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N-Fold Darboux Transformation for a Nonlinear Evolution Equation

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ABSTRACT

In this paper, we present a *N*-fold Darboux transformation (DT) for a nonlinear evolution equation. Comparing with other types of DTs, we give the relationship between new solutions and the trivial solution. The DT presented in this paper is more direct and universal to obtain explicit solutions.

Keywords: Darboux Transformation; Derivative Nonlinear Schrödinger Equation; Explicit Solution

1. Introduction

There are many methods to obtain explicit solutions of nonlinear evolution equations, such as the inverse scattering transformation (IST) [1], Bäcklund transformation [2], Darboux transformation [3], Painlevé analysis method [4], etc. [5-8]. Among these methods, DT is a useful method which is a special gauge transformation transforming a linear problem into itself. Many different forms of DTs have been considered in [9-19]. Generally, there are two kinds for the DTs of Lax pairs. One is to give 1-fold form Darboux matrix and obtain the *N*-th solution by iterating *N* times [14-16]. The other is to directly construct *N*-fold form Darboux matrix and obtain the *N*-th solution without iteration [17-19].

In this paper, according to the form of a Lax pair and the properties of solutions, we consider the relations between the above two kinds of DTs by considering the Lax pair given in [16]. We construct the *N*-fold DT for the Lax pair, which simplifies the complicated process of iterating 1-fold form Darboux matrix and gives the relationship between the trivial solution and the general solutions. The main idea of this paper comes from the reduction of DT in [20] and the determinant representation of Darboux transformation for AKNS system in [21].

In Section 2, from a Lax pair in [22], we deduce several nonlinear evolution equations. For a special case, we give the *N*-fold DT. In Section 3, we give the *N*-fold Darboux matrix and the relationship between new and old potentials. In Section 4, we obtain exact solutions of the nonlinear evolution equations and discuss the properties of these solutions. In Section 5, we make our conclusion.

2. Soliton Equations

We consider the isospectral problem introduced in [22]

$$\Phi_x = U\Phi, U = \begin{pmatrix} \lambda q & \lambda^2 + \lambda r + \mu\left(q^2 + r^2\right) \\ -\lambda^2 + \lambda r - \mu\left(q^2 + r^2\right) & -\lambda q \end{pmatrix} \tag{2.1}$$

and the auxiliary spectral problem

$$\Phi_t = V\Phi,\ V = \begin{pmatrix} V_{11}(\lambda) & V_{12}(\lambda) \\ -V_{12}(-\lambda) & V_{11}(-\lambda) \end{pmatrix}, \tag{2.2}$$

where

$$V_{11}(\lambda) = \lambda^3 q + \lambda\left(-\frac{1}{2}r_x + \left(\frac{1}{2} - \mu\right)q\left(q^2 + r^2\right)\right),$$

$$V_{12}(\lambda) = \lambda^4 + \lambda^3 r + \frac{1}{2}\lambda^2\left(q^2 + r^2\right) + \lambda\left(\frac{1}{2}q_x + \left(\frac{1}{2} - \mu\right)r\left(q^2 + r^2\right)\right) + \mu\left(rq_x - qr_x\right) + \left(\frac{3}{4}\mu - 2\mu^2\right)\left(q^2 + r^2\right)^2.$$

By using of the zero curvature equation

$$U_t - V_x + UV - VU = 0,$$

We get a new nonlinear evolution equation

$$\begin{cases} q_t = -\frac{1}{2} r_{xx} + \left(\frac{1}{2} - \mu\right)\left(q\left(q^2+r^2\right)\right)_x - \mu q_x\left(q^2+r^2\right) + 2\mu r\left(rq_x - qr_x\right) + \left(\frac{1}{2}\mu - 2\mu^2\right) r\left(q^2+r^2\right)^2, \\ r_t = \frac{1}{2} q_{xx} + \left(\frac{1}{2} - \mu\right)\left(r\left(q^2+r^2\right)\right)_x - \mu r_x\left(q^2+r^2\right) - 2\mu q\left(rq_x - qr_x\right) - \left(\frac{1}{2}\mu - 2\mu^2\right) q\left(q^2+r^2\right)^2. \end{cases} \tag{2.3}$$

