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L^p p-Harmonic 1-Forms on δ-Stable Hypersurface in Space Form with Nonnegative Bi-Ricci Curvature

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Abstract

In this paper, we investigate the space of L^p p-harmonic 1-forms on a complete noncompact orientable δ-stable hypersurface M^m that is immersed in space form $\mathbb{N}_c^{m+1}$ with nonnegative BiRic curvature. We prove the nonexistence of L^p p-harmonic 1-forms on M^m. Moreover, we obtain some vanishing properties for this class of harmonic 1-forms.

Keywords

L^p p-Harmonic 1-Forms, δ-Stable Hypersurface, BiRic Curvature, Space Form

1. Introduction

Let $x: M^m \to \mathbb{N}_c^{m+1}$, be a complete noncompact orientable stable hypersurface M^m immersed in space form $\mathbb{N}_c^{m+1}$ with nonnegative BiRic curvature bounded from below. Fix a point $x \in M$ and let $\{e_1, \cdots, e_{m+n}\}$ be local orthogonal frame of $\mathbb{N}_c^{m+1}$ such that $\{e_1, \cdots, e_m\}$ are tangent fields of M^m. Now we will use the following convention on the ranges of induces: $1 \le i, j, k, \cdots \le m$ and $m+1 \le \alpha \le m+n$. Let A denote the second fundamental form of x, is define by

$$A(X,Y) = \sum_{\alpha} \left\langle \bar{\nabla}_X Y, e_\alpha \right\rangle e_\alpha, \quad \forall X, Y \in T_x M, \tag{1}$$

where $\bar{\nabla}$ is the Levi-Civita connection on the ambient manifold $\mathbb{N}_c^{m+1}$. Here, we denote $h_{ij}^\alpha = \left\langle \bar{\nabla}_{e_i} e_j, e_\alpha \right\rangle$, then $|A|^2 = \sum_\alpha \sum_{i,j} \left(h_{ij}^\alpha\right)^2$ denote the square length of the norm of A and the mean curvature vector field H is define by

$$H = \sum_\alpha H^\alpha e_\alpha = \frac{1}{m} \sum_\alpha \sum_i h_{ii}^\alpha e_\alpha. \tag{2}$$

The traceless second fundamental form Φ is defined by

$$\Phi(X,Y)=A(X,Y)-\langle X,Y\rangle H,\quad \forall X,Y\in T_xM, \tag{3}$$

where $\langle,\rangle$ is the metric of M^m. A simple computational shows that

$$|\Phi|^2=|A|^2-m|H|^2. \tag{4}$$

In particular, if $\|\Phi\|\equiv 0$, then M^m is totally umbilical see ([1] [2] [3] [4]).

Definition 1.1. [5], Let M^m be an m-dimensional Riemannian manifold, μ, ν be orthonormal tangent vectors at a point $p\in M^m$ and D be the 2-plane generated by μ and ν. The bi-Ricci curvature of the plane D is defined by

$$BiRic(D)=BiRic(\mu,\nu):=Ric(\mu,\mu)+\delta Ric(\nu,\nu)-R(\mu,\nu,\mu,\nu), \tag{5}$$

where $\delta>0$, $R(\mu,\nu,\mu,\nu)$ denotes the sectional curvature and $BiRic(\mu,\nu)$, denotes the BiRic curvature in the direction μ,ν. Observe that when $m=3$, we have that

$$2BiRic(\mu,\nu)=R(\mu,\nu,\mu,\nu). \tag{6}$$

In general, BiRic is the sum of the sectional curvatures overall mutually orthogonal 2-planes containing at least one of these tangent vectors (see [6]).

