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A Family of the Global Attractor for Higher Order Nonlinear Kirchhoff Equation

Guoguang Lin*, Yingguo Wang

Department of Mathematics, Yunnan University, Kunming, China
Email: *gglin@ynu.edu.cn, 790612351@qq.com

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Abstract

In this paper, we study the wellness and long time dynamic behavior of the solution of the initial boundary value problem for a class of higher order Kirchhoff equations $u_{tt}+M\left(\left\|D^m u\right\|_p^p\right)(-\Delta)^{2m}u+\beta(-\Delta)^{2m}u_t+g(u_t)=f(x)$ with strong damping terms. We will properly assume the stress term $M(s)$ and nonlinear term $g(u_t)$. First, we can prove the existence and uniqueness of the solution of the equation via a prior estimate and Galerkin's method, then the existence of the family of global attractor is obtained. At last, we can obtain that the Hausdorff dimension and Fractal dimension of the family of global attractors are finite.

Keywords

Kirchhoff-Type Equations, Prior Estimation, Galerkin's Method, The Family of Global Attractor, Hausdorff Dimension, Fractal Dimension

1. Introduction

This paper intends to study the initial-boundary value problem of higher-order Kirchhoff-type equation

$$u_{tt}+M\left(\left\|D^m u\right\|_p^p\right)(-\Delta)^{2m}u+\beta(-\Delta)^{2m}u_t+g(u_t)=f(x), \tag{1.1}$$

$$u(x,t)=0, \frac{\partial^i u}{\partial v^i}=0, i=1,2,\cdots,2m-1, x\in\partial\Omega, t>0, \tag{1.2}$$

$$u(x,0)=u_0(x), u_t(x,0)=u_t(x), x\in\Omega\subset R^n. \tag{1.3}$$

where $m>1$, and $m\in N^+$, $\Omega\in R^n(n\geq 1)$ is a bounded domain, $\partial\Omega$ denotes the boundary of Ω, $g(u_t)$ is a nonlinear source term, $\beta(-\Delta)^{2m}u_t$ is a strongly dissipative term, $\beta>0$, $f(x)$ is an external force term.

Kirchhoff-type equation model is one of the hot topics in mathematical physics equation research in recent years, which shows the importance of its position and influence. There have been many achievements in the study of the long-term behavior of the solution of Kirchhoff-type equation, for details, refer to references ([1] [2] [3] [4] [5]). Cheng Jianling and Yang Zhijian studied the asymptotic behavior of the solution of Kirchhoff-type equation in reference [6]:

$$u_{tt} - M\left(\|\Delta u\|^2\right)\Delta u - \Delta u_t + h(u_t) + g(u) = f(x), (x,t) \in \Omega \times R^+,$$

$$u(x,t) = 0, \frac{\partial^i u}{\partial v^i} = 0, i = 1, 2, \cdots, m-1, x \in \partial\Omega, t > 0,$$

$$u(x,0) = u_0, u_t = u_1(x), x \in \Omega.$$

The existence of the global attractor of the corresponding operator semigroup $S(t)$ in phase space is proved.

Recently, Lin Guoguang *et al.* studied the existence of global attractors for higher order Kirchhoff type equations with nonlinear strong damping terms in reference [7]:

$$u_{tt} + (-\Delta)^m u_t + \phi\left(\|\nabla^m u\|^2\right)(-\Delta)^m u + h(u_t) = f(x),$$

$$u(x,t) = 0, \frac{\partial^i u}{\partial v^i} = 0, i = 1, 2, \cdots, m-1, x \in \partial\Omega, t > 0,$$

$$u(x,0) = u_0, u_t = u_1(x), x \in \Omega.$$

They proved the existence and uniqueness of the solution of the equation by using prior estimation and Galerkin's method, and then obtained that the attractor exists in space $H^{2m}(\Omega) \times H^m(\Omega)$.

Guoguang Lin and Changqing Zhu studied asymptotic state of solutions for a class of nonlinear higher order Kirchhoff type equations in reference [8]:

$$u_{tt} + M\left(\|D^m u\|^2\right)(-\Delta)^m u + \beta(-\Delta)^m u_t + g(x, u_t) = f(x),$$

$$u(x,t) = 0, \frac{\partial^i u}{\partial v^i} = 0, i = 1, 2, \cdots, m-1, x \in \partial\Omega, t > 0,$$

$$u(x,0) = u_0, u_t = u_1(x), x \in \Omega.$$

For more results, please refer to references ([9]-[15]).

2. Basic Assumptions

For convenience, space and notations are defined as follows:

$H = L^2(\Omega)$, $D = \nabla$, $H_0^m(\Omega) = H^m(\Omega) \cap H_0^1(\Omega)$, $H_0^{2m+k}(\Omega) = H^{2m+k}(\Omega) \cap H_0^1(\Omega)$, $E_0 = H^{2m}(\Omega) \times L^2(\Omega)$, $E_k = H_0^{2m+k}(\Omega) \times H_0^k(\Omega)$. Remember that A_k is a family of global attractors from E_0 to E_k, B_{0k} is a bounded absorption set in E_k. In which $k = 1, 2, \cdots, 2m$, $C_i (i = 0, 1, 2, \cdots)$ is a constant; $(\cdot,\cdot)$, $\|\cdot\|$ represent the inner product and norm on space H, namely $(u,v) = \int_\Omega u(x)v(x)\mathrm{d}x$, $(u,u) = \|u\|^2$.

Kirchhoff type stress term $M(s)$ satisfies the following conditions:

(H$_1$) $M(s)\in C^2([0,+\infty),R)$;

(H$_2$) $\varepsilon+1=\mu_0\le M(s)\le\mu_1$,

where μ_0,μ_1 are constant, $0<\varepsilon<\min\left\{\frac{2\mu_0\lambda_1^m}{1+\beta\lambda_1^m},\frac{\sqrt{4+2\beta\lambda_1^m-2}}{2},\frac{2}{\beta-1+\lambda_1^{-m}}\right\}$.

The nonlinear term $g(u_t)$ satisfies the following conditions:

(H$_3$) $(g(s),s)\ge 0,\forall s\in H^{2m}(\Omega);\|g(s)\|\le(g(s),s)^{\partial},\partial\in(0,1),\partial+\partial'=1$;

(H$_4$) There is $\sigma\in(0,1)$, $\|g(s)\|\le C_4(R)\left(1+\varepsilon\left\|(-\Delta)^k s\right\|\right)^{1-\sigma}$, $\forall s\in H^{4m}(\Omega)\cap H_0^{2m}(\Omega)$;

(H$_5$) $\|g'(s)\|_\infty\le C_3$.

