu(\tau)}b_{21}(t)\right)^{\alpha_1}\left(\omega_2(t)-b_{21}(t)\right)^{\alpha_2}$$

Assuming symmetry, *i.e.* $b_{21}(\tau)=b_{21}(t)$, for $\tau\neq t$, $t\in[0,1]$, one obtains:

$$b_{21}(t)=\frac{\alpha_1}{\alpha_1+\alpha_2}\omega_2(t),t\in[0,1]$$

$$-\frac{\partial\Gamma}{\partial x_1}+\frac{\sum_{-i}b_{12}(a_{-i})+\sum_{i=1}^m b_{12}(a_i)}{\left(\sum_{i=1}^n b_{12}(a_i)\right)^2}\int_0^1 b_{21}(t)\mathrm{d}\mu(t)\frac{\partial\Gamma}{\partial x_2}=0.$$

The second equation defines implicitly the best-response $\psi^i\left(\mathbf{b}^1,\mathbf{b}^2_{-i},\mathbf{b}^3\right)$ of follower $i$, $i=m+1,\cdots,n$, where $\mathbf{b}^1\in\prod_{i=1}^m S_{a_i}$ is the vector of leaders' strategies, $\mathbf{b}^2_{-i}\in\prod_{-i}S_{a_{-i}}$ is the vector of all followers strategies but $i$, while $\mathbf{b}^3\in S_t^{[0,1]}$ represents the strategy profile of the small traders. Note that from (A1) $\mathbf{b}^3$ depends neither of $\mathbf{b}^1$ nor on $\mathbf{b}^2_{-i}$. In the symmetric SCE, one gets $b_{12}\left(a_i\right)=b_{12}\left(a_{-i}\right)$ for all $i$ and all $-i$, with $i\neq -i$, so $b_{12}\left(a_i\right)=\boldsymbol{\psi}\left(\mathbf{b}^1\right)$, $i=m+1,\cdots,n$.

The second strategic step consists in determining the equilibrium strategy of any leader $i$, $i=1,\cdots,m$, whom program may then be written:

$$b_{12}\left(a_i\right)\in$$

$$\operatorname{argmax}\Gamma\left(\omega_1-b_{12}(a_i),\frac{\frac{\alpha_1}{\alpha_1+\alpha_2}\int_0^1\omega_2(t)\mathrm{d}\mu(t)}{\sum_{i=1}^m b_{12}(a_i)+\sum_{i=m+1}^n\boldsymbol{\psi}\left(\boldsymbol{b}^1\right)}b_{12}(a_i)\right)$$

At the symmetric SCE, we get $b_{12}\left(a_i\right)=b_{12}\left(a_{-i}\right)$ for all $i$ and all $-i$, with $i\neq -i$, $i=1,\cdots,m$, so:

$$-\frac{\partial\Gamma}{\partial x_1}+\left(\frac{(m-1)b_{12}\left(a_i\right)+(n-m)\boldsymbol{\psi}\left(\mathbf{b}^1\right)(1-\left|\nu^i\right|)}{\left(mb_{12}\left(a_i\right)+(n-m)\boldsymbol{\psi}\left(\mathbf{b}^1\right)\right)^2}\right)\frac{\partial\Gamma}{\partial x_2}=0 \quad \text{(C1)}$$

where $\nu^i\equiv\dfrac{\partial\boldsymbol{\psi}\left(\mathbf{b}^1\right)}{\partial b_{12}(a_i)}\dfrac{b_{12}(a_i)}{\boldsymbol{\psi}\left(\mathbf{b}^1\right)}$, $i=1,\cdots,m$, represents the elasticity of the best response function of follower $i$, $i=m+1,\cdots,n$, (correctly) perceived by $i$, $i=1,\cdots,m$. Equations (C1) yield the equilibrium strategy $\hat{b}_{12}\left(a_i\right)$ of leader $i$, $i=1,\cdots,m$, and then $\hat{b}_{12}\left(a_i\right)$, $i=m+1,\cdots,n$, $\hat{\mathbf{p}}$ and $\hat{\mathbf{x}}\left(a_i\right)$, $i=1,\cdots,n$, and $\hat{\mathbf{x}}(t)$, $t\in[0,1]$.