The vanishing theorems for L^p p-harmonic 1-forms on complete noncompact submanifolds have been studied extensively by many mathematicians from various points of views. There are some relations between the geometry and topology of a manifold and the space of L^p p-harmonic 1-forms. According to the decomposition theorem by Hodge-Rham [7], L^p p-harmonic 1-forms completely represent the L^p cohomology of the underlying manifold. The nonexistence of nontrivial L^p p-harmonic 1-forms on M^m implies that any codimension one cycle on M^m must disconnect M^m, also the uniqueness of the non-parabolic ends of the underlying manifold. In [8], Li considers hypersurface $M^m(2\le m\le 5)$ with constant means curvature and then drives the same vanishing properties. In [9], Dung studied immersed hypersurface in a weighted Riemannian manifold with weighted BiRici curvature and proved that if such hypersurfaces are weighted stable then the space of L^2 weighted harmonic 1-forms is trivial. In [10], Tanno studied a complete noncompact oriented stable minimal hypersurface immersed in a Riemannian manifold with nonnegative BiRic curvature and proved that there are no nontrivial L^2 harmonic 1-forms on M^m. In [11], Cheng generalized Li's results by assuming that $BiRic\ge\frac{m-5}{4}H^2$, where H is the mean curvature of M^m, and is normalized to be equal to the second fundamental form. In [5], the Author proves that there are no nontrivial L^2 harmonic 1-forms on a strongly stable hypersurface M^m of a general Riemannian manifold $\mathbb{N}$ when the bi-Ricci curvature of $\mathbb{N}$ is no less than certain lower bound, which gives a topological obstruction for the stability of M^m. In [12], Palmer considered L^2 harmonic forms on a complete oriented stable minimal hypersurface M^m in $\mathbb{R}^{m+1}$, and proved that there exist no nontrivial L^2 harmonic 1-forms on M^m. In this direction, many Authors give us various results for L^2 harmonic 1-forms on stable minimal hypersurfaces (see [13] [14]). In [15], the Author proved that the nonexistence of L^2 harmonic 1-forms on a complete super stable minimal submanifold M^m in hyperbolic space.

The aim of this work is to investigate some vanishing theorems for L^p p-harmonic 1-forms on a complete noncompact orientable stable hypersurface that is immersed in space form with nonnegative BiRic curvature bounded from below.

2. Preliminaries

Let M^m be an m-dimensional Riemannian manifold and the Riemannian structure under a local coordinate system given by

$$ds^2 = g_{ij} dx^i \otimes dx^j, \tag{7}$$

where g is the Riemannian metric. We shall make use of the following conventions about indices:

$$1 = i, j, k, \cdots = m, \tag{8}$$

and shall agree that repeated indices are summed over their ranges. Denote $\frac{\partial}{\partial x^i}$ by ∂_i. The Riemannian curvature tensor R_{ijkl}, the Ricci curvature tensor Ric_{ij} and scalar curvature $\bar{R}$ are defined by (see [16] [17])

$$R(X,Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]} Z, \tag{9}$$

where $\bar{\nabla}$ denotes the Levi-Civita connection on M^m and

$$R_{ijkl} = \left\langle R(\partial_i, \partial_j)\partial_l, \partial_k \right\rangle, \quad Ric_{ij} = \sum_k g^{pq} R_{ipjq}, \quad \bar{R} = \sum_{1 \le i, j \le n} g^{ij} Ric_{ij}. \tag{10}$$

The Weyl conformal curvature tensor W_{ijkl} and Einstein tensor A_{ij} are defined respectively by

$$\begin{aligned} W_{ijkl} = R_{ijkl} &- \frac{1}{m-2}\left(Ric_{jk} g_{il} - Ric_{jl} g_{ik} - Ric_{il} g_{jk}\right) \\ &+ \frac{1}{(m-1)(m-2)} \bar{R}\left(g_{ik} g_{jl} - g_{il} g_{jk}\right), \end{aligned} \tag{11}$$

and

$$A_{ij} = Ric_{ij} - \frac{1}{m} g_{ij} \bar{R} \tag{12}$$

By direct computations, we obtain

$$|A|^2 = |Ric|^2 - \frac{1}{m}\bar{R}^2, \tag{13}$$

$$|W|^2 = |R|^2 - \frac{4}{m-2}|Ric|^2 + \frac{2}{(m-1)(m-2)}\bar{R}^2. \tag{14}$$