3. The Existence of the Family of Global Attractor

Lemma 3.1. set $M(s)$ satisfy assumption (H$_1$), $(u_0,u_1)\in E_0=H^{2m}(\Omega)\times L^2(\Omega)$, $f(x)\in L^2(\Omega)$, and $u\in L^\infty(0,+\infty;H^{2m}(\Omega))$, $v\in L^\infty(0,+\infty;L^2(\Omega))\cap L^2(0,T;H^{2m}(\Omega))$, then the smooth solution of problems (1)-(3) $u(x,t)$ and $v(t)=u_t(x,t)+\varepsilon u(x,t)$ satisfy

$$\|(u,v)\|_{E_0}^2=\|D^{2m}u\|^2+\|v\|^2\le w(0)\mathrm{e}^{-bt}+\frac{C_1}{b}\left(1-\mathrm{e}^{-bt}\right).$$

where $v=u_t+\varepsilon u$, $w(0)=\|v_0\|^2+\varepsilon^2\|u_0\|^2+u\|\nabla^{2m}u_0\|^2$, where constants t_0 and R_0, so that

$$\|(u,v)\|_{E_0}^2\le\frac{2C_1}{b}=R_0\,(t>t_0).$$

Proof. It is proved that the inner product of $v=u_t+\varepsilon u$ and Equation (1.1) can be obtained

$$\left(u_{tt}+M\left(\|D^m u\|_p^p\right)(-\Delta)^{2m}u+\beta(-\Delta)^{2m}u_t+g(u_t),v\right)=(f(x),v). \tag{3.1}$$

$$(u_{tt},v)=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2+\frac{\varepsilon^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2-\varepsilon\|v\|^2+\varepsilon^3\|u\|^2. \tag{3.2}$$

$$\begin{aligned}\left(M\left(\|\nabla^m u\|_p^p\right)(-\Delta)^{2m}u,v\right)&=\left(M\left(\|\nabla^m u\|_p^p\right)\nabla^{2m}u,\nabla^{2m}(u_t+\varepsilon u)\right)\\&=\frac{M\left(\|\nabla^m u\|_p^p\right)}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^{2m}u\|^2+\varepsilon M\left(\|\nabla^m u\|_p^p\right)\|\nabla^{2m}u\|^2\\&\ge\frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^{2m}u\|^2+\varepsilon\mu_0\|\nabla^{2m}u\|^2.\end{aligned} \tag{3.3}$$

By using the Poincare's inequality, we obtain

$$\begin{aligned}\left(\beta(-\Delta)^{2m}u_t,v\right)&=\beta\|\nabla^{2m}v\|^2-\left(\beta\varepsilon(-\Delta)^{2m}u,v\right)\\&\ge\frac{\beta}{2}\|\nabla^{2m}v\|^2-\frac{\beta\varepsilon^2}{2}\|\nabla^{2m}u\|^2\ge\frac{\beta\lambda_1^{2m}}{2}\|v\|^2-\frac{\beta\varepsilon^2}{2}\|\nabla^{2m}u\|^2\\&\ge\frac{\beta}{4}\|D^{2m}v\|^2+\frac{\beta\lambda_1^{2m}}{4}\|v\|^2-\frac{\beta\varepsilon^2}{2}\|D^{2m}u\|^2.\end{aligned} \tag{3.4}$$

By using the hypothesis (H_3) and Young's inequality, we obtain the following estimation

$$\begin{aligned}(g(u_t),v)&=(g(u_t),u_t+\varepsilon u)=(g(u_t),u_t)+(g(u_t),\varepsilon u)\\&\ge(g(u_t),u_t)-\varepsilon\|g(u_t)\|_{-2m}\|D^{2m}u\|\\&\ge(g(u_t),u_t)-\varepsilon(g(u_t),u_t)^{\partial}\|D^{2m}u\|\\&\ge(1-\partial)(g(u_t),u_t)-\partial'\varepsilon^{\frac{1}{\partial'}}\|D^{2m}u\|^{\frac{1}{\partial'}}\\&\ge(1-\partial)(g(u_t),u_t)-\frac{\varepsilon^{\frac{1}{\partial'}}}{2}\|D^{2m}u\|^2-\frac{\partial'\varepsilon^{\frac{1}{\partial'}}}{2\partial'-1}.\end{aligned}\tag{3.5}$$

$$(f(x),v)\le\|f\|\|v\|\le\frac{\varepsilon^2}{2}\|v\|^2+\frac{1}{2\varepsilon^2}\|f\|^2.\tag{3.6}$$

Substitute Inequality (3.2)-(3.6) into Equation (3.1), therefore,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|v\|^2+\varepsilon^2\|u\|^2+\mu\|\nabla^{2m}u\|^2\right)+\left(\frac{\beta\lambda_1^{2m}}{2}-\varepsilon^2-2\varepsilon\right)\|v\|^2+2\varepsilon^3\|u\|^2\\&+\left(2\varepsilon\mu_0-\beta\varepsilon^2-\varepsilon^{\frac{1}{\partial'}}\right)\|\nabla^{2m}u\|^2+\frac{\beta}{2}\|D^{2m}v\|^2\le\frac{\|f\|^2}{\varepsilon^2}+\frac{2\partial'\varepsilon^{\frac{1}{\partial'}}}{2\partial'-1}.\end{aligned}\tag{3.7}$$

let $b_1=\frac{\beta\lambda_1^{2m}}{2}-\varepsilon^2-2\varepsilon$, $b_2=2\varepsilon\mu_0-\beta\varepsilon^2-\varepsilon^{\frac{1}{\partial'}}$, and let $b=\min\left\{b_1,\frac{b_2}{\varepsilon},\varepsilon^2\right\}$, $C_1=\frac{\|f\|^2}{\varepsilon^2}+\frac{2\partial'\varepsilon^{\frac{1}{\partial'}}}{2\partial'-1}$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}w(t)+bw(t)+\frac{\beta}{2}\|D^{2m}v\|^2\le C_1,\tag{3.8}$$

where

$$w(t)=\|v\|^2+\varepsilon^2\|u\|^2+\mu\|\nabla^{2m}u\|^2.\tag{3.9}$$

By using the Gronwall's inequality, we get

$$w(t)\le w(0)\mathrm{e}^{-bt}+\frac{C_1}{b}\left(1-\mathrm{e}^{-bt}\right).\tag{3.10}$$

where

$$w(0)=\|v_0\|^2+\varepsilon^2\|u_0\|^2+\mu\|\nabla^{2m}u_0\|^2.\tag{3.11}$$

Then

$$\|(u,v)\|^2_{H^{2m}\times L^2}=\|D^{2m}u\|^2+\|v\|^2\le w(0)\mathrm{e}^{-bt}+\frac{C_1}{b}\left(1-\mathrm{e}^{-bt}\right),\tag{3.12}$$

and

$$\frac{\beta}{2}\int_0^T\|\Delta^m v\|^2\,\mathrm{d}t\le C_1T.\tag{3.13}$$

There are constants $t_0(\Omega)$ and R_0, we, get

$$\|(u,v)\|_{E_0}^2 \le \frac{2C_1}{b} = R_0\left(t > t_0\right). \tag{3.14}$$

Lemma 3.1 is proved.

Lemma 3.2. If lemma 3.1 holds, and the condition is (H_4), set $\left(u_0, v_0\right) \in H^{2m+k}(\Omega) \times H^k(\Omega)$, $k = 1, 2, \cdots, m$. $f \in H^k(\Omega)$, then the smooth solution of Problems (1.1)-(1.3) $u(x,t)$ and $v(t) = u_t(x,t) + \varepsilon u(x,t)$ satisfy

$$\|(u,v)\|_{H^{2m+k}\times H^k}^2 = \left\|D^{2m+k}u(t)\right\|^2 + \left\|D^k v(t)\right\|^2 \le M(0)\mathrm{e}^{-\alpha t} + \frac{C_2}{\alpha}\left(1 - \mathrm{e}^{-\alpha t}\right),$$

where constants $t_1(\Omega)$ and R_1, then

$$\|(u,v)\|_{H^{2m+k}\times H^k}^2 \le \frac{2C_2}{\alpha} = R_1^2, \left(t > t_1(\Omega)\right).$$

Proof. Set $(-\Delta)^k v = (-\Delta)^k u_t + \varepsilon(-\Delta)^k u$. It is obtained by inner product of $(-\Delta)^k v$ and Formula (1.1).

