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On *P*-Regularity of Acts

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ABSTRACT

By a regular act we mean an act that all its cyclic subacts are projective. In this paper we introduce *P*-regularity of acts over monoids and will give a characterization of monoids by this property of their right (Rees factor) acts.

Keywords: *P*-Regularity; Rees Factor Act

1. Introduction

Throughout this paper S will denote a monoid. We refer the reader to ([1]) and ([2]) for basic results, definitions and terminology relating to semigroups and acts over monoids and to [3,4] for definitions and results on flatness which are used here.

A monoid S is called *left* (*right*) *collapsible* if for every $s,s'\in S$ there exists $z\in S$ such that $zs=zs'$ $(sz=s'z)$. A submonoid P of a monoid S is called *weakly left collapsible* if for all $s,s'\in P$, $z\in S$ the equality $(sz=s'z)$ implies that there exists an element $u\in P$ such that $us=us'$.

A monoid S is called *right* (*left*) *reversible* if for every $s,s'\in S$, there exist $u,v\in S$ such that $us=vs'$ $(su=s'v)$. A right ideal K of a monoid S is called *left stabilizing* if for every $k\in K$, there exists $l\in K$ such that $lk=k$ and it is called *left annihilating* if,

$$(\forall t\in S)(\forall x,y\in S\setminus K)(xt,yt\in K\Rightarrow xt=yt).$$

If for all $s,t\in S\setminus K$ and all homomorphisms $f:{}_S(Ss\bigcup St)\to {}_SS$

$$f(s),f(t)\in K\Rightarrow f(s)=f(t)$$

then K is called *strongly left annihilating*.

A right S-act A satisfies *Condition* (P) if $as=a's'$ for $a,a'\in A,$ $s,s'\in S,$ implies the existence of $a''\in A,$ $u,v\in S$ such that $a=a''u, a'=a''v$ and $us=vs'$.

A right S-act A is called *connected* if for $a,a'\in A$ there exist $s_1,t_1,\cdots,s_n,t_n\in S$ and $a_1,\cdots,a_{n-1}\in A$ such that

$$as_1=a_1t_1$$
$$a_1s_2=a_2t_2$$
$$\cdots$$
$$a_{n-1}s_n=a't_n$$

We use the following abbreviations:
Strong flatness = SF;
Pullback flatness = PF;
Weak pullback flatness = WPF;
Weak kernelflatness = WKF;
Principal weak kernelflatness = $PWKF$;
Translation kernelflatness = TKF;
Weak homoflatness = (WP);
Principal weak homoflatness = (PWP);
Weak flatness = WF;
Principal weak flatness = PWF;
Torsion freeness = TF.

2. Characterization by *P*-Regularity of Right Acts

Definition 2.1. Let S be a monoid. A right S-act A is called *P-regular* if all cyclic subacts of A satisfy Condition (P).

We know that a right S-act A is regular if every cyclic subact of A is projective. It is obvious that every regular right act is *P*-regular, but the converse is not true, for example if S is a non trivial group, then S is right reversible, and so by ([2, III, 13.7]), Θ_S is *P*-regular, but by ([2, III, 19.4]), Θ_S is not regular, since S has no left zero element.

Theorem 2.1. Let S be a monoid. Then:

1) Θ_S is *P*-regular if and only if S is right reversible.

2) S_S is *P*-regular if and only if all principal right ideals of S satisfy Condition (P).

3) If A is a right S-act and $A_i, i\in I$, are subacts of A, then $\bigcup_{i\in I}A_i$ is *P*-regular if and only if A_i is *P*-regular for every $i\in I$.

4) Every subact of a *P*-regular right S-act is *P*-regular.

Proof. It is straightforward. q.e.d.

Here we give a criterion for a right S-act to be *P*-

regular.

Theorem 2.2. Let S be a monoid and A a right S-act. Then A is P-regular if and only if for every $a \in A$ and $x, y \in S$, $ax = ay$ implies that there exist $u, v \in S$ such that $a = au = av$ and $ux = vy$.

Proof. Suppose that A is a P-regular right S-act and let $ax = ay$, for $a \in A$ and $x, y \in S$. Then aS satisfies Condition (P). But $aS \cong S/\ker\lambda_a$, and so by ([2, III, 13.4]), we are done.

Conversely, we have to show that aS satisfies Condition (P) for every $a \in A$. Since $aS \cong S/\ker\lambda_a$, then it suffices to show that $S/\ker\lambda_a$ satisfies condition (P) and this is true by ([2, III, 13.4]). q.e.d.