Now we define a new tensor B_{ijkl} of type (0,4) as follows:

$$B_{ijkl} = (m-3) R_{ijkl} - (m-2) W_{ijkl} + \frac{1}{m-1}\bar{R}\left(g_{ik} g_{jl} - g_{il} g_{jk}\right). \tag{15}$$

It is clear that B_{ijkl} has all the symmetries of the curvature tensor R_{ijkl} and the Weyl curvature W_{ijkl}.

$$B_{ijkl} = -B_{jikl} = -B_{ijlk} = B_{jilk} = B_{klij}. \tag{16}$$

$$B_{ijkl} = B_{iklj} + B_{iljk} = 0. \tag{17}$$

By direct computations, the BiRici curvature of the plane generated by ∂_i, ∂_j

$$\frac{1}{\left|\partial_i \wedge \partial_j\right|^2} B_{ijij} = \frac{1}{g_{ii}g_{jj} - g_{ij}^2}\left(R_{ii}g_{jj} + g_{ii}R_{jj} - 2R_{ij}g_{ij} - R_{ijij}\right). \tag{18}$$

So BiRic behaves like a "sectional curvature" of the tensor B_{ijkl}.

$$B_{ijkl} = R_{ik}g_{jl} + R_{jl}g_{ik} - R_{il}g_{jk} - R_{jk}g_{il} - R_{ijkl}. \tag{19}$$

From (19), we obtain

$$|B|^2 = |R|^2 + 4(m-1)|Ric|^2 + \bar{R}^2. \tag{20}$$

And

$$\left|B_{ijkl} - \frac{(2m-3)\bar{R}}{m(m-1)}\left(g_{ik}g_{jl} - g_{il}g_{jk}\right)\right|^2 = |B|^2 - \frac{2(2m-3)^2}{m(m-1)}\bar{R}^2. \tag{21}$$

Combining (13), (14) and (20), we obtain

$$|B|^2 = |W|^2 + \frac{4(m-3)^2}{m-2}|A|^2 + \frac{2(2m-3)^2}{m(m-1)}\bar{R}^2. \tag{22}$$

From (21) and (22), we obtain

$$\left|B_{ijkl} - \frac{(2m-3)\bar{R}}{m(m-1)}\left(g_{ik}g_{jl} - g_{il}g_{jk}\right)\right|^2 = |W|^2 + \frac{4(m-3)^2}{m-2}|A|^2. \tag{23}$$

When the BiRic curvatures of all 2 planes are the same at a point, by the argument of polarization, we have

$$B_{ijkl} = c\left(g_{ik}g_{jl} - g_{il}g_{jk}\right). \tag{24}$$

We get $c = \frac{(2m-3)\bar{R}}{m(m-1)}$. Therefore, $W = A = 0$ by (24) and the Riemannian curvature is constant.

3. The Estimation of the BiRic Curvature

Let $M^m \to \mathbb{N}_c^{m+1}$ be a complete noncompact orientable stable hypersurface M^m immersed in space form $\mathbb{N}_c^{m+1}$. We shall make use of the following conventions about indices:

$$1 \le i, j, k, \cdots \le m, m+1 \le \alpha, \beta \le m+n.$$

Denote by $\bar{\nabla}$, $\bar{R}$, Ric and $BiRic$ the Levi-Civita connection, sectional curvature, Ric curvature and BiRic curvature of $\mathbb{N}_c^{m+1}$ respectively.

The Gauss equation is

$$R_{ijkl}=\overline{R}_{ijkl}+\sum_{\alpha}\left(h_{ik}^{\alpha}h_{jl}^{\alpha}-h_{il}^{\alpha}h_{jk}^{\alpha}\right). \tag{25}$$

we have

$$B_{klkl}=Ric_{kk}+Ric_{ll}-R_{klkl}=\sum_{i}\left(\overline{R}_{ikik}+\overline{R}_{ilil}\right)-\overline{R}_{klkl}. \tag{26}$$

By the Gauss Equation (25), we have

$$Ric(X,X)=\sum_{i}\overline{R}(X,e_i,X,e_i)+h(X,X)H-\sum_{i}h(e_i,X)^2. \tag{27}$$

Lemma 3.2. [9] Let $(h_{ij})_{i,j=1}^{m}$ be a symmetric matrix $m\times m$, $m\geq 3$.