$$\left(u_{tt} + M\left(\|D^m u\|_p^p\right)(-\Delta)^{2m} u + \beta(-\Delta)^{2m} u_t + g\left(u_t\right), (-\Delta)^k v\right) = \left(f(x), (-\Delta)^k v\right). \tag{3.15}$$

$$\begin{aligned}\left(u_{tt}, (-\Delta)^k v\right) &= \left(v_t - \varepsilon u_t, (-\Delta)^k v\right) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^k v\right\|^2 - \varepsilon\left\|D^k v\right\|^2 + \varepsilon^2\left(u, (-\Delta)^k v\right) \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^k v\right\|^2 - \varepsilon\left\|D^k v\right\|^2 + \frac{\varepsilon^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^k u\right\|^2 + \varepsilon^3\left\|D^k u\right\|^2.\end{aligned} \tag{3.16}$$

$$\begin{aligned}&\left(\beta(-\Delta)^{2m} u_t, (-\Delta)^k v\right) = \left(\beta(-\Delta)^{2m}(v - \varepsilon u), (-\Delta)^k v\right) \\ &= \left(\beta(-\Delta)^{2m} v, (-\Delta)^k v\right) - \varepsilon\beta\left((-\Delta)^{2m} u, (-\Delta)^k v\right) \\ &\ge \beta\left\|D^{2m+k} v\right\|^2 - \frac{\beta\varepsilon^2}{2}\left\|D^{2m+k} u\right\|^2 - \frac{\beta\lambda_1^{2m}}{2}\left\|D^k v\right\|^2 \\ &\ge \frac{\beta}{3}\left\|D^{2m+k} v\right\|^2 - \frac{\varepsilon^2\beta}{2}\left\|D^{2m+k} u\right\|^2 + \frac{\lambda_1^{2m}\beta}{6}\left\|D^k v\right\|^2.\end{aligned} \tag{3.17}$$

$$\left(M\left(\|D^m u\|_p^p\right)(-\Delta)^{2m} u, (-\Delta)^k v\right) \ge \frac{\delta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m+k} u\right\|^2 + \varepsilon\delta_0\left\|D^{2m+k} u\right\|^2. \tag{3.18}$$

From hypothesis (H_4), we get

$$\begin{aligned}&\left(g\left(u_t\right), (-\Delta)^k v\right) \\ &\le \left\|g\left(u_t\right)\right\|\left\|D^{2k} v\right\| \\ &\le C_4\left\|D^{2k} v\right\|\left(1 + \varepsilon\left\|(-\Delta)^k u_t\right\|\right)^{1-\sigma} \\ &\le C_4\left\|D^{2k} v\right\|\left(1 + \varepsilon\left\|D^{2k} v - \varepsilon D^{2k} u\right\|\right)^{1-\sigma} \\ &\le C_4\left\|D^{2k} v\right\|\left(1 + \varepsilon\left\|D^{2k} v\right\| + \varepsilon^2\left\|D^{2k} u\right\|\right)^{1-\sigma} \\ &\le C_4\left\|D^{2k} v\right\|\left(1 + \varepsilon\left\|D^{2k} v\right\| + \varepsilon^2\left\|D^{2k} u\right\|\right) \\ &\le C_4\left\|D^{2k} v\right\| + C_4\varepsilon\left\|D^{2k} v\right\|^2 + C_4\varepsilon^2\left\|D^{2k} v\right\|\left\|D^{2k} u\right\|\end{aligned}$$

$$\le C_5 + C_4\left(\varepsilon+\varepsilon^2\right)\left\|D^{2k}v\right\|^2 + \frac{C_4}{2}\varepsilon^2\left\|D^{2k}u\right\|^2$$
$$\le C_5 + C_4\left(\varepsilon+\varepsilon^2\right)\lambda_1^{2m-k}\left\|D^{2m+k}v\right\|^2 + C_4\varepsilon^2\lambda_1^{2m-k}\left\|D^{2m+k}u\right\|^2.$$

Then

$$\left(g\left(u_t\right),(-\Delta)^k v\right) \ge -C_5 - C_4\left(\varepsilon+\varepsilon^2\right)\lambda_1^{2m-k}\left\|D^{2m+k}v\right\|^2 - C_4\varepsilon^2\lambda_1^{2m-k}\left\|D^{2m+k}u\right\|^2. \tag{3.19}$$

$$\left(f(x),(-\Delta)^k v\right) = \left(\nabla^k f(x),\nabla^k v\right) \le \frac{\left\|\nabla^k f\right\|^2}{2\varepsilon^2} + \frac{\varepsilon^2}{2}\left\|\nabla^k v\right\|^2. \tag{3.20}$$

Substitute Inequality (3.16)-(3.20) into Equation (3.15), therefore,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|D^k v\right\|^2 + \varepsilon^2\left\|D^k u\right\|^2 + \delta\left\|D^{2m+k}u\right\|^2\right) + \left(\frac{\beta\lambda_1^{2m}}{3} - 2\varepsilon - \varepsilon^2\right)\left\|D^k v\right\|^2\\ &+ 2\varepsilon^3\left\|D^k u\right\|^2 + \left(2\varepsilon\delta_0 - \beta\varepsilon^2 - C_4\varepsilon^2\lambda_1^{2m-k}\right)\left\|D^{2m+k}u\right\|^2\\ &+\left[\frac{\beta}{3} - C_4\left(\varepsilon+\varepsilon^2\right)\lambda_1^{2m-k}\right]\left\|D^{2m+k}v\right\|^2 + \frac{\beta}{3}\left\|D^{2m+k}v\right\|^2 \le \frac{\left\|\nabla^k f\right\|^2}{\varepsilon^2} + C_5.\end{aligned} \tag{3.21}$$

Let $\alpha_1 = \frac{\beta\lambda_1^{2m}}{3} - 2\varepsilon - \varepsilon^2 \ge 0$,

$$\alpha_2 = 2\varepsilon\delta_0 - \beta\varepsilon^2 - C_4\varepsilon^2\lambda_1^{2m-k} \ge 0,$$

$\frac{\beta}{3} - C_4\left(\varepsilon+\varepsilon^2\right)\lambda_1^{2m-k} \ge 0$, and let $\alpha = \min\left\{\alpha_1, \frac{\alpha_2}{\delta}, \varepsilon^3\right\}$, $C_2 = \frac{\left\|\nabla^k f\right\|^2}{\varepsilon^2} + C_5$,

then

$$\frac{\mathrm{d}}{\mathrm{d}t}M(t) + \alpha M(t) + \frac{2\beta}{3}\left\|D^{2m+k}v\right\|^2 \le C_2. \tag{3.22}$$

where

$$M(t) = \left\|D^k v\right\|^2 + \varepsilon^2\left\|D^k u\right\|^2 + \delta\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m+k}u\right\|^2. \tag{3.23}$$

By using the Gronwall's inequality, we get

$$M(t) \le M(0)\mathrm{e}^{-\alpha t} + \frac{C_2}{\alpha}\left(1-\mathrm{e}^{-\alpha t}\right), \tag{3.24}$$

where

$$M(0) = \left\|D^k v_0\right\|^2 + \varepsilon^2\left\|D^k u_0\right\|^2 + \delta\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m+k}u_0\right\|^2. \tag{3.25}$$

Then

$$\left\|(u,v)\right\|^2_{H^{2m+k}\times H^k} = \left\|D^{2m+k}u(t)\right\|^2 + \left\|D^k v(t)\right\|^2 \le M(0)\mathrm{e}^{-\alpha t} + \frac{C_2}{\alpha}\left(1-\mathrm{e}^{-\alpha t}\right), \tag{3.26}$$

and

$$\frac{\beta}{3}\int_0^T\left\|D^{2m+k}v\right\|^2\mathrm{d}t \le C_2 T. \tag{3.27}$$

There are constants $t_1(\Omega)$ and R_1, then

$$\|(u,v)\|^2_{H^{2m+k}\times H^k} \le \frac{2C_2}{\alpha} = R_1^2. \tag{3.28}$$

Lemma 3.2 is proved.