We now give a characterization of monoids for which all right S-acts are P-regular.

Theorem 2.3. For any monoid S the following statements are equivalent:

1) All right S-acts are P-regular.
2) All finitely generated right S-acts are P-regular.
3) All cyclic right S-acts are P-regular.
4) All monocyclic right S-acts are P-regular.
5) All right Rees factor S-acts are P-regular.
6) S is a group or a group with a zero adjoined.

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) and (3) $\Rightarrow$ (5) are obvious.

(4) $\Rightarrow$ (6). By assumption all monocyclic right S-acts satisfy Condition (P), and so by ([2, IV, 9.9]), S is a group or a group with a zero adjoined.

(5) $\Rightarrow$ (6). By assumption all right Rees factor S-acts satisfy Condition (P) and again by ([2, IV, 9.9]), S is a group or a group with a zero adjoined.

(6) $\Rightarrow$ (1). By ([2, IV, 9.9]), all cyclic right S-acts satisfy condition (P), and so by definition all right S-acts are P-regular as required. q.e.d.

Notice that freeness of acts does not imply P-regularity, for if $S = \{0,1,x\}$, with $x^2 = 0$, then S_S is free, but S_S is not P-regular, otherwise $xS = \{0,x\}$ satisfies Condition (P) as a cyclic subact of S_S, and so $x.x = x.0$, implies the existence of $u, v \in S$ such that $x = xu = xv$ and $ux = v0$, and this is a contradiction.

Theorem 2.4. For any monoid S the following statements are equivalent:

1) All right S-acts satisfying Condition (E) are P-regular.
2) All finitely generated right S-acts satisfying Condition (E) are P-regular.
3) All cyclic right S-acts satisfying Condition (E) are P-regular.
4) All *SF* right S-acts are P-regular.
5) All *SF* finitely generated right S-acts are P-regular.
6) All *SF* cyclic right S-acts are P-regular.
7) All projective right S-acts are P-regular.
8) All finitely generated projective right S-acts are P-regular.
9) All projective cyclic right S-acts are P-regular.
10) All projective generators in Act-S are P-regular.
11) All finitely generated projective generators in Act-S are P-regular.
12) All cyclic projective generators in Act-S are P-regular.
13) All free right S-acts are P-regular.
14) All finitely generated free right S-acts are P-regular.
15) All free cyclic right S-acts are P-regular.
16) All principal right ideals of S satisfy Condition (P).
17) $(\forall s,t,z \in S)$
$$\big(zs = zt \Rightarrow (\exists u, v \in S)(z = zu = zv \wedge us = vt)\big).$$

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (6) $\Rightarrow$ (9) $\Rightarrow$ (12) $\Rightarrow$ (15), (1) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6), (4) $\Rightarrow$ (7) $\Rightarrow$ (8) $\Rightarrow$ (9), (7) $\Rightarrow$ (10) $\Rightarrow$ (11) $\Rightarrow$ (12) and (10) $\Rightarrow$ (13) $\Rightarrow$ (14) $\Rightarrow$ (15) are obvious.

(15) $\Rightarrow$ (16). As a free cyclic right S-act S_S is P-regular, and so by (2) of Theorem 2.1, all principal right ideals of S satisfy Condition (P).

(16) $\Leftrightarrow$ (17). By ([2, III, 13.10]), it is obvious.

(17) $\Rightarrow$ (1). Suppose the right S-act A satisfies Condition (E) and let $ax = ay$, for $a \in A$ and $x, y \in S$. Then there exist $a' \in A$ and $u \in S$ such that $a = a'u$ and $ux = uy$. Thus by assumption there exist $s, t \in S$ such that $u = us = ut$ and $sx = ty$. Therefore $a = a'u = a'us = as$, $a = a'u = a'ut = at$, $sx = ty$, and so by Theorem 2.2, A is P-regular. q.e.d.

Notice that cofreeness does not imply P-regularity, otherwise every act is P-regular, since by ([2, II, 4.3]), every act can be embedded into a cofree act. But if $S = \{0,1,x\}$, with $x^2 = 0$, then as we saw before, S_S is not P-regular, and so we have a contradiction.

Theorem 2.5. For any monoid S the following statements are equivalent:

1) All divisible right S-acts are P-regular.
2) All principally weakly injective right S-acts are P-regular.
3) All *fg*-weakly injective right S-acts are P-regular.
4) All weakly injective right S-acts are P-regular.
5) All injective right S-acts are P-regular.
6) All injective cogenerators in Act-S are P-regular.
7) All cofree right S-acts are P-regular.
8) All right S-acts are P-regular.
9) S is a group or a group with a zero adjoined.