Let's now determine the CE. Given strategy profiles $\mathbf{b}_{12}\in\prod_{i=1}^n S_{a_i}$ and $\mathbf{b}_{21}\in S_t^{[0,1]}$ one deduces:

$$p_2\left(\int_0^1 b_{21}(t)\mathrm{d}\mu(t)\right)=p_1\left(\sum_{i=1}^n b_{12}(a_i)\right).$$

The first strategic step consists in determing the best-responses of the followers $\{a_i\}$, $i=m+1,\cdots,n$, and $t$, $t\in[0,1]$, which are the solutions to:

$$b_{12}\left(a_i\right)\in\operatorname{argmax}\Gamma\left(\omega_1-b_{12}(a_i),\frac{\int_0^1 b_{21}(t)\mathrm{d}\mu(t)}{\sum_{i=1}^n b_{12}(a_i)}b_{12}(a_i)\right)$$

$$b_{21}(t)\in\operatorname{argmax}\left(\frac{\sum_{i=1}^n b_{12}(a_i)}{\int_0^1 b_{21}(\tau)d\mu(\tau)}b_{21}(t)\right)^{\alpha_1}\left(\omega_2(t)-b_{21}(t)\right)^{\alpha_2}$$

Assuming symmetry, *i.e.* $b_{21}(\tau)=b_{21}(t)$, $t\in[0,1]$, one gets:

$$b_{21}(t)=\frac{\alpha_1}{\alpha_1+\alpha_2}\omega_2(t),t\in[0,1]$$

 

and:

$$-\frac{\partial\Gamma}{\partial x_1}+\frac{\sum_{-i}b_{12}(a_{-i})+\sum_{i=1}^{m}b_{12}(a_i)}{\left(\sum_{i=1}^{n}b_{12}(a_i)\right)^2}\int_0^1 b_{21}(t)\mathrm{d}\mu(t)\frac{\partial\Gamma}{\partial x_2}=0.$$

The second equation defines implicitly the best-response of trader $\{a_i\}$, $i=1,\cdots,n$. Assuming symmetry among the atoms $b_{12}(a_i)=b_{12}(a_{-i})$ for all $i$ and all $-i$, with $i\neq -i$, $i=1,\cdots,n$, one deduces:

$$-\frac{\partial\Gamma}{\partial x_1}+\left(\frac{n-1}{n}\frac{1}{\bar{b}_{12}(a_i)}\right)\frac{\partial\Gamma}{\partial x_2}=0 \qquad \text{(C2)}$$

where $\bar{b}_{12}(a_i)$ represents the equilibrium strategy of atom $\{a_i\}$, $i=1,\cdots,n$.

If for any $i$, $i=m+1,\cdots,n$,

$\hat{b}_{12}(a_i)=\boldsymbol{\psi}\left(\hat{b}_{12}(a_1),\cdots,\hat{b}_{12}(a_i),\cdots,\hat{b}_{12}(a_m)\right)=\bar{b}_{12}(a_i)$,

$i=1,\cdots,n$, then $\nu^i=0$, $i=1,\cdots,m$. In addition, if $\nu^i=0$, $i=1,\cdots,m$, then (C1) and (C2) are equivalent. So, one concludes that (C1) and (C2) lead to the same equilibrium strategies, prices and allocations. QED.