Theorem 3.1. Assume that the nonlinear function $M(s)$ satisfies (H_1), (H_2), $(u_0,u_i)\in E_0$, $f(x)\in L^2(\Omega)$, then the Problems (1.1)-(1.3) have a unique global smooth solution $(u,v)\subset L^\infty([0,+\infty),E_0)$, and $v\in L^2(0,T;H_0^{2m}(\Omega))$.

Proof: The proof of existence is divided into the following three steps by Galekin's method:

Step 1: Approximate solution

Suppose the eigenvector w_j of $(-\Delta)^{2m}\omega_j=\lambda_j^{2m}\omega_j$ generates an orthonormal basis for H^{2m}, where λ_j is the eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary on Ω, define k order approximation $u_k(t)$:

$$u_k(t)\in span[\omega_1,\omega_2,\cdots,\omega_k], u_k(t)=\sum_{j=1}^{k} g_{jk}(t)\omega_j.$$

$$\begin{aligned}&\left(u_k''(t)+M\left(\|D^m u_k(t)\|_P^P\right)(-\Delta)^{2m}u_k(t)+\beta(-\Delta)^{2m}u_k'(t)+g(u_k),\omega_j\right)\\&=\left(f(x),\omega_j\right).\end{aligned} \tag{3.29}$$

where $1<j<k$,

$$u_k(0)=v_{0k}, u_k'(0)=u_{1k}, u_{0k},u_{1k}\in span[\omega_1,\omega_2,\cdots,\omega_k].$$

with in H,

$$u_{0k}\to u_0, u_{1k}\to u_1. \tag{3.30}$$

This system of ordinary differential equations about $g_{jk}(t)$ can determine $u_k(t)$ in the interval $[0,t_k]$; need to prove $t_k=T$.

Step 2: Prior estimation

According to the conclusion and proof method of lemma 3.1, $(u_k(t),u_k'(t))$ is uniformly bounded on E_0, then

$$\|u_k(t)\|\le R \tag{3.31}$$

$$\|D^{2m}u_k(t)\|\le R \tag{3.32}$$

$$\|u_k'(t)\|\le R \tag{3.33}$$

thus it can be seen $t_k=T$, Inequality (3.32)-(3.33) shows $u_k(t)$ is bounded in $L^\infty(0,T;H^{2m}(\Omega))$, and $u_k'(t)$ is bounded in $L^\infty(0,T;L^2(\Omega))$.

And it's actually available $u_k(t)$ in $L^\infty(0,+\infty;H^{2m}(\Omega))$ and $u_k'(t)$ in $L^\infty(0,+\infty;L^2(\Omega))$.

Step 3: Limit process

According to Danford-Pttes theorem, Space $L^\infty(0,T;H^{2m}(\Omega))$ conjugate to space $L'(0,T;H^{-2m}(\Omega))$; Space $L^\infty(0,T;L^2(\Omega))$ conjugate to space $L'(0,T;L^2(\Omega))$, select the subsequence $u_h(t)$ from the sequence $u_k(t)$, such that

$u_h(t) \to u$ weakly * converges in $L^\infty\left(0,T;H^{2m}(\Omega)\right)$,
$u_h'(t) \to u'$ weakly * converges in $L^\infty\left(0,T;L^2(\Omega)\right)$,
$u_h'(t) \to u'$ weakly converges in $L^2\left(0,T;H^{2m}(\Omega)\right)$.

According to Pellich-Kohdarachov theorem, $H^{2m}(\Omega)$ is compact embedded in $L^P(\Omega)$ and $L^2(\Omega)$, $u_k(t) \to u$ is strong convergence almost everywhere in $L^P(0,T)$ and $L^2(0,T)$.

$M\left(\|u_k(t)\|_p^p\right) \to M\left(\|u(t)\|_p^p\right)$ converges in R^+.

$M\left(\|u_k(t)\|_p^p\right)\left((-\Delta)^{2m} u_k(t), D^{2m} w_j(t)\right) \to M\left(\|u(t)\|_p^p\right)\left((-\Delta)^{2m} u_k(t), D^{2m} w_j(t)\right)$ is weak * convergence in $L^\infty(0,T)$.

$\beta\left(\nabla^{2m} u_k'(t), \nabla^{2m} w_j(t)\right) \to \beta\left(\nabla^{2m} u'(t), \nabla^{2m} w_j\right)$ is weak * convergence in $L^\infty(0,T)$.

$\left(u_h(t), \omega_j\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(u_k'(t), \omega_j\right) \to \left(u''(t), \omega_j\right)$ converges in $D'(0,T)$.

$\left(g\left(u_k'(t), \omega_j\right)\right) \to \left(g\left(u'(t), \omega_j\right)\right)$ is weak * convergence in $L^\infty(0,T)$.

From the Formula (3.30), we get

$$\begin{aligned}&\left(u''(t), \omega_j\right) + \left(M\left(\|D^m u\|_p^p\right)\right)\left(\nabla^{2m} u, \nabla^{2m} \omega_j\right) + \beta\left(\nabla^{2m} u'(t), \nabla^{2m} w_j\right)\\ &+\left(g(u_t), \omega_j\right) = \left(f(x), \omega_j\right)\end{aligned}$$

For $\forall j$, according to the density of $\omega_1, \omega_2, \cdots, \omega_k, \cdots$.

$$\begin{aligned}&\left(u'', \varphi\right) + \left(M\left(\|D^m u\|_p^p\right)\right)\left(\nabla^{2m} u, \nabla^{2m} \varphi\right) + \beta\left(\nabla^{2m} u'(t), \nabla^{2m} \varphi\right) + \left(g(u_t), \varphi\right)\\ &= \left(f(x), \varphi\right)\end{aligned} \tag{3.34}$$

$\forall \varphi \in H^{2m}(\Omega)$ and $u_k(0) \to u(0)$ weakly converges in $L^2(\Omega)$, and in $H^{2m}(\Omega)$.

$$u_k(0) \to u(0),$$

$$\left(u_k'(0), \omega_j\right) \to \left(u', \omega_j\right)\Big|_{t=0} = \left(u'(0), \omega_j\right),$$

$$\left(u_k'(0), \omega_j\right) \to \left(u_1, \omega_j\right),$$

then $\left(u'(0), \omega_j\right) \to \left(u_1, \omega_j\right)$ is satisfied for all j, so that existence can be proved.

Then prove the uniqueness of the solution.

Set u^*, v^* be two solutions of the Problem (1.1)-(1.3), let $\omega = u^* - v^*$, then

$$\begin{aligned}&\omega_{tt} + \beta(-\Delta)^{2m} \omega_t + M\left(\|D^m u^*(t)\|_p^p\right)(-\Delta)^{2m} u^* - M\left(\|D^m v^*(t)\|_p^p\right)(-\Delta)^{2m} v^*\\ &+ g\left(u_t^*\right) - g\left(v_t^*\right) = 0,\end{aligned} \tag{3.35}$$

$$\omega(0) = 0, \omega'(0) = 0. \tag{3.36}$$

Take the inner product with ω_t and (3.34).

$$\begin{aligned}&\left(\omega_{tt} + \beta(-\Delta)^{2m} \omega_t + M\left(\|D^m u^*(t)\|_p^p\right)(-\Delta)^{2m} u^*, \omega_t\right)\\ &-\left(M\left(\|D^m v^*(t)\|_p^p\right)(-\Delta)^{2m} v^* + g\left(u_t^*\right) - g\left(v_t^*\right), \omega_t\right) = 0.\end{aligned} \tag{3.37}$$

$$\left(\omega_{tt}, \omega_t\right) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\omega_t\right\|^2. \tag{3.38}$$

$$\beta\left(\left(-\Delta\right)^{2m}\omega_t, \omega_t\right) \ge \frac{\beta}{2}\left\|D^{2m}\omega_t\right\|^2. \tag{3.39}$$