Letting $u = r + iq$, the above system reduces to a generalized derivative nonlinear SchrÖdinger (GDNS) equation

$$iu_t = \frac{1}{2} u_{xx} + \left(\frac{1}{2} - \mu\right)\left(u|u|^2\right)_x - i\mu|u|^2 u_x + 2\mu u \operatorname{Im}\left(uu_x^*\right) - \left(\frac{1}{2}\mu - 2\mu^2\right) u|u|^4. \tag{2.4}$$

If $\mu = 0$, the above equation reduces to the derivative nonlinear SchrÖdinger (DNS) equation which describes the propagation of circular polarized nonlinear Alfvén waves in plasmas [23].

We consider the Darboux transformation of the Lax pair (2.1) and (2.2). In this paper, we find the Darboux transformation for the case of $\mu = 1/2$. In this case, from (2.3), we have

$$\begin{cases} q_t = -\frac{1}{2} r_{xx} - \frac{1}{2} q_x\left(q^2+r^2\right) + r\left(rq_x - qr_x\right) - \frac{1}{4} r\left(q^2+r^2\right)^2, \\ r_t = \frac{1}{2} q_{xx} - \frac{1}{2} r_x\left(q^2+r^2\right) - q\left(rq_x - qr_x\right) + \frac{1}{4} q\left(q^2+r^2\right)^2, \end{cases} \tag{2.5}$$

and when $u = r + iq$, the above system becomes

$$iu_t = \frac{1}{2} u_{xx} - \frac{i}{2}|u|^2 u_x + u \operatorname{Im}\left(uu_x^*\right) + \frac{1}{4} u|u|^4, \tag{2.6}$$

In which u^* means the conjugate of *u*.

In [16], 1-fold Darboux matrix has been given and by applying it *N* times, a series of explicit solutions are obtained. Also, the relationship between $\left(q[N-1], r[N-1]\right)$ and $\left(q[N], r[N]\right)$ is given. By using of this relationship, if we want to get $\left(q[N], r[N]\right)$, we have to deduce $\left(q[i], r[i]\right)$ for $i = 1, \cdots N-1$. This is very complicated. The purpose of this paper is to improve this process and obtain the relationship between the trivial solution $\left(q[0], r[0]\right)$ and the new solution $\left(q[N], r[N]\right)$ by constructing *N*-fold Darboux matrix.

3. Darboux Transformation

We first introduce a transformation

$$\bar{\Phi} = T\Phi \tag{3.1}$$

for the spectral problem (2.1), where *T* satisfies

$$T_x + TU - \bar{U}T = 0. \tag{3.2}$$

Note that $\bar{U}$ and *U* have the same form except that *q* and *r* are replaced by $\bar{q}$ and $\bar{r}$, respectively, in (2.1).

According to the forms of (2.1) and (2.2), we find that if $(\alpha, \beta)^T$ is a solution with $\lambda = \lambda_1$, $(\beta, -\alpha)^T$ is a solution with $\lambda = -\lambda_1$. Then we suppose that *T* has the following form

$$T = \begin{pmatrix} \lambda^N + \sum_{k=0}^{N-1} (-1)^{N-k} a_k \lambda^k & \sum_{k=0}^{N-1} b_k \lambda^k \\ \sum_{k=0}^{N-1} (-1)^{N-k-1} b_k \lambda^k & \lambda^N + \sum_{k=0}^{N-1} a_k \lambda^k \end{pmatrix}, \tag{3.3}$$

where a_k and b_k $(0 \le k \le N-1)$ are functions of *x* and *t*.