The equivalence result stipulates that Stackelberg competition in interrelated markets can lead to Cournot outcomes. Provided symmetry assumptions regarding the primitives, this equivalence holds if and only if the consistent conjectures are zero. Why? Any leader rationally expects that a change in his strategy will elicit no reaction from the followers. Consequently, it mimics the case where the traders take the decisions of their rivals as given when optimizing, thereby behaving as if they played a simultaneous move game (with the belief that their rivals behave following a Cournotian reaction function). In such a case, the value of the elasticity of the best response functions coincides with the true slope of the best response functions (here zero): conjectures are fulfilled and are thus consistent. This means that the strategies are neither substitutes nor complements in equilibrium. This result may be explained as follows. The shape of the reaction functions and their slopes at equilibrium depend notably on the market demand function. The Cobb-Douglas specification leads to an isoelastic aggregate market demand function (constant unitary price elasticity). So, when all atoms have the same endowments and preferences, their market powers are equal, which implies that their (Cournotian) equilibrium strategies are identical. Our proposition extends a result obtained in partial equilibrium by Julien [17] to cover a general equilibrium framework. In addition, it spreads the result obtained in Julien [16] to cover mixed markets exchange in which *all* traders behave strategically.

## 5. An Example

Consider the case for which $\ell=2$. The price system is $\mathbf{p}=(p_1,p_2)$. The economy embodies two atoms $a_1$ (the leader) and $a_2$ (the follower), each of measure $\mu(\{a_i\})=1$, $i=1,2$, and an atomless continuum of traders represented by the unit interval $[0,1]$, with the Lebesgue measure $\mu(t)=0$, $t\in[0,1]$.

Assume the following specification for endowments:

$$\boldsymbol{\omega}(a_i)=(1,0), i=1,2 \qquad (9)$$

$$\boldsymbol{\omega}(t)=(0,1), t\in[0,1]$$

The preferences of any trader are represented by the following utility functions:

$$U_{ai}(\mathbf{x}(a_i))=\beta x_1(a_i)+x_2(a_i), \beta\in(0,1), i=1,2 \qquad (10)$$

$$U_t(\mathbf{x}(t))=x_1^{\alpha_1}(t)\cdot x_2^{\alpha_2}(t), \alpha_h>0, h=1,2, t\in[0,1]$$

The strategy sets are given by:

$$S_{a_i}=\{b_{12}(a_i)\in\mathbb{R}_+:0\le b_{12}(a_i)\le 1\}, i=1,2 \qquad (11)$$

$$S_t=\{b_{21}(t)\in\mathbb{R}_+:0\le b_{21}(t)\le 1\}, t\in[0,1].$$

### 5.1. The SCE

Given $\mathbf{b}_{12}\in S_{a_1}\times S_{a_2}$ and $\mathbf{b}_{21}(t)\in S_t^{[0,1]}$, the Sahi and Yao [9] price mechanism yields $p_1(b_{12}(a_1)+b_{12}(a_2))=p_2\int_0^1 b_{21}(t)\mathrm{d}\mu(t)$, so one deduces the price system:

$$\mathbf{p}=\left(\frac{\int_0^1 b_{21}(t)\mathrm{d}\mu(t)}{b_{12}(a_1)+b_{12}(a_2)},1\right) \qquad (12)$$

The best-responses $b_{12}(a_2)$ and $b_{21}(t)$ of the followers $\{a_2\}$ and $t\in[0,1]$ are solutions to the following system of equations:

$$\begin{aligned} b_{12}(a_2)\in \operatorname{argmax} &\ \beta(1-b_{12}(a_2)) \\ &+\frac{\int_0^1 b_{21}(t)\mathrm{d}\mu(t)}{b_{12}(a_1)+b_{12}(a_2)}b_{12}(a_2) \end{aligned} \qquad (13)$$

$$b_{21}(t)\in \operatorname{argmax}\left(\frac{b_{12}(a_1)+b_{12}(a_2)}{\int_0^1 b_{21}(\tau)d\mu(\tau)}b_{21}(t)\right)^{\alpha_1}(1-b_{21}(t))^{\alpha_2}$$