By using hypothesis (H_5), there are $\left\|g'(s)\right\|_\infty \le C_3$, then

$$\left|\left(\left(g\left(u_t^*\right) - g\left(v_t^*\right)\right), w_t\right)\right| = \left|\left(g'(s)w_t, w_t\right)\right| \le C_3\left\|w_t\right\|^2. \tag{3.40}$$

By using hypothesis (H_1), Lemma 3.1 and Differential Mean Value Theorem

$$\begin{aligned}
&\left(M\left(\left\|D^m u^*\right\|_p^p\right)\left(-\Delta\right)^{2m}u^* - M\left(\left\|D^m v^*(t)\right\|_p^p\right)\left(-\Delta\right)^{2m}v^*, w_t\right)\\
&= \frac{1}{2}\left(M\left(\left\|D^m u^*\right\|_p^p\right)\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m}w\right\|^2 + M\left(\left\|D^m u^*(t)\right\|_p^p - M\left(\left\|D^m v^*\right\|_p^p\right)\right)\left(-\Delta\right)^{2m}v^*, w_t\right)\\
&\ge \frac{1}{2}\mu\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m}w\right\|^2 - \left\|M'(\xi)\right\|_\infty C_6\left(\left\|D^m w\right\|_{p-1}^{p-1}\right)\cdot\left\|D^m w\right\|\cdot\left\|D^{2m}v^*\right\|\left\|D^{2m}w_t\right\|\\
&\ge \frac{1}{2}\mu\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m}w\right\|^2 - C_7\left\|D^m w\right\|\left\|D^{2m}w_t\right\|\\
&\ge \frac{1}{2}\mu\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m}w\right\|^2 - \frac{\beta}{2}\left\|D^{2m}w_t\right\|^2 - \frac{C_7^2}{2\beta}\left\|D^m w\right\|^2.
\end{aligned} \tag{3.41}$$

To sum up, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|w_t\right\|^2 + \mu\left\|D^{2m}w\right\|^2\right) \le \frac{C_7^2}{\beta}\left\|D^m w\right\|^2 \le \frac{C_7^2}{\mu\beta\lambda_1^m}\left(\left\|w_t\right\|^2 + \mu\left\|D^{2m}w\right\|^2\right).$$

From Gronwall's inequality, we get

$$\left\|w_t\right\|^2 + \mu\left\|D^{2m}w(t)\right\|^2 \le \mathrm{e}^{\frac{C_7^2}{\mu\beta\lambda_1^m}t}\left(\left\|w(0)\right\|^2 + \mu\left\|D^{2m}w_1(0)\right\|^2\right) = 0. \tag{3.42}$$

Therefore $u^* = v^*$, the uniqueness is proved.

Theorem 3.2. [9] Let E be a Banach space, $S(t): E \to E$ Semigroups satisfy the following conditions

1) Semigroup $S(t)$ is uniformly bounded in E, then $\forall r > 0$, there is constant $C(r)$, so that when $\left\|u\right\|_E \le r$, there is $\left\|S(t)u\right\|_E \le C(r), \left(\forall t \in [0, +\infty)\right)$;

2) There is a bounded absorption set B_0 in E;

3) $S(t)$ is a fully continuous operator. That is, semigroups $S(t)$ have compact global attractors A_0.

The Banach space E in theorem 3.2 is changed into Hilbert space E_k, there are the following existence theorems of global attractor's families.

Theorem 3.3. Under the hypothesis of lemma 3.1 and lemma 3.2, Then there is a family of global attractor $A_k\left(k = 1, 2, \cdots, 2m\right)$ for Problems (1.1)-(1.3). That is, there is a compact set $A_k \subset E_k \subset E_0$, make:

1) $S(t)A_k = A_k, \forall t > 0$;

2) $\lim\limits_{t\to\infty} dist\left(S(t)B_k, A_k\right) = 0$ ($\forall B_k \subset E_k$ is a bounded set), among $\lim\limits_{t\to\infty} dist\left(S(t)B_k, A_k\right) = \sup\limits_{x\in B_k}\inf\limits_{y\in A_k}\left\|S(t)x - y\right\|_{E_k}$, $S(t)$ is the solution semigroup generated by Problems (1.1)-(1.3).

Proof: It is necessary to verify the conditions (1), (2) and (3) of Theorem 3.2. Solution Semigroup $S(t): E_k \to E_k$ generated by Theorem 3.1 and Lemma 3.2 to know Problems (1.1)-(1.3).

1) Knowing the arbitrary bounded set $B_k \subset E_k$ by Lemma 3.2, have $\|(u,v)\|^2_{H^{2m+k}\times H^k} \le R_1^2$ and

$$\|S(t)(u_0,v_0)\|^2_{E_k} = \|u\|^2_{H_0^{2m+k}(\Omega)} + \|v\|^2_{H_0^k(\Omega)} \le \|u_0\|^2_{H_0^{2m+k}(\Omega)} + \|v_0\|^2_{H_0^k(\Omega)} \le R_k^2. \tag{3.43}$$

where $t \ge 0$, $(u_0,v_0) \in B_k$, this shows that $\{S(t)\}(t \ge 0)$ is uniformly bounded with in E_k.

2) Further, about $\forall (u_0,v_0) \in E_k$, there is

$$\|S(t)(u_0,v_0)\|^2_{E_k} = \|u\|^2_{H_0^{2m+k}(\Omega)} + \|v\|^2_{H_0^k(\Omega)} \le R_1^2 + R_0^2 \le R_k^2. \tag{3.44}$$

Then B_{0k} is a bounded absorbing set of semigroup $S(t)$.

3) According to Rellich-Kondrachov compact embedding theorem $E_k \subset E_0$, then the bounded set in E_k is the compact set in E_0, therefore, the solution semigroup $S(t)$ is a fully continuous operator. Therefore, the family of global attractor A_k of the solution semigroup $S(t)$ can be obtained, where

$$A_k = W(B_{0k}) = \bigcap_{0\le\tau\le t} \overline{\bigcup S(t) B_{0k}}. \tag{3.45}$$

The theorem is completed.

4. Dimension Estimation of the Family of Global Attractor

First consider the linearization of Problems (1.1)-(1.3)

$$\begin{aligned} &U_{tt} + M\left\|D_u^m\right\|_p^p (-\Delta)^{2m} U + M\left\|D_u^m\right\|_p^p \left(\left\|D_u^m\right\|_p^p\right)' (-\Delta)^{2m} U \\ &+ \beta(-\Delta)^{2m} U_t + g'(u_t) U_t = 0. \end{aligned} \tag{4.1}$$

$$U(x,t)\big|_{\partial\Omega} = (-\Delta)^k U(x,t)\big|_{\partial\Omega} = 0,\ k = 1,\cdots,2m-1,\ t > 0, \tag{4.2}$$

$$U(x,0) = \xi_1, U_t(x,0) = \xi_2. \tag{4.3}$$

where $(\xi_1,\xi_2) \in E_k$, $(u,u_t) = S(t)(u_0,u_1)$ is the solution of Problems (1.1)-(1.3), known $(u_0,u_1) \in A_k$, $S(t): E \to E$, can prove $\forall (\xi_1,\xi_2) \in E_k$, the linearization Problems (4.1)-(4.3) have unique solutions $(U(t),U_t(t)) \in L^\infty((0,+\infty); E_k)$.

Theorem 4.1. $\forall t > 0, R > 0$, the mapping $S(t): E_k \to E_k$ is Frechet differentiable on E_k, and differentiate the linear operator of $G(t): (\xi_1,\xi_2)^{\mathrm{T}} \to (U(t),U_t(t))^{\mathrm{T}}$, where $U(t), U_t(t)$ is the solution of problems (4.1)-(4.3).