Let

$$\Phi_+\left(\lambda_j\right) = \left(\varphi_1\left(\lambda_j\right), \varphi_2\left(\lambda_j\right)\right)^T$$

and

$$\Psi_+\left(\lambda_j\right) = \left(\psi_1\left(\lambda_j\right), \psi_2\left(\lambda_j\right)\right)^T$$

be two basic solutions of (2.1) and (2.2) with $\lambda = \lambda_j$. Then

$$\Phi_-\left(\lambda_j\right) = \left(\varphi_2\left(\lambda_j\right), -\varphi_1\left(\lambda_j\right)\right)^T$$

and

$$\Psi_-\left(\lambda_j\right) = \left(\psi_2\left(\lambda_j\right), -\psi_1\left(\lambda_j\right)\right)^T$$

are two basic solutions of (2.1) and (2.2) with $\lambda = -\lambda_j$ $\left(j = 1, 2, \cdots, N\right)$. From (3.3), we find that

$$\det T = \prod_{j=1}^{N}\left(\lambda^2 - \lambda_j^2\right), \tag{3.4}$$

which means that λ_j and $-\lambda_j$ are roots of $\det T = 0$.

From (3.1), we find that a_k and b_k satisfy the following linear algebraic system

$$\begin{cases} \sum_{k=0}^{N-1}\left((-1)^{N-k} a_k + \sigma_j b_k\right)\lambda_j^k = -\lambda_j^N, \\ \sum_{k=0}^{N-1}\left(\sigma_j a_k + (-1)^{N-k-1} b_k\right)\lambda_j^k = -\lambda_j^N \sigma_j, \end{cases} \tag{3.5}$$

where

$$\sigma_j = \frac{l_{1j}\varphi_2(\lambda_j) - l_{2j}\psi_2(\lambda_j)}{l_{1j}\varphi_1(\lambda_j) - l_{2j}\psi_1(\lambda_j)} \tag{3.6}$$

with l_{1j} and l_{2j} $(j = 1, 2, \cdots, N)$ are constants.

Proposition 3.1 Through the transformation (3.1) and (3.2), $\Phi_x = U\Phi$ becomes $\bar{\Phi}_x = \bar{U}\bar{\Phi}$ with

$$\bar{U} = \begin{pmatrix} \lambda\bar{q} & \lambda^2 + \lambda\bar{r} + \mu\left(\bar{q}^2 + \bar{r}^2\right) \\ -\lambda^2 + \lambda\bar{r} - \mu\left(\bar{q}^2 + \bar{r}^2\right) & -\lambda\bar{q} \end{pmatrix} \tag{3.7}$$

and

$$\mu = 1/2, \bar{q} = q - 2b_{N-1}, \bar{r} = r - 2a_{N-1}. \tag{3.8}$$

Proof. Let $T^{-1} = T^*/\det T$ (T^* means the adjoint matrix of T) and

$$(T_x + TU)T^* = \begin{pmatrix} f_{11}(\lambda) & f_{12}(\lambda) \\ f_{21}(\lambda) & f_{22}(\lambda) \end{pmatrix}. \tag{3.9}$$

It is easy to see that $f_{11}(\lambda)$ and $f_{22}(\lambda)$ are $(2N + 1)$th-order polynomials of λ and $f_{22}(-\lambda) = f_{11}(\lambda)$. Also, $f_{12}(\lambda)$ and $f_{21}(\lambda)$ are $(2N + 2)$th-order polynomials of λ and $f_{21}(-\lambda) = -f_{12}(\lambda)$. When $\lambda = \lambda_j$ $(1 \le j \le N)$, together with (2.1) and (3.6), we obtain a Riccati equation

$$\begin{aligned} \sigma_{jx} = &-\lambda_j^2 + \lambda_j r - \mu\left(q^2 + r^2\right) - 2\lambda_j q\sigma_j \\ &-\left(\lambda_j^2 + \lambda_j r + \mu\left(q^2 + r^2\right)\right)\sigma_j^2. \end{aligned} \tag{3.10}$$