Assuming symmetry among the small traders, *i.e.* $b_{21}(\tau)=b_{21}(t)$, $\tau\neq t$, $t\in[0,1]$, one gets the best-responses functions:

$$b_{12}(a_2)=-b_{12}(a_1)+\sqrt{\frac{\int_0^1 b_{21}(t)\mathrm{d}\mu(t)}{\beta}b_{12}(a_1)} \qquad (14)$$

$$b_{21}(t)=\frac{\alpha_1}{\alpha_1+\alpha_2},t\in[0,1]$$

The former best response satisfies: $\frac{\mathrm{d}b_{12}(a_2)}{\mathrm{d}b_{12}(a_1)}>0$ when $b_{12}(a_1)<\frac{\alpha_1}{4\beta(\alpha_1+\alpha_2)}$, reflecting strategic complementarities, while $\frac{\mathrm{d}b_{12}(a_2)}{\mathrm{d}b_{12}(a_1)}<0$ when $b_{12}(a_1)>\frac{\alpha_1}{4\beta(\alpha_1+\alpha_2)}$, reflecting strategic substituabilities. In addition, $\frac{\mathrm{d}^2 b_{12}(a_2)}{\mathrm{d}\left(b_{12}(a_1)\right)^2}<0$.

The program of the leader becomes:

$$b_{12}(a_1)\in\operatorname{argmax}\beta\left(1-b_{12}(a_1)\right)+\frac{\frac{\alpha_1}{\alpha_1+\alpha_2}\int_0^1\mathrm{d}\mu(t)}{b_{12}(a_1)+b_{12}(a_2)\left[b_{12}(a_1)\right]}b_{12}(a_1). \tag{15}$$

Little algebra lead to the SCE strategy for the leader:

$$\hat{b}_{12}(a_1)=\frac{\alpha_1}{(\alpha_1+\alpha_2)\beta}\frac{1-\nu}{(2-\nu)^2} \tag{16}$$

From (14), the equilibrium strategies of the followers are given by:

$$\hat{b}_{12}(a_2)=\frac{\alpha_1}{(\alpha_1+\alpha_2)\beta}\varphi(\nu) \tag{17}$$

$$\hat{b}_{21}(t)=\frac{\alpha_1}{\alpha_1+\alpha_2},t\in[0,1]$$

where $\varphi(\nu)\equiv\sqrt{\frac{1-\nu}{(2-\nu)^2}}-\frac{1-\nu}{(2-\nu)^2}$.

The SCE price system and allocations are given by:

$$\hat{\mathbf{p}}=\left(\beta\sqrt{\frac{(2-\nu)^2}{1-\nu}},1\right) \tag{18}$$

$$\hat{\mathbf{x}}(a_1,\hat{\mathbf{p}})=\left(1-\frac{\alpha_1}{(\alpha_1+\alpha_2)\beta}\frac{1-\nu}{(2-\nu)^2},\frac{\alpha_1}{\alpha_1+\alpha_2}\sqrt{\frac{1-\nu}{(2-\nu)^2}}\right) \tag{19}$$

$$\hat{\mathbf{x}}(a_2,\hat{\mathbf{p}})=\left(1-\frac{\alpha_1}{(\alpha_1+\alpha_2)\beta}\varphi(\nu),\frac{\alpha_1}{\alpha_1+\alpha_2}\sqrt{\frac{1-\nu}{(2-\nu)^2}}\varphi(\nu)\right)$$

$$\hat{\mathbf{x}}(t,\hat{\mathbf{p}})=\left(\frac{\alpha_1}{2\beta(\alpha_1+\alpha_2)}\sqrt{\frac{1-\nu}{(2-\nu)^2}},\frac{\alpha_2}{\alpha_1+\alpha_2}\right),t\in[0,1]$$

## 5.2. The CE

Given $\mathbf{b}_{12}\in S_{a_1}\times S_{a_1}$ and $\mathbf{b}_{21}(t)\in S_t^{[0,1]}$, the same price mechanism yields:

$$\frac{p_1}{p_2}=\frac{\int_0^1 b_{21}(t)\mathrm{d}\mu(t)}{b_{12}(a_1)+b_{12}(a_2)} \tag{20}$$