Proof: Set $\phi_0 = (u_0,u_1)^{\mathrm{T}} \in E_k$, $\overline{\phi_0} = (u_0+\xi_1, u_1+\xi_2)^{\mathrm{T}} \in E_k$, then $\|\phi_0\|_{E_k} \le R, \|\overline{\phi_0}\|_{E_k} \le R$, from this we can get the Lipchitz property of $S(t)$ on the bounded set of E_k, that is,

$$\left\|S(t)\overline{\phi_0} - S(t)\phi_0\right\|^2_{E_k} \le \mathrm{e}^{ct} \left\|(\xi_1,\xi_2)\right\|^2_{E_k}. \tag{4.4}$$

Set $\sigma = \tilde{u} - u - U$, then

$$\sigma_{tt} + M\left(\left\|D^m u\right\|_p^p\right)(-\Delta)^{2m}\sigma + \beta(-\Delta)^{2m}\sigma_t = h. \tag{4.5}$$

$$\sigma(0) = \xi_1, \sigma_t(0) = \xi_2, \tag{4.6}$$

let $s = \left\|\nabla^m u\right\|_p^p, \tilde{s} = \left\|\nabla^m \tilde{u}\right\|_p^p$,

then

$$\begin{aligned} h = &\left(M(s) - M(\tilde{s})\right)(-\Delta)^{2m}\tilde{u} + pM'(s)\left\|\nabla^m u\right\|^{p-2}\left\|\nabla^m u\right\|\left\|D^m U\right\|(-\Delta)^{2m} u \\ &+ g(u_t) - g\left(\overline{u_t}\right) + g'(u_t)U_t. \end{aligned} \tag{4.7}$$

Take the inner product of $(-\Delta)^k \sigma_t$ and both sides of Formula (4.5)

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^k\sigma_t\right\|^2 + \frac{1}{2}M(s)\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m+k}\sigma\right\|^2 + \beta\left\|D^{2m+k}\sigma_t\right\|^2 = \left(h, (-\Delta)^k\sigma_t\right). \tag{4.8}$$

Let $\overline{u} = u - \tilde{u}$, according to lemma 3.2, differential mean value theorem and poincare's inequality

$$\begin{aligned} &\left(\left(M(s) - M(\tilde{s})\right)(-\Delta)^{2m}\tilde{u} + pM'(s)\left\|\nabla^m u\right\|^{p-2}\left\|\nabla^m u\right\|\left\|D^m U\right\|(-\Delta)^{2m} u, (-\Delta)^k\sigma_t\right) \\ &= \left(M'\left(as + (1-a)\tilde{s}\right)(s - \tilde{s})(-\Delta)^{2m}\tilde{u} + pM'(s)\left\|\nabla^m u\right\|^{p-2}\left\|\nabla^m u\right\|\left\|D^m U\right\|(-\Delta)^{2m} u, (-\Delta)^k\sigma_t\right) \\ &\geq \left(M'\left(as + (1-a)\tilde{s}\right)\left(\nabla^m\overline{u}, \left\|\nabla^m u\right\|^{p-1} + \left\|\nabla^m\tilde{u}\right\|^{p-1}\right)(-\Delta)^{2m}\tilde{u}, (-\Delta)^k\sigma_t\right) \\ &\quad + \left(pM'(s)\left\|\nabla^m u\right\|^{p-2}\left\|\nabla^m u\right\|\left\|D^m U\right\|(-\Delta)^{2m} u, (-\Delta)^k\sigma_t\right) \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{4.9}$$

$$\begin{aligned} I_1 &= \left(M'\left(as + (1-a)\tilde{s}\right)\left(\nabla^m\overline{u}, \left\|\nabla^m u\right\|^{p-1} + \left\|\nabla^m\tilde{u}\right\|^{p-1}\right)(-\Delta)^{2m}\tilde{u}, (-\Delta)^k\sigma_t\right) \\ &\quad - \left(M'(s)\left(\nabla^m\overline{u}, \left\|\nabla^m u\right\|^{p-1} + \left\|\nabla^m\tilde{u}\right\|^{p-1}\right)(-\Delta)^{2m}\tilde{u}, (-\Delta)^k\sigma_t\right) \\ &= \left(M''(\xi)\left(as + (1-a)\tilde{s} - s\right)\right)\left(\nabla^m\overline{u}, \left\|\nabla^m u\right\|^{p-1} + \left\|\nabla^m\tilde{u}\right\|^{p-1}\right)\left((-\Delta)^{2m}\tilde{u}, (-\Delta)^k\sigma_t\right) \\ &= M''(\xi)(1-a)(\tilde{s} - s)\left(\nabla^m\overline{u}, \left\|\nabla^m u\right\|^{p-1} + \left\|\nabla^m\tilde{u}\right\|^{p-1}\right)\left((-\Delta)^{2m}\tilde{u}, (-\Delta)^k\sigma_t\right) \\ &\leq C_8\left\|\nabla^{2m+k}\overline{u}\right\|^2\left\|\nabla_{2m+k}\sigma_t\right\|. \end{aligned} \tag{4.10}$$

$$\begin{aligned} I_2 &= M'(s)\left(\nabla^m\overline{u}, \left\|\nabla^m u\right\|^{p-1} + \left\|\nabla^m\tilde{u}\right\|^{p-1}\right)\left((-\Delta)^{2m}\tilde{u}, (-\Delta)^k\sigma_t\right) \\ &\quad - pM'(s)\left(\nabla^m\overline{u} + \nabla^m\sigma, \left\|\nabla^m u\right\|^{p-1}\right)\left((-\Delta)^{2m}\tilde{u}, (-\Delta)^k\sigma_t\right) \\ &= M'(s)\left(\nabla^m\overline{u}, \left\|\nabla^m\tilde{u}\right\|^{p-1}\right)\left((-\Delta)^{2m}\tilde{u}, (-\Delta)^k\sigma_t\right) \\ &\quad - pM'(s)\left(\nabla^m\sigma, \left\|\nabla^m u\right\|^{p-1}\right)\left((-\Delta)^{2m}\tilde{u}, (-\Delta)^k\sigma_t\right) \end{aligned}$$

$$\begin{aligned}&\le M'(s)\left\|\nabla^{2m+k}\bar{u}\right\|^2\left\|\nabla^m\tilde{u}\right\|^{p-1}\left\|\nabla^{2m}\tilde{u}\right\|\left\|\nabla^{2m+k}\sigma_t\right\|\\&\quad+pM'(s)\left\|\nabla^{m+k}\sigma\right\|\left\|\nabla^m u\right\|^{p-1}\left\|\nabla^{2m}\tilde{u}\right\|\left\|\nabla^{2m+k}\sigma_t\right\|\\&\le C_9\left(\left\|\nabla^{2m+k}\bar{u}\right\|^2+\left\|\nabla^{2m+k}\sigma\right\|\right)\left\|\nabla^{2m+k}\sigma_t\right\|.\end{aligned} \tag{4.11}$$

$$\begin{aligned}I_3&=-pM'(s)\left\|\nabla^m u\right\|^{p-2}\left\|\nabla^m u\right\|\left\|\nabla^m U\right\|\left((-\Delta)^{2m}\tilde{u},(-\Delta)^k\sigma_t\right)\\&\quad+pM'(s)\left\|\nabla^m u\right\|^{p-2}\left\|\nabla^m u\right\|\left\|\nabla^m U\right\|\left((-\Delta)^{2m}u,(-\Delta)^k\sigma_t\right)\\&=pM'(s)\left\|\nabla^m u\right\|^{p-2}\left(\nabla^m\tilde{u}-\nabla^m u-\nabla^m\sigma,\nabla^m u\right)\left((-\Delta)^{2m}\bar{u},(-\Delta)^k\sigma_t\right)\\&\le pM'(s)\left\|\nabla^{2m+k}\bar{u}\right\|^2\left\|\nabla^m u\right\|^{p-2}\left\|\nabla^{2m+k}\sigma_t\right\|\\&\quad+\left\|\nabla^{m+k}\sigma\right\|\left\|\nabla^m u\right\|^{p-1}\left\|\nabla^{2m}\bar{u}\right\|\left\|\nabla^{2m+k}\sigma_t\right\|\\&\le C_{10}\left(\left\|\nabla^{2m+k}\bar{u}\right\|^2+\left\|\nabla^{2m+k}\sigma\right\|\right)\left\|\nabla^{2m+k}\sigma_t\right\|.\end{aligned} \tag{4.12}$$