And let $H=\sum_{i=1}^{m}h_{ii}$ and $S=|A|^2=\sum_{i,j=1}^{m}(h_{ij})^2$ then

$$\begin{aligned}&h(X,X)H-\sum_{i}h(X,e_i)^2\\&\geq\frac{|X|^2}{n^2}\left\{2(m-1)H^2-(m-2)H\sqrt{(m-1)(mS-H^2)}-m(m-1)S\right\}.\end{aligned} \tag{28}$$

Assume that $X\neq 0$. By the definition of the BiRic in Equation (5), we obtain

$$Ric(X,X)\geq\sum_{i}\overline{R}(X,e_i,X,e_i)-(\delta S+\varphi(H,S))|X|^2. \tag{29}$$

Let us first assume that $X\neq 0$ everywhere. By the definition, we have

$$\sum_{i}\overline{R}(X,e_i,X,e_i)=\left(BiRic\left(\frac{X}{|X|},N\right)-\delta Ric(N,N)\right)|X|^2. \tag{30}$$

Combining (29) with (30), we obtain

$$Ric(X,X)\geq\left\{BiRic\left(\frac{X}{|X|},N\right)-\varphi(H,S)-\delta\left(Ric(N,N)+S\right)\right\}|X|^2, \tag{31}$$

where

$$\varphi(H,S)=\left(\frac{m-1}{m}-\delta\right)S-\frac{1}{m^2}\left\{2(m-1)H^2-(m-2)H\sqrt{(m-1)(mS-H^2)}\right\}. \tag{32}$$

From the Bochner formula [18], we have

$$\Delta|\omega|^2=2\left(|\nabla\omega|^2+Ric(\omega,\omega)\right). \tag{33}$$

Since

$$\Delta|\omega|^2=2\left(|\omega|\Delta|\omega|+|\nabla|\omega||^2\right). \tag{34}$$

Combining (33) with (34), we get

$$|\omega|\Delta|\omega|-Ric(\omega,\omega)=|\nabla\omega|^2-|\nabla|\omega||^2\geq\frac{1}{m-1}|\nabla|\omega||^2. \tag{35}$$

Inparticular, we know

$$Ric(\omega,\omega)\geq\left(BiRic\left(\frac{X}{|X|},N\right)-\varphi(H,S)-\delta\left(Ric(N,N)+S\right)\right)|\omega|^2. \tag{36}$$

We set $q=Ric(N,N)+S$, thus

$$Ric(\omega,\omega) \geq \left(BiRic\left(\frac{X}{|X|},N\right) - \left(\delta q + \varphi(H,S)\right)\right)|\omega|^2. \tag{37}$$

4. The Structure of δ-Stable Hypersurfaces in $\mathbb{N}_c^{m+1}$

In this section, we assume that $\mathbb{N}_c^{m+1}$ is a complete noncompact oriented space form and M^m is a complete noncompact oriented stable hypersurface of $\mathbb{N}_c^{m+1}$. Adapt the same notations as in the previous section and the second fundamental form can be written as $h = \sum_{i,j} h_{ij}\omega_i \otimes \omega_j$. We assume that the mean curvature vector is in the same direction as in e_{m+1}. We have

$$H = \frac{1}{m}\sum_i h_{ii} \geq 0. \tag{38}$$

Definition 4.1. [19], Let $x: M^m \to \mathbb{N}^{m+1}$, $m \geq 3$, be a complete noncompact hypersurface immersed in a Riemannian manifold $\mathbb{N}^{m+1}$. Then the first eigenvalue of the Laplacian of M is defined by

$$\lambda_1(M)\int_M \varphi^2 \leq \int_M |\nabla\varphi|^2, \tag{39}$$

for all smooth function $\varphi \in C_0^\infty(M)$.