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) and (5) $\Rightarrow$ (7) are obvious.

(6) $\Rightarrow$ (8). Suppose that A is a right S-act, B is an injective cogenerator in Act-S and C is an injective envelope of A (C exists by [2, III, 1.23]). By ([5, Theorem 2]), $D = B \coprod C$ is an injective cogenerator in Act-S, and so by assumption D is P-regular. Since $A \subseteq C$, we have $A \subseteq D$, and so by Theorem 2.1, A is P-regular.

(7) $\Rightarrow$ (8). Let A be a right *S*-act. Then by ([2, II, 4.3]), *A* can be embedded into a cofree right *S*-act. Since *A* is a subact of a cofree right *S*-act, by assumption *A* is a subact of a *P*-regular right *S*-act, and so by Theorem 2.1, A is *P*-regular.

(8) $\Leftrightarrow$ (9). By Theorem 2.3, it is obvious.

(8) $\Rightarrow$ (1). It is obvious. q.e.d.

Theorem 2.6. Let *S* be a monoid. Then every strongly faithful right *S*-act is *P*-regular.

Proof. By Theorem 2.2, it is obvious. q.e.d.

Although strong faithfulness implies *P*-regularity, but faithfulness does not imply *P*-regularity, since every monoid as an act is faithful, $S=\{0,1,x\}$ with $x^2=0$ is faithful, but as we saw before, S_S is not *P*-regular. Now see the following theorem.

Theorem 2.7. For any monoid *S* the following statements are equivalent:

1) All faithfull right *S*-acts are *P*-regular.

2) All finitely generated faithfull right *S*-acts are *P*-regular.

3) All faithfull right *S*-acts generated by at most two elements are *P*-regular.

4) *S* is a group or a group with a zero adjoined.

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4). By Theorem 2.3, it suffices to show that every cyclic right *S*-act is *P*-regular. Thus we consider a cyclic right *S*-act aS and let $A_S=aS\coprod S_S$. Since S_S is faithful, A_S is faithful, also A_S is generated by at most two elements, thus by assumption A_S is *P*-regular. Since aS is a subact of A_S, by (4) of Theorem 2.1, aS is *P*-regular as required.

(4) $\Rightarrow$ (1). By Theorem 2.3, it is obvious. q.e.d.

Since regularity does not imply flatness in general, *P*-regularity also does not imply flatness in general, but as the following theorem shows, for regular monoids *P*-regularity implies flatness.

Theorem 2.8. Let *S* be a regular monoid. Then every *P*-regular right *S*-act is flat.

Proof. Suppose that *S* is a regular monoid, ${}_SM$ is a left *S*-act and A_S is a *P*-regular right *S*-act. Let $a\otimes m=a'\otimes m'$ in $A\otimes_S M$ for $a,a'\in A_S$ and $m,m'\in {}_SM$. We show $a\otimes m=a'\otimes m'$ holds also in $A\otimes_S(Sm\cup Sm')$. Since $a\otimes m=a'\otimes m'$ in $A\otimes_S M$, we have a tossing

$$\begin{array}{ll} & s_1m_1=m\\ as_1=a_1t_1 & s_2m_2=t_1m_1\\ a_1s_2=a_2t_2 & s_3m_3=t_2m_2\\ \cdots & \cdots\\ a_{k-1}s_k=a't_k & m'=t_km_k\end{array}$$

of length k, where $s_1,\cdots,s_k,t_1,\cdots,t_k\in S$, $a_1,\cdots,a_{k-1}\in A_S$, $m_1,\cdots,m_k\in {}_SM$.

If $k=1$, then we have

$$\begin{array}{ll} & s_1m_1=m\\ as_1=a't_1 & m'=t_1m_1.\end{array}$$

Since S is regular, the equality $as_1=a't_1$ implies that $a't_1=a't_1s_1's_1$, for $s_1'\in V(s_1)$. Since A_S is *P*-regular, there exist $a''\in A_S$ and $u,v\in S$ such that $a'=a''u=a''v$ and $ut_1=vt_1s_1's_1$. From the last equality we obtain $um'=ut_1m_1=vt_1s_1's_1m_1=vt_1s_1'm.$ Since $m=s_1m_1$, we get $s_1s_1'm=m,$ and so we have

$$\begin{aligned}a\otimes m&=a\otimes s_1s_1'm=as_1\otimes s_1'm=a't_1\otimes s_1'm\\&=a''ut_1\otimes s_1'm=a''vt_1\otimes s_1'm=a''\otimes vt_1s_1'm\\&=a''\otimes um'=a''u\otimes m'=a'\otimes m'\end{aligned}$$

in $A\otimes_S(Sm\cup Sm')$.