It can be obtained from hypothesis (H1), Holder's inequality, Young's inequality, Poincare's inequality and differential mean value theorem, let $w_t=u_t-\overline{u_t}$

$$\begin{aligned}&\left(g(u_t)-g\left(\overline{u_t}\right)+g'(u_t)U_t,(-\Delta)^k\sigma_t\right)\\&=\left(g'(x)w_t,(-\Delta)^k\sigma_t\right)-\left(g'(u_t)w_t,(-\Delta)^k\sigma_t\right)-\left(g'(u_t)\sigma,(-\Delta)^k\sigma_t\right)\\&\le(\theta-1)\left(g''(\xi)\left\|w_t\right\|^2,(-\Delta)^k\sigma_t\right)-\left(g'(u_t)\sigma,(-\Delta)^k\sigma_t\right)\\&\le C_{11}\left(\nabla^{2m+k}\left\|w_t\right\|^2+\nabla^k\left\|\sigma_t\right\|\right)\nabla^{2m+k}\left\|\sigma_t\right\|.\end{aligned} \tag{4.13}$$

Substitute (4.9)-(4.13) into Formula (4.8), then it can be obtained by Young's inequality and Poincare's inequality

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\mu D^{2m+k}\sigma\right\|^2+\left\|D^k\sigma_t\right\|^2\right)\\&\le C_{12}\left(\left\|\mu D^{2m+k}\sigma\right\|^2+\left\|D^k\sigma_t\right\|^2\right)+C_{13}\left(\left\|\mu D^{2m+k}\sigma\right\|^4\right)+C_{14}\left(\left\|\nabla^{2m+k}w_t\right\|^2\right).\end{aligned}$$

According to Gronwall's inequality

$$\left\|D^k\sigma_t\right\|^2+\mu\left\|D^{2m+k}\sigma_t\right\|^2\le C_{15}\cdot\mathrm{e}^{C_{12}t}\left\|(\xi_1,\xi_2)\right\|_{E_k}^4. \tag{4.14}$$

When $\left\|(\xi_1,\xi_2)\right\|_{E_k}^2\to 0$, there is

$$\frac{\left\|\overline{\phi(t)}-\phi(t)-U(t)\right\|}{\left\|(\xi_1,\xi_2)\right\|_{E_k}^2}\le C_{15}\mathrm{e}^{C_{12}t}\left\|(\xi_1,\xi_2)\right\|_{E_k}^2\to 0. \tag{4.15}$$

The theorem is proved.

Theorem 4.2. Under the condition of theorem 3.3, the family of global attractor A_k of Problems (1.1)-(1.3) have finite Hansdorff dimension and Fractal dimension, and $d_H(A_k)<\frac{2n}{5}, d_F(A_k)<\frac{7n}{5}$.

Proof: assume $\Psi=R_\xi\Phi=(u,v)^{\mathrm{T}},\Phi=(u,v_t),v=u_t+\varepsilon u$, then $R_\varepsilon:(u,u_t)^{\mathrm{T}}\to(u,u_t+\varepsilon u)$ is an isomorphic mapping.

Linearize Equation (4.1)

$$\Psi_t + \Lambda_\varepsilon \Psi + \bar{g}(\Psi) = \bar{f}. \tag{4.16}$$

$$\Psi(0) = (u_0, u_1 + \varepsilon u_0)^{\mathrm{T}}. \tag{4.17}$$

where $\Psi = \{u, u_t + \varepsilon u\}^{\mathrm{T}}, \bar{g}(\Psi) = \{0, g(u_t)\}^{\mathrm{T}}, \bar{f} = \{0, f(x)\}^{\mathrm{T}}$

$$\Lambda_\varepsilon = \Lambda(\Psi) = \begin{pmatrix} \varepsilon I & -I \\ M\left(\left\|A^{\frac{m}{2}}u\right\|_p^p - \varepsilon\beta\right)A^{2m} + \varepsilon^2 I & \beta A^{2m} - \varepsilon I \end{pmatrix}. \tag{4.18}$$

$$\Psi_t := F(\Psi) = \bar{f} - \Lambda_\varepsilon \Psi - \bar{g}(\Psi). \tag{4.19}$$

$$P_t = F_t(\Psi)P. \tag{4.20}$$

$$P_t + \Lambda_\varepsilon P + \bar{h}(\Psi) = 0. \tag{4.21}$$

where $P = \{U, U_t + \varepsilon U\}^{\mathrm{T}}, \overline{g_t}(\Psi)P = \{0, g_t(u_t)U\}^{\mathrm{T}}$. $U$is the solution of (4.16).

For a fixed $(u_0, v_0) \in E_k$, Let $\xi_1, \xi_2, \cdots, \xi_n$ be n elements in E_k. letting $U_1(t), U_2(t), \cdots, U_n(t)$ is several solutions of linear Equation (4.1) with initial value of $U_1(0) = \xi_1, U_2(0) = \xi_2, \cdots, U_n(0) = \xi_n$. Available by direct calculation

$$\begin{aligned} &\left\|U_1(t)\Lambda U_2(t)\Lambda\cdots\Lambda U_n(t)\right\|_{\Lambda E_k}^2 \\ &= \left\|\xi_1\Lambda\xi_2\Lambda\cdots\Lambda\xi_n\right\|_{\Lambda E_k}^2 \exp\left(-\int_0^t trF_t'(\Psi)Q_n(\tau)\mathrm{d}\tau\right). \end{aligned} \tag{4.22}$$

where Λ represents the outer product, tr stands for trace, $Q_n(\tau)$ represents the orthogonal projection from E_k to the subspace generated by $U_1(t), U_2(t), \cdots, U_n(t)$.

For a given moment τ, let $w_j(\tau) = (\xi_j(\tau), \eta_j(\tau))^{\mathrm{T}}, j = 1, 2, \cdots, n$ be the standard orthogonal basis of space $U_1(t), U_2(t), \cdots, U_n(t)$.