The best-response functions of any atom $\{a_i\}$, $i=1,2$ and of any trader $t\in[0,1]$ are the solutions to:

$$b_{12}(a_i)\in\operatorname{argmax}\beta\left(1-b_{12}(a_i)\right)+\frac{\int_0^1 b_{21}(t)\mathrm{d}\mu(t)}{b_{12}(a_1)+b_{12}(a_2)}b_{12}(a_i),i=1,2 \tag{21}$$

$$b_{21}(t)\in\operatorname{argmax}\left(\frac{b_{12}(a_1)+b_{12}(a_2)}{\int_0^1 b_{21}(\tau)\mathrm{d}\mu(\tau)}b_{21}(t)\right)^{\alpha_1}\left(1-b_{21}(t)\right)^{\alpha_2}$$

Little algebra lead to the CE strategies:

$$\bar{b}_{12}(a_i)=\frac{\alpha_1}{4\beta(\alpha_1+\alpha_2)},i=1,2 \tag{22}$$

$$\bar{b}_{21}(t)=\frac{\alpha_1}{\alpha_1+\alpha_2},t\in[0,1]$$

The CE equilibrium price system and allocations are then:

$$\bar{\mathbf{p}}=(2\beta,1) \tag{23}$$

$$\bar{\mathbf{x}}(a_i,\bar{\mathbf{p}})=\left(\frac{\alpha_1(4\beta-1)+4\alpha_2\beta}{4\beta(\alpha_1+\alpha_2)},\frac{\alpha_1}{2(\alpha_1+\alpha_2)}\right),\quad i=1,2 \tag{24}$$

$$\bar{\mathbf{x}}(t,\bar{\mathbf{p}})=\left(\frac{\alpha_1}{2\beta(\alpha_1+\alpha_2)},\frac{\alpha_2}{\alpha_1+\alpha_2}\right),t\in[0,1]$$

Consider (16)-(19) with (22)-(24). The SCE and the CE relative price and allocations coincide if and only if $\nu=0$. In addition, note that (15) may be written as

$$b_{12}(a_1)\in\operatorname{argmax}\beta\left(1-b_{12}(a_1)\right)+\sqrt{\frac{\alpha_1}{\alpha_1+\alpha_2}\beta b_{12}(a_1)},$$

which leads to $\hat{b}_{12}(a_i)=\frac{\alpha_1}{4\beta(\alpha_1+\alpha_2)}$, $i=1,2$. From (14), one gets

$\nu=-1+\frac{1}{2}\sqrt{\frac{\alpha_1}{\beta(\alpha_1+\alpha_2)}}\left(\frac{1}{4\beta}\frac{\alpha_1}{\alpha_1+\alpha_2}\right)^{-\frac{1}{2}}=0$, so zero conjectures are consistent.

## 6. Conclusion

In this paper, we consider a general equilibrium concept - the Stackelberg-Cournot equilibrium—where all traders behave strategically. One side of the market includes negligible traders, while the other side embodies atoms. In this economy, the strategic interactions recover from sequential decisions.

The framework used belongs to the class of mixed markets exchange models. Traders have not the same "weight": this idea is captured with a mixed measure space of traders. Such a specification notably enables to model asymmetries in the working of market power in interrelated markets. It also gives some insights regarding the consequences of market power in a general equilibrium perspective. Finally, it facilitates comparisons between general equilibrium concepts in economies where all agents behave strategically.

Within this framework, it is shown that the set of Stackelberg-Cournot equilibria and the set of Cournot equilibria can have a nonempty intersection. When the preferences of the small traders are represented by Cobb-Douglas utility functions, and when the atoms have the same endowments and utility functions, the SCE and the CE coincide if and only if the followers' best responses functions have a zero slope at the SCE. Provided conjectures of atoms are consistent, the traders behave as if they played a simultaneous move game. So, the equivalence result stems from consistent conjectures formed by leaders.

## 7. Acknowledgements

I am grateful to an anonymous referee for her/his remarks and suggestions. All remaining deficiencies are mine.