We now suppose that $k\ge 2$ and that the required equality holds for every tossing of length less than k. From $as_1=a_1t_1$ we obtain equalities $a_1t_1=a_1t_1s_1's_1$ for $s_1'\in V(s_1)$ and $as_1=as_1t_1't_1$ for $t_1'\in V(t_1)$. Since A_S is *P*-regular, there exist $a_1'',a_2''\in A_S$ and $u_1,u_2,$ $v_1,v_2\in S$ such that $a_1=a_1''u_1=a_1''v_1,$ $u_1t_1=v_1t_1s_1's_1$ and $a=a_2''u_2=a_2''v_2, u_2s_1=v_2s_1t_1't_1.$ Thus we have the following tossing

$$\begin{array}{ll} & u_2s_1m_1=u_2m\\ a_2''u_2s_1=a_1''u_1t_1 & u_1s_2m_2=u_1t_1m_1\end{array}$$

of length 1 and

$$\begin{array}{ll} & u_1s_2m_2=u_1t_1m_1\\ a_1''u_1s_2=a_2t_2 & s_3m_3=t_2m_2\\ \cdots & \cdots\\ a_{k-1}s_k=a't_k & m'=t_km_k\end{array}$$

of length $k-1$.

From the tossing of length 1, we have $a_2''\otimes u_2m=a_1''\otimes u_1s_2m_2$ in $A\otimes_S M$, and so we have $a_2''\otimes u_2m=a_1''\otimes u_1s_2m_2$ in $A\otimes_S(Su_2m\cup Su_1s_2m_2)$.

Since

$$u_1s_2m_2=u_1t_1m_1=v_1t_1s_1's_1m_1=v_1t_1s_1'm\in Sm,$$

we have $a_2''\otimes u_2m=a_1''\otimes u_1s_2m_2$ in $A\otimes_S(Sm\cup Sm')$.

Also from the tossing of length $k-1$, we have $a_1''\otimes u_1t_1m_1=a'\otimes m'$ in $A\otimes_S M$. Thus we have $a_1''\otimes u_1t_1m_1=a'\otimes m'$ in $A\otimes_S(Su_1t_1m_1\cup Sm')$ Since

$$u_1t_1m_1=v_1t_1s_1'm\in Sm,$$

we have $a_1''\otimes u_1t_1m_1=a'\otimes m'$ in $A\otimes_S(Sm\cup Sm'),$ and so

$$\begin{aligned}a\otimes m&=a_2''u_2\otimes m=a_2''\otimes u_2m=a_1''\otimes u_1s_2m_2\\&=a_1''\otimes u_1t_1m_1=a'\otimes m'\end{aligned}$$

in $A\otimes_S(Sm\cup Sm')$ as required. q.e.d.

3. Characterization by *P*-Regularity of Right Rees Factor Acts

In this section we give a characterization of monoids by *P*- regularity of right Rees factor acts.

Theorem 3.1. Let S be a monoid and K_S a right ideal of S. Then S/K_S is *P*-regular if and only if $K_S = S$ and S is right reversible or $|K_S| = 1$ and all principal right ideals of S satisfy Condition (P).

Proof. Let K_S be a right ideal of S and suppose that S/K_S is *P*-regular. Then S/K_S satisfies Condition (P) If $K_S = S$, then by ([2, III, 13.7]), S is right reversible, otherwise by ([2, III, 13.9]), $|K_S| = 1$, and so $S/K_S \cong S$. Thus by (2) of Theorem 2.1, all principal right ideals of S satisfy Condition (P).

Conversely, suppose that K_S is a right ideal of S. If $K_S = S$ and S is right reversible, then by (1) of Theorem 2.1, $S/K_S \cong \Theta_S$ is *P*-regular. If $|K_S| = 1$ and all principal right ideals of S satisfy Condition (P), then by (2) of Theorem 2.1, $S/K_S \cong S$ is *P*-regular. q.e.d.