Define the inner product over E_k

$$\left((\xi,\eta),(\bar{\xi},\bar{\eta})\right) = \left(D^{2m+k}\xi, D^{2m+k}\bar{\xi}\right) + \left(D^k\eta, D^k\bar{\eta}\right). \tag{4.23}$$

To sum up, there are

$$trF_t(\Psi(\tau))Q(\tau) = \sum_{j=1}^n \left(F_t(\Psi(\tau))Q_n(\tau)w_j(\tau), w_j(\tau)\right)_{E_k}. \tag{4.24}$$

where

$$\left(F_t(\Psi(\tau))w_j(\tau), w_j(\tau)\right)_{E_k} = -\left(\Lambda_\varepsilon w_j, w_j\right)_{E_k} - \left(g_t(u_t)w_j, w_j\right). \tag{4.25}$$

$$\begin{aligned} &\left(\Lambda_\varepsilon \omega_j, \omega_j\right)_{E_k} \\ &= \left(\left(\varepsilon\xi_j - \eta_j, \left(M\left\|D^m u\right\|_p^p - \beta\varepsilon\right)A^{2m}\xi_j + \varepsilon^2\xi_j + D\xi_j + \beta A^{2m}\eta_j - \varepsilon\eta_j\right), (\xi_j, \eta_j)\right) \\ &= \varepsilon\left\|D^{2m+k}\xi_j\right\|^2 + \left(M\left\|D^m u\right\|_p^p - \beta\varepsilon - 1\right)\left(D^{2m+k}\xi_j, D^{2m+k}\eta_j\right) \\ &\quad + \varepsilon^2\left(D^k\xi_j, D^k\eta_j\right) + \left(D^kD\xi_j, D^k\eta_j\right) + \beta\left\|D^{2m+k}\eta_j\right\|^2 - \varepsilon\left\|D^k\eta_j\right\|^2 \end{aligned}$$

$$
\begin{aligned}
&\geq \varepsilon\left\|D^{2m+k}\xi_j\right\|^2-(M-\beta\varepsilon-1)\left\|D^{2m+k}\xi_j\right\|\left\|D^{2m+k}\eta_j\right\|+\beta\left\|D^{2m+k}\eta_j\right\|^2 \\
&\quad -\varepsilon\left\|D^k\eta_j\right\|^2-\frac{\varepsilon^2}{\lambda_j^m}\left\|D^{2m+k}\xi_j\right\|\left\|D^k\eta_j\right\| \\
&\quad -pM'\left\|D^m u\right\|_p^p\left\|D^m u\right\|_{L^{2p-2}(\Omega)}^{p-1}\left\|D^m\xi_j\right\|\left\|D^{2m+k}u\right\|\left\|D^{2m+k}\eta_j\right\| \\
&\geq\left(\varepsilon-\frac{\varepsilon}{2}\right)\left\|D^{2m+k}\xi_j\right\|^2+\left(\frac{\beta}{2}-\frac{1}{2\varepsilon}\right)\left\|D^{2m+k}\eta_j\right\|^2-\varepsilon\left\|D^k\eta_j\right\|^2 \\
&\quad -\frac{\varepsilon^2}{2\lambda_j^m}\left(\left\|D^{2m+k}\xi_j\right\|^2+\left\|D^k\eta_j\right\|^2\right)-\frac{C_{16}}{\beta}\left\|D^m\xi_j\right\|^2 \\
&\geq\left(\frac{\varepsilon}{2}-\frac{\varepsilon^2}{2\lambda_j^m}-\frac{C_{16}}{\beta\lambda_j^{\frac{m+k}{2}}}\right)\left\|D^{2m+k}\xi_j\right\|^2+\left(\left(\frac{\beta}{2}-\frac{1}{2\varepsilon}\right)\lambda_1^m-\varepsilon-\frac{\varepsilon^2}{2\lambda_j^m}\right)\left\|D^k\eta_j\right\|^2 \\
&\quad -\frac{C_{16}}{\beta}\left\|D^m\xi_j\right\|^2
\end{aligned} \tag{4.26}
$$

$$
a=\min\left\{\frac{\varepsilon}{2}-\frac{\varepsilon^2}{2\lambda_j^m}-\frac{C_{16}}{\beta\lambda_j^{\frac{m+k}{2}}},\left(\frac{\beta}{2}-\frac{1}{2\varepsilon}\right)\lambda_1^m-\varepsilon-\frac{\varepsilon^2}{2\lambda_j^m}\right\}.
$$

Present hypothesis $\{u_0,u_1\}\in A$, according to theorem 3.3, A is a bounded absorption set in E_1.

$\psi(t)=\{u(t),u_t(t)+\varepsilon u(t)\}\in D(A)$; $D(A)=\{u\in v, Au\in H\}$, there is $s\in[0,1]$ with mapping $h_1:D(A)\to p(v_s,H)$

$$
R_A=\sup_{(w,\eta)\in A}|A\xi|<\infty, \tag{4.27}
$$

$$
\sup_{u\in D(A),|A_u<R_A|}\left|g_t(u_t)\right|_{p(v_s,H)}\leq R<\infty, \tag{4.28}
$$

where $\left\|g_t(u_t)\omega_j,\eta_j\right\|$ satisfy

$$
\left\|g_t(u_t)\omega_j,\eta_j\right\|\leq r\left\|\omega_j\right\|_s\left\|\eta_j\right\|. \tag{4.29}
$$

Comprehensive the above contents are as follows

$$
\begin{aligned}
\left(F_t(\psi)\omega_j,\omega_j\right)_{E_k}&\leq -a\left(\left\|D^{2m+k}\xi_j\right\|^2+\left\|D^k\eta_j\right\|^2\right)-\frac{C_{14}}{\beta}\left\|D^m\xi_j\right\|^2+r\left\|\omega_j\right\|_s\left\|\eta_j\right\| \\
&\leq-\frac{a}{2}\left(\left\|D^{2m+k}\xi_j\right\|^2+\left\|D^k\eta_j\right\|^2\right)+\frac{r}{2}\left\|\xi_j\right\|^2.
\end{aligned} \tag{4.30}
$$

where

$$
\left\|D^{2m+k}\xi_j\right\|^2+\left\|D^k\eta_j\right\|^2=1. \tag{4.31}
$$

$$
\sum_{j=1}^n\left(F_t(\psi(\tau))\omega_j(\tau),\omega_j(\tau)\right)_{E_k}\leq-\frac{na}{2}+\frac{r}{2}\left\|\xi_j\right\|^2. \tag{4.32}
$$

Almost any t has

$$
\sum_{j=1}^n\left\|\xi_j\right\|_s^2\leq\sum_{j=1}^{n-1}\lambda_j^{s-1}. \tag{4.33}
$$

So

$$T_r F_t\left(\psi(\tau) Q_n(\tau)\right) \le -\frac{na}{2} + \frac{r}{2}\sum_{j=1}^{n-1}\lambda_j^{s-1}. \tag{4.34}$$

Set

$$q_n(t) = \sup_{\psi_0 \in A} \sup_{D_j \in E_K, \left|D^K \eta_j\right| \le 1} \left(\frac{1}{t}\int_0^t trF_t\left(s(\tau)\psi_0\right) Q_n(\tau)\right) \mathrm{d}\tau. \tag{4.35}$$

$$q_N = \limsup_{t\to\infty} q_N. \tag{4.36}$$

It can be known from (4.32)

$$q_n \le -\frac{na}{2} + \frac{r}{2}\sum_{j=1}^{n-1}\lambda_j^{s-1}. \tag{4.37}$$

Therefore, the Lyapunov exponent $\mu_j\left(j \in N\right)$ of B_{0k} is uniformly bounded

$$\mu_1 + \mu_2 + \cdots + \mu_n \le -\frac{na}{2} + \frac{r}{2}\sum_{j=1}^{n-1}\lambda_j^{s-1}. \tag{4.38}$$

From the above knowledge, there are $n > 1$ and $s \in [0,1]$,

$$\left(q_j\right)_+ \le -\frac{na}{2} + \frac{r}{2}\sum_{j=1}^{n-1}\lambda_j^{s-1} \le \frac{r}{2}\sum_{j=1}^{n-1}\lambda_j^{s-1} \le \frac{na}{7}. \tag{4.39}$$

$$q_n \le -\frac{na}{2}\left(1 - \frac{r}{na}\sum_{j=1}^{n-1}\lambda_j^{s-1}\right) \le -\frac{5na}{14}. \tag{4.40}$$

So

$$\max_{1\le j\le n}\frac{\left(q_j\right)_+}{\left|q_n\right|} \le \frac{2}{5}. \tag{4.41}$$

Thus, we can get the conclusion

$$d_H\left(A_k\right) < \frac{2n}{5}, d_F\left(A_k\right) < \frac{7n}{5}. \tag{4.42}$$

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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