After calculation, we find that all λ_j $(1 \le j \le 2N-1)$ are roots of $f_{ij}(\lambda) = 0$ $(i, j = 1, 2)$. Hence we have

$$(T_x + TU)T^* = (\det T)P(\lambda), \tag{3.11}$$

where

$$P(\lambda) = \begin{pmatrix} P_{11}^{(1)}\lambda + P_{11}^{(0)} & P_{12}^{(2)}\lambda^2 + P_{12}^{(1)}\lambda + P_{12}^{(0)} \\ -P_{12}^{(2)}\lambda^2 + P_{12}^{(1)}\lambda - P_{12}^{(0)} & -P_{11}^{(1)}\lambda + P_{11}^{(0)} \end{pmatrix}$$

and $P_{ij}^{(k)}$ $(i, j = 1, 2, k = 0, 1, 2)$ are independent of λ. Then we have

$$T_x + TU = P(\lambda)T. \tag{3.12}$$

Comparing the coefficients, we have

$$\lambda^{N+2} : P_{12}^{(2)} = 1;$$

$$\lambda^{N+1} : P_{11}^{(1)} = q - 2b_{N\text{-}1} = \bar{q},\ P_{12}^{(1)} = r - 2a_{N\text{-}1} = \bar{r};$$

$$\begin{aligned} \lambda^N\ : P_{11}^{(0)} &= qa_{N\text{-}1} + P_{11}^{(1)}a_{N\text{-}1} - b_{N\text{-}2} + P_{12}^{(2)}b_{N\text{-}2} \\ &\quad + rb_{N\text{-}1} - P_{12}^{(1)}b_{N\text{-}1} = 0; \end{aligned}$$

$$P_{12}^{(0)} = \mu q^2 + \mu r^2 - ra_{N\text{-}1} - P_{12}^{(1)}a_{N\text{-}1} - qb_{N\text{-}1} - P_{11}^{(1)}b_{N\text{-}1}.$$

We find that $\bar{U} = P(\lambda)$, that is $P_{12}^{(0)} = \mu\left(\bar{q}^2 + \bar{r}^2\right)$ if and only if $\mu = 1/2$. The proof is complete.

Remark 3.1 The proof of Proposition 3.1 is similar to that in [18]. Due to the property of basic solutions of the Lax pair (2.1) and (2.2), the proof here is more tricky.

Proposition 3.2 From the transformation (3.1) and $T_t + TV - \bar{V}T = 0$ together with $\mu = 1/2$, $\bar{q} = q - 2b_{N\text{-}1}$, $\bar{r} = r - 2a_{N\text{-}1}$. $\Phi_t = V\Phi$ is transformed into $\bar{\Phi}_t = \bar{V}\bar{\Phi}$, where $\bar{V}$ has the same form as V with q and r replaced by $\bar{q}$ and $\bar{r}$, respectively.

Remark 3.2 The proof of Proposition 3.2 is similar to Proposition 3.1 and we omit it here for brevity.

According to Proposition 3.1 and 3.2, from the zero curvature equation $\bar{U}_t - \bar{V}_x + \bar{U}\bar{V} - \bar{V}\bar{U} = 0$, we find that both $(\bar{q}, \bar{r})$ and (q, r) satisfy (2.5). The transformation (3.1) and (3.8) is called the Darboux transformation of (2.5). Then we have the following theorem.

Theorem 3.1 The solution (q, r) of (2.5) are mapped into the new solution $(\bar{q}, \bar{r})$ through the Darboux transformation (3.1) and (3.8), where a_{N-1} and b_{N-1} are determined by (3.5). And $\bar{u} = \bar{r} + i\bar{q}$ is a new solution of (2.6).