Although freeness of acts implies Condition (P) in general, but notice that freeness of Rees factor acts does not imply *P*-regularity, for if $S = \{0,1,x\}$, with $x^2 = 0$, and $K_S = 0S$, then $S/K_S = S/0S \cong S_S$ as a Rees factor act is free, but as we saw before, S_S is not *P*-regular.

Now let see the following theorem.

Theorem 3.2. Let S be a monoid and (U) be a property of *S*-acts implied by freeness. Then the following statements are equivalent:

1) All right Rees factor *S*-acts satisfying property (U) are *P*-regular.

2) All right Rees factor *S*-acts satisfying property (U) satisfy Condition (P) and either S contains no left zero or all principal right ideals of S satisfy Condition (P).

Proof. (1) $\Rightarrow$ (2). By definition all right Rees factor *S*-acts satisfying property (U) satisfy Condition (P). Suppose now that S contains a left zero z_0. Then $K_S = z_0S = \{z_0\}$ is a right ideal of S, and so $S/K_S \cong S_S$. Since S_S is free, S_S is *P*-regular, by assumption, and so all principal right ideals of S satisfy Condition (P).

(2) $\Rightarrow$ (1). Let S/K_S satisfies property (U) for the right ideal K_S of S. Then by assumption S/K_S satisfies Condition (P). Now there are two cases as follows:

Case 1. $K_S = S$. Then $S/K_S = \Theta_S$, and so by ([2, III, 13.7]), S is right reversible, thus by (1) of Theorem 2.1, $S/K_S = \Theta_S$ is *P*-regular.

Case 2. K_S is a proper right ideal of S. Then by ([2, III, 13.9]), $|K_S| = 1$. Thus $K_S = \{z_0\}$, for some $z_0 \in S$, and so z_0 is left zero. Thus by assumption all principal right ideals of S satisfy Condition (P), that is $S/K_S \cong S_S$ is *P*-regular. q.e.d.

Corollary 3.1. For any monoid S the following statements are equivalent:

1) All right Rees factor *S*-acts satisfying Condition (P) are *P*-regular.

2) All WPF right Rees factor *S*-acts are *P*-regular.

3) All PF right Rees factor *S*-acts are *P*-regular.

4) All SF right Rees factor *S*-acts are *P*-regular.

5) All projective right Rees factor *S*-acts are *P*-regular.

6) All Rees factor projective generators in Act-*S* are *P*-regular.

7) All free right Rees factor *S*-acts are *P*-regular.

8) S contains no left zero or all principal right ideals of S satisfy Condition (P).

Proof. By Theorem 3.2, it is obvious. q.e.d.

Corollary 3.2. For any monoid S the following statements are equivalent:

1) All WF right Rees factor *S*-acts are *P*-regular.

2) All flat right Rees factor *S*-acts are *P*-regular.

3) S is not right reversible or no proper right ideal K_S, $|K_S| \geq 2$ of S is left stabilizing, and if S contains a left zero, then all principal right ideals of S satisfy Condition (P).

Proof. It follows from Theorem 3.2, ([2, IV, 9.2]), and that for Rees factor acts weak flatness and flatness are coinside. q.e.d.

Corollary 3.3. For any monoid S the following statements are equivalent:

1) All PWF right Rees factor *S*-acts are *P*-regular.

2) S is right reversible, no proper right ideal K_S, $|K_S| \geq 2$ of S is left stabilizing, and if S contains a left zero, then all principal right ideals of S satisfy Condition (P).

Proof. It follows from Theorem 3.2, and ([2, IV, 9.7]). q.e.d.

Corollary 3.4. For any monoid S the following statements are equivalent:

1) All TF right Rees factor *S*-acts are *P*-regular.

2) Either S is a right reversible right cancellative monoid or a right cancellative monoid with a zero adjoined, and if S contains a left zero, then all principal right ideals of S satisfy Condition (P).

Proof. It follows from Theorem 3.2, and ([2, IV, 9.8]). q.e.d.

Corollary 3.5. For any monoid S the following statements are equivalent:

1) All right Rees factor *S*-acts satisfying Condition (WP) are *P*-regular.

2) S is not right reversible or no proper right ideal K_S, $|K_S| \geq 2$ of S is left stabilizing and strongly left annihilating, and if S contains a left zero, then all principal right ideals of S satisfy Condition (P).

Proof. It follows from Theorem 3.2, and ([3, Proposition 3.26]). q.e.d.

Corollary 3.6. For any monoid S the following statements are equivalent:

1) All right Rees factor S-acts satisfying Condition (PWP) are P-regular.