## REFERENCES

[1] J. J. Gabszewicz and J. P. Vial, "Oligopoly 'à la Cournot' in General Equilibrium Analysis," *Journal of Economic Theory*, Vol. 4, No. 3, 1972, pp. 381-400. doi:10.1016/0022-0531(72)90129-9

[2] G. Codognato and J. J. Gabszewicz, "Equilibres de Cournot-Walras dans une Économie d'Échange," *Revue Economique*, Vol. 42, No. 6, 1991, pp. 1013-1026. doi:10.2307/3502021

[3] G. Codognato and J. J. Gabszewicz, "Cournot-Walras Equilibria in Markets with a Continuum of Traders," *Economic Theory*, Vol. 3, No. 3, 1993, pp. 453-464. doi:10.1007/BF01209696

[4] J. J. Gabszewicz and P. Michel, "Oligopoly Equilibria in Exchange Economies," In: B. C. Eaton and R. G. Harris, Eds., *Trade, Technology and Economics. Essays in Honour of R.G. Lipsey*, Edward-Elgar, Cheltenham, 1997, pp. 217-240.

[5] F. Busetto, G. Codognato and S. Ghosal, "Noncooperative Oligopoly in Markets with a Continuum of Traders," *Games and Economic Behavior*, Vol. 72, No. 1, 2011, pp. 38-45. doi:10.1016/j.geb.2010.06.009

[6] F. Busetto, G. Codognato and S. Ghosal, "Cournot-Walras Equilibrium as a Subgame Perfect Equilibrium," *International Journal of Game Theory*, Vol. 37, No. 3, 2008, pp. 371-386. doi:10.1007/s00182-008-0123-8

[7] L. Shapley and M. Shubik, "Trade Using One Commodity as a Means of Payment," *Journal of Political Economy*, Vol. 85, No. 5, 1977, pp. 937-967. doi:10.1086/260616

[8] P. Dubey and M. Shubik, "The Non Cooperative Equilibria of a Closed Trading Economy with Market Supply and Bidding Strategies," *Journal of Economic Theory*, Vol. 17, No. 1, 1978, pp. 1-20. doi:10.1016/0022-0531(78)90119-9

[9] S. Sahi and S. Yao, "The Noncooperative Equilibria of a Trading Economy with Complete Markets and Consistent Prices," *Journal of Mathematical Economics*, Vol. 18, No. 4, 1989, pp. 325-346. doi:10.1016/0304-4068(89)90010-4

[10] R. Amir, S. Sahi, M. Shubik and S. Yao, "A Strategic Market Game with Complete Markets," *Journal of Economic Theory*, Vol. 51, No. 1, 1990, pp. 126-143. doi:10.1016/0022-0531(90)90054-N

[11] G. Codognato, "Cournot-Walras Equilibria in Mixed Markets: A Comparison," *Economic Theory*, Vol. 5, No. 2, 1995, pp. 361-370. doi:10.1007/BF01215210

[12] G. Codognato, "Cournot-Nash Equilibria in Limit Exchange Economies with Complete Markets: A Comparison between Two Models," *Games and Economic Behavior*, Vol. 31, No. 1, 2000, pp. 136-146. doi:10.1006/game.1999.0735

[13] L. A. Julien and F. Tricou, "Oligopoly Equilibria '*à la* Stackelberg' in Pure Exchange Economies," *Louvain Economic Review*, Vol. 76, No. 2, 2010, pp. 239-252.

[14] L. A. Julien and F. Tricou, "Market Price Mechanisms and Stackelberg General Equilibria: An Example," *Bulletin of Economic Research*, Vol. 64, No. 2, 2012, pp. 175-194. doi:10.1111/j.1467-8586.2010.00362.x

[15] B. Shitovitz, "Oligopoly in Markets with a Continuum of Traders," *Econometrica*, Vol. 41, No. 3, 1973, pp. 467-501. doi:10.2307/1913371

[16] L. A. Julien, "Stackelberg-Walras and Cournot-Walras Equilibria in Mixed Markets: A Comparison," *Theoretical Economics Letters*, Vol. 2, No. 1, 2012, pp. 69-74. doi:10.4236/tel.2012.21013

[17] L. A. Julien, "A Note on Stackelberg Competition," *Journal of Economics*, Vol. 103, No. 2, 2011, pp. 171-187. doi:10.1007/s00712-010-0187-3

 