Proof. On one hand, according to the Proposition 3.1 and 3.2, together with the transformation (3.1), we know that (q, r) is a solution of (2.5), and $(\bar{q}, \bar{r})$ is another solution of (2.5). On the other hand, if (q, r) is a solution of (2.5), then $u = r + iq$ is a solution of (2.6). So, $\bar{u} = \bar{r} + i\bar{q}$ is a new solution of (2.6).

4. Explicit Solutions

In this section, we apply the Darboux transformation (3.1) and (3.8) to get explicit solutions of (2.5).

To compare with the solutions obtained in [16], we start from the same trivial solution (q[0], r[0]) = (0,0) and select basic solutions

$$\Phi(\lambda_j) = \left(c_{1j}e^{\theta_j} + c_{2j}e^{-\theta_j}, -\mathrm{i}\left(c_{1j}e^{\theta_j} - c_{2j}e^{-\theta_j}\right)\right)^T \tag{4.1}$$

and

$$\Psi(\lambda_j) = \left(c_{1j}e^{\theta_j} - c_{2j}e^{-\theta_j}, -\mathrm{i}\left(c_{1j}e^{\theta_j} + c_{2j}e^{-\theta_j}\right)\right)^T, \tag{4.2}$$

where $\lambda_j = r_j e^{\frac{\pi}{4}i}, \theta_j = r_j^2 x + i r_j^4 t$ and c_{1j}, c_{2j}, r_j are constants. Then

$$\sigma_j = -i\frac{\left(l_{1j} - l_{2j}\right)c_{1j}e^{\theta_j} - \left(l_{1j} + l_{2j}\right)c_{2j}e^{-\theta_j}}{\left(l_{1j} - l_{2j}\right)c_{1j}e^{\theta_j} + \left(l_{1j} + l_{2j}\right)c_{2j}e^{-\theta_j}}, \tag{4.3}$$

$$j = 1, 2, \cdots, N.$$

From the linear algebraic system (3.5), we have

$$a_{N-1}=\frac{\left|\Delta_{a_{N-1}}\right|}{\left|\Delta_{2N}\right|},\quad b_{N-1}=\frac{\left|\Delta_{b_{N-1}}\right|}{\left|\Delta_{2N}\right|},\tag{4.4}$$

where

$$\Delta_{2N}=\begin{pmatrix} -\lambda_1^{N-1} & \lambda_1^{N-2} & \cdots & (-1)^N & \lambda_1^{N-1}\sigma_1 & \lambda_1^{N-2}\sigma_1 & \cdots & \sigma_1 \\ -\lambda_2^{N-1} & \lambda_2^{N-2} & \cdots & (-1)^N & \lambda_2^{N-1}\sigma_2 & \lambda_2^{N-2}\sigma_2 & \cdots & \sigma_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -\lambda_N^{N-1} & \lambda_N^{N-2} & \cdots & (-1)^N & \lambda_N^{N-1}\sigma_N & \lambda_N^{N-2}\sigma_N & \cdots & \sigma_N \\ \lambda_1^{N-1}\sigma_1 & \lambda_1^{N-2}\sigma_1 & \cdots & \sigma_1 & \lambda_1^{N-1} & -\lambda_1^{N-2} & \cdots & (-1)^{N-1} \\ \lambda_2^{N-1}\sigma_2 & \lambda_2^{N-2}\sigma_2 & \cdots & \sigma_2 & \lambda_2^{N-1} & -\lambda_2^{N-2} & \cdots & (-1)^{N-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda_N^{N-1}\sigma_N & \lambda_N^{N-2}\sigma_N & \cdots & \sigma_N & \lambda_N^{N-1} & -\lambda_N^{N-2} & \cdots & (-1)^{N-1} \end{pmatrix}$$

and $\Delta_{a_{N-1}},\Delta_{b_{N-1}}$ are obtained by replacing 1-st and ($N+1$)-th columns with

$$\left(-\lambda_1^N,-\lambda_2^N,\cdots,-\lambda_N^N,-\lambda_1^N\sigma_1,-\lambda_2^N\sigma_2,\cdots,-\lambda_N^N\sigma_N\right)^T$$

in Δ_{2N}, respectively.