2) S is right reversible and no proper right ideal K_S, $|K_S| \geq 2$ of S is left stabilizing and left annihilating, and if S contains a left zero, then all principal right ideals of S satisfy Condition (P).

Proof. It follows from Theorem 3.2, and ([3, Corollary 3.27]). q.e.d.

Here we consider monoids over which P-regularity of Rees factor acts implies other properties.

Theorem 3.3. Let S be a monoid and (U) be a property of S-acts implied by freeness. Then all P-regular right Rees factor S-acts satisfy property (U) if and only if S is not right reversible or Θ_S satisfies property (U).

Proof. Suppose that S is right reversible. By (1) of Theorem 2.1, $\Theta_S \cong S/S_S$ is P-regular, and so by assumption Θ_S satisfies property (U).

Conversely, suppose S/K_S is P-regular for the right ideal K_S of S. Then there are two cases as follows:

Case 1. $K_S = S$. Then $S/K_S = \Theta_S$ is P-regular, and so by (1) of Theorem 2.1, S is right reversible.

Thus by assumption $S/K_S \cong \Theta_S$ satisfies property (U).

Case 2. K_S is a proper right ideal of S. By Theorem 3.1, $|K_S| = 1$, and so $S/K_S \cong S_S$. Thus S/K_S is free, and so satisfies property (U). q.e.d

Corollary 3.7. Let S be a monoid. Then all P-regular right Rees factor S-acts are free if and only if S is not right reversible or $S = \{1\}$.

Proof. It follows from Theorem 3.3, and ([2, I, 5.23]). q.e.d.

Corollary 3.8. Let S be a monoid. Then all P-regular right Rees factor S-acts are projective if and only if S is not right reversible or S contains a left zero.

Proof. It follows from Theorem 3.3, and ([2, III, 17.2]). q.e.d.

Corollary 3.9. Let S be a monoid. Then all P-regular right Rees factor S-acts are strongly flat if and only if S is not right reversible or S is left collapsible.

Proof. It follows from Theorem 3.3, and ([2, III, 14.3]). q.e.d

Theorem 3.4. For any monoid S the following statements are equivalent:

1) All P-regular right Rees factor S-acts are *WPF*.

2) All P-regular right Rees factor S-acts are *WKF*.

3) All P-regular right Rees factor S-acts are *PWKF*

4) All P-regular right Rees factor S-acts are *TKF*.

5) S is not right reversible or S is weakly left collapsible.

6) S is not right reversible or for every left ideal I of S, $\ker f$ is connected for every homomorphism $f : {}_SI \to {}_SS$.

7) S is not right reversible or for every $z \in S$, $ker\,\rho_z$ is connected as a left S-act.

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(1) $\Leftrightarrow$ (5). By Theorem 3.3, and ([4, Corollary 24]) it is obvious.

(2) $\Leftrightarrow$ (6). By Theorem 3.3, and ([6, Proposition 8]) it is obvious.

(4) $\Leftrightarrow$ (7). By Theorem 3.3, and ([6, Proposition 7]) it is obvious.

(4) $\Rightarrow$ (1). By ([6, Proposition 28]), $WPF \Leftrightarrow (P) \wedge TKF$. Now if A_S is a P-regular right Rees factor S-act, then it is obvious that A_S satisfies Condition (P), also by assumption A_S is TKF, and so A_S is WPF. q.e.d.

Corollary 3.10. For any monoid S the following statements are equivalent:

1) Θ_S is WPF.

2) Θ_S is WKF.

3) S is right reversible and weakly left collapsible.

4) S is right reversible and for every left ideal I of S, $\ker f$ is connected for every homomorphism $f : {}_SI \to {}_SS$.

5) S is right reversible and for every $z \in S$, $ker\,\rho_z$ is connected as a left S-act.

Proof. Implication (1) $\Rightarrow$ (2) is obvious.

(1) $\Leftrightarrow$ (3). It is obvious by ([6, Corollary 24]).

(3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5). It is obvious by Theorem 3.4.

(3) $\Leftrightarrow$ (4). It is obvious by ([6, Proposition 8]). q.e.d.

Corollary 3.11. Let S be a right reversible monoid. Then Θ_S is WPF if and only if Θ_S is TKF.

Proof. It is obvious that every WPF right S-act is TKF. If Θ_S is TKF, then by ([6, Proposition 7]), for every $z \in S$, $ker\,\rho_z$ is connected as a left S-act, and so by Corollary 3.10 Θ_S is WPF. q.e.d.

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