Definition 4.2. [11], Let M^m be a complete noncompact manifold and let $H \neq 0$, M^m is said to be strongly stable if

$$I(\varphi) = \int_M \left(|\nabla\varphi|^2 - \left(Ric(N,N)+S\right)\varphi^2\right)\mathrm{d}v \geq 0, \quad \forall \varphi \in C_0^\infty(M), \tag{40}$$

where C_0^∞ is the smooth functions and $\mathrm{d}v$ is the volume form.

Definition 4.3. [11], For some number $0 < \delta \leq 1$, M^m is δ-stable if

$$I(\varphi) = \int_M \left(|\nabla\varphi|^2 - \delta\left(Ric(N,N)+S\right)\varphi^2\right)\mathrm{d}v \geq 0, \quad \forall \varphi \in C_0^\infty(M), \tag{41}$$

where S is the square norm of the second fundamental form of M^m. Obviously, given $\delta_1 > \delta_2$, δ_1-stable implies δ_2-stable. So, that M^m is stable implies that M^m is δ-stable.

M^m is said to be δ-stable or weakly δ-stable if $I(\varphi) \geq 0$, $\forall \varphi \in C_0^\infty$ satisfying

$$\int_M \varphi = 0. \tag{42}$$

Remark. When $H = 0$, *i.e.* M^m is minimal, then the immersion is called stable if it is in the strong sense, which is different from the stability of the hypersurfaces with constant mean curvature as said above.

5. The Vanishing Theorems

In this section, we presented some vanishing theorems as follows.

Theorem 5.1. Let $x: M^m \to \mathbb{N}_c^{m+1}$, $m \geq 3$, be a complete noncompact orientable δ-stable minimal hypersurface M^m immersed in space form $\mathbb{N}_c^{m+1}$ with nonnegative BiRic curvature bounded from below. If

$$BiRic(Y,N) \geq \left(\frac{m-1}{m} - \delta\right)S.$$

Then there is no nontrivial L^p p-harmonic 1-form on M^m.

Proof: Using (35) and (37), we obtain

$$|\omega|\Delta|\omega| \geq \frac{1}{m-1}|\nabla|\omega||^2 + \left(BiRic\left(\frac{X}{|X|}, N\right) - \left(\delta q + \varphi(H,S)\right)\right)|\omega|^2. \tag{43}$$

Since

$$|\omega|^p \Delta|\omega|^p = \frac{p-1}{p}\left|\nabla|\omega|^p\right|^2 + p|\omega|^{2p-2}|\omega|\Delta|\omega| \tag{44}$$

for any $p > 0$. Combining (43) with (44), we get

$$\begin{aligned}|\omega|^p \Delta|\omega|^p \geq{}& \frac{p-1}{p}\left|\nabla|\omega|^p\right|^2 + \frac{p}{m-1}|\omega|^{2p-2}|\nabla|\omega||^2 \\ &+ p\left(BiRic\left(\frac{X}{|X|}, N\right) - \left(\delta q + \varphi(H,S)\right)\right)|\omega|^{2p}\end{aligned} \tag{45}$$

Let $\eta \in C_0^\infty(M)$ be a smooth function with compact supported. Multiplying both sides of (45) by η^2 and integrating over M, we obtain

$$\begin{aligned}\int_M \eta^2|\omega|^p \Delta_f|\omega|^p \geq{}& \left(1 - \frac{m-2}{p(m-1)}\right)\int_M \eta^2\left|\nabla|\omega|^p\right|^2 \\ &+ p\int_M\left(BiRic\left(\frac{X}{|X|}, N\right) - \left(\delta q + \varphi(H,S)\right)\right)\eta^2|\omega|^{2p}\end{aligned} \tag{46}$$