Then, according to the Theorem 3.1 and above analysis, the solution of nonlinear evolution Equation (2.5) is

$$q[N]=-2b_{N-1},r[N]=-2a_{N-1},\tag{4.5}$$

and the solution of (2.6) is

$$u[N]=-2a_{N-1}-2\mathrm{i}b_{N-1}.\tag{4.6}$$

In [16], the relationship of $\left(q[N-1],r[N-1]\right)$ and $\left(q[N],r[N]\right)$ is given. However, the Darboux transformation obtained here gives the relationship between $\left(q[0],r[0]\right)$ and $\left(q[N],r[N]\right)$, which is more direct and universal to get solutions.

For $N = 1$, we have

$$a_0=\frac{\lambda_1\left(1-\sigma_1^2\right)}{1+\sigma_1^2},\quad b_0=\frac{-2\lambda_1\sigma_1}{1+\sigma_1^2},\tag{4.7}$$

then

$$q[1]=-\frac{i\lambda_1\left(c_{11}^2\left(l_{11}-l_{21}\right)^2e^{2\theta_1}-c_{21}^2\left(l_{11}+l_{21}\right)^2e^{-2\theta_1}\right)}{c_{11}c_{21}\left(l_{11}^2-l_{21}^2\right)},\tag{4.8}$$

$$r[1]=-\frac{\lambda_1\left(c_{11}^2\left(l_{11}-l_{21}\right)^2e^{2\theta_1}+c_{21}^2\left(l_{11}+l_{21}\right)^2e^{-2\theta_1}\right)}{c_{11}c_{21}\left(l_{11}^2-l_{21}^2\right)}.\tag{4.9}$$

For $N = 2$, we have

$$a_1=\frac{\left(\lambda_1^2-\lambda_2^2\right)\left(\lambda_1\left(1-\sigma_1^2\right)\left(1+\sigma_2^2\right)-\lambda_2\left(1+\sigma_1^2\right)\left(1-\sigma_2^2\right)\right)}{\left(\lambda_1-\lambda_2\right)^2\left(1+\sigma_1^2\right)\left(1+\sigma_2^2\right)+4\lambda_1\lambda_2\left(\sigma_1-\sigma_2\right)^2},\tag{4.10}$$

$$b_1=\frac{2\left(\lambda_1^2-\lambda_2^2\right)\left(\lambda_2\sigma_2\left(1+\sigma_1^2\right)-\lambda_1\sigma_1\left(1+\sigma_2^2\right)\right)}{\left(\lambda_1-\lambda_2\right)^2\left(1+\sigma_1^2\right)\left(1+\sigma_2^2\right)+4\lambda_1\lambda_2\left(\sigma_1-\sigma_2\right)^2},\tag{4.11}$$

then

$$q[2]=\frac{-i\left(\lambda_1^2-\lambda_2^2\right)\left(\alpha_1+\alpha_2+\alpha_3+\alpha_4\right)}{\beta_1+\beta_2+\beta_3},\tag{4.12}$$

$$r[2]=\frac{\left(\lambda_1^2-\lambda_2^2\right)\left(\alpha_1+\alpha_2-\alpha_3-\alpha_4\right)}{\beta_1+\beta_2+\beta_3},\tag{4.13}$$

where

$$\alpha_1=\lambda_2c_{22}^2c_{11}c_{21}\left(l_{11}^2-l_{21}^2\right)\left(l_{12}+l_{22}\right)^2e^{2\theta_1},$$

$$\alpha_2=-\lambda_1c_{21}^2c_{12}c_{22}\left(l_{12}^2-l_{22}^2\right)\left(l_{11}+l_{21}\right)^2e^{2\theta_2},$$

$$\alpha_3=\lambda_1c_{11}^2c_{12}c_{22}\left(l_{12}^2-l_{22}^2\right)\left(l_{11}-l_{21}\right)^2e^{4\theta_1+2\theta_2},$$

$$\alpha_4=-\lambda_2c_{12}^2c_{11}c_{21}\left(l_{11}^2-l_{21}^2\right)\left(l_{12}-l_{22}\right)^2e^{2\theta_1+4\theta_2},$$

$$\beta_1=-\lambda_1\lambda_2c_{11}^2c_{22}^2\left(l_{11}-l_{21}\right)^2\left(l_{12}+l_{22}\right)^2e^{4\theta_1},$$

$$\beta_2 = -\lambda_1\lambda_2 c_{12}^2 c_{21}^2 \left(l_{11}+l_{21}\right)^2 \left(l_{12}-l_{22}\right)^2 e^{4\theta_2},$$

$$\beta_3 = \left(\lambda_1^2-\lambda_2^2\right) c_{11}c_{12}c_{21}c_{22} \left(l_{11}^2-l_{21}^2\right)\left(l_{12}^2-l_{22}^2\right) e^{2(\theta_1+\theta_2)}.$$

If we let $c_{11}=c_1, c_{21}=c_2, c_{12}=c_3, c_{22}=c_4, l_{11}=1$, $l_{21}=0, l_{12}=0, l_{22}=1$, solutions $\left(q[1], r[1]\right)$ and $\left(q[2], r[2]\right)$ are exactly the same as the solutions in [16].

In general, according to [21], we know that the N-fold Darboux transformation is an action of the n-times repeated 1-fold Darboux transformation. The solution $\left(q[N], r[N]\right)$ are the same as the solution in [16] in essence. The matrix T can be expressed by the determinant of basic solutions

$$T = \frac{1}{|\Delta_{2N}|}\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \tag{4.14}$$

where

$$T_{11} = \begin{vmatrix} \lambda^n & \xi_{2N-1} \\ \eta_{2N-1} & \Delta_{2N} \end{vmatrix}, \quad T_{12} = \begin{vmatrix} 0 & \hat{\xi}_{2N-1} \\ \eta_{2N-1} & \Delta_{2N} \end{vmatrix},$$

$$T_{21} = \begin{vmatrix} 0 & \xi_{2N-1} \\ \hat{\eta}_{2N-1} & \Delta_{2N} \end{vmatrix}, \quad T_{22} = \begin{vmatrix} \lambda^n & \hat{\xi}_{2N-1} \\ \hat{\eta}_{2N-1} & \Delta_{2N} \end{vmatrix}$$

with

$$\xi_{2N-1} = \left(\lambda^{n-1}, \lambda^{n-2}, \cdots, \lambda, 1, 0, 0, \cdots, 0\right),$$

$$\hat{\xi}_{2N-1} = \left(0, 0, \cdots, 0, \lambda^{n-1}, \lambda^{n-2}, \cdots, \lambda, 1\right),$$

$$\eta_{2N-1} = \left(-\lambda_1^N, -\lambda_2^N, \cdots, -\lambda_N^N, -\lambda_1^N\sigma_1, -\lambda_2^N\sigma_2, \cdots, -\lambda_N^N\sigma_N\right)^T,$$

$$\hat{\eta}_{2N-1} = \left(-\lambda_1^N\sigma_1, -\lambda_2^N\sigma_2, \cdots, -\lambda_N^N\sigma_N, -\lambda_1^N, -\lambda_2^N, \cdots, -\lambda_N^N\right)^T.$$

The matrix T is the same as (3.3), which is also consistent with the Darboux matrix in [21].

5. Conclusion

In this paper, for a Lax pair which is not the AKNS system, we give a N-fold Darboux transformation, coefficients of this matrix can be obtained from a algebraic system and expressed with rank-2N determinants. The Darboux transformation gives the relationship between $\left(q[0], r[0]\right)$ and $\left(q[N], r[N]\right)$ The advantage of this method is that it is more direct and universal to get explicit solutions of (2.5) and (2.6).

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