Applying the divergence theorem, we obtain

$$\begin{aligned}&\int_M \eta^2|\omega|^p \Delta_f|\omega|^p \\ &= \int_M div\left(\eta^2|\omega|^p \nabla|\omega|^p\right) - \int_M \eta^2\left|\nabla|\omega|^p\right|^2 - 2\int_M \eta|\omega|^p\left\langle\nabla\eta, \nabla|\omega|^p\right\rangle \\ &= -\int_M \eta^2\left|\nabla|\omega|^p\right|^2 - 2\int_M \eta|\omega|^p\left\langle\nabla\eta, \nabla|\omega|^p\right\rangle.\end{aligned} \tag{47}$$

Combining (46) with (47), we get

$$\begin{aligned}&\left(\frac{2p(m-1)-(m-2)}{p(m-1)}\right)\int_M \eta^2\left|\nabla|\omega|^p\right|^2 \\ &\leq -p\int_M\left(BiRic\left(\frac{X}{|X|}, N\right) - \left(\delta q + \varphi(H,S)\right)\right)\eta^2|\omega|^{2p} \\ &\quad - 2\int_M \eta|\omega|^p\left\langle\nabla\eta, \nabla|\omega|^p\right\rangle.\end{aligned} \tag{48}$$

$$\begin{aligned}&\left(\frac{2p(m-1)-(m-2)}{p(m-1)}\right)\int_M \eta^2\left|\nabla|\omega|^p\right|^2 \\ &\leq -p\int_M\left(BiRic\left(\frac{X}{|X|}, N\right) - \varphi(H,S)\right)\eta^2|\omega|^{2p} \\ &\quad - 2\int_M \eta|\omega|^p\left\langle\nabla\eta, \nabla|\omega|^p\right\rangle + p\delta\int_M q\eta^2|\omega|^{2p}.\end{aligned} \tag{49}$$

From definition (4.2), we obtain

$$\int_M|\nabla\varphi|^2 \geq \int_M q\varphi^2 \mathrm{d}v. \tag{50}$$

Replacing φ by $\eta|\omega|^p$, we obtain

$$\int_M \left|\nabla\left(\eta|\omega|^p\right)\right|^2 \geq \int_M q\eta^2|\omega|^{2p}\,\mathrm{dv}. \tag{51}$$

Combining (49) with (51), we obtain

$$\begin{aligned}&\left(\frac{2p(m-1)-(m-2)}{p(m-1)}\right)\int_M \eta^2\left|\nabla|\omega|^p\right|^2\\ &\leq -p\int_M\left(BiRic\left(\frac{X}{|X|},N\right)-\varphi(H,S)\right)\eta^2|\omega|^{2p}\\ &\quad -2\int_M \eta|\omega|^p\left\langle\nabla\eta,\nabla|\omega|^p\right\rangle + p\delta\int_M\left|\nabla\left(\eta|\omega|^p\right)\right|^2.\end{aligned} \tag{52}$$

$$\begin{aligned}&\left(\frac{2p(m-1)-(m-2)}{p(m-1)}+p\delta\right)\int_M \eta^2\left|\nabla|\omega|^p\right|^2\\ &\leq -p\int_M\left(BiRic\left(\frac{X}{|X|},N\right)-\varphi(H,S)\right)\eta^2|\omega|^{2p}\\ &\quad -2(p\delta+1)\int_M \eta|\omega|^p\left\langle\nabla\eta,\nabla|\omega|^p\right\rangle + p\delta\int_M|\nabla\eta|^2|\omega|^{2p}.\end{aligned} \tag{53}$$

Note that

$$-2\int_M \eta|\omega|^p\left\langle\nabla\eta,\nabla|\omega|^p\right\rangle \leq \varepsilon\int_M\eta^2\left|\nabla|\omega|^p\right|^2+\frac{1}{\varepsilon}\int_M|\nabla\eta|^2|\omega|^{2p}, \tag{54}$$

for some constant $\varepsilon>0$.

$$\begin{aligned}&\left(\frac{2p(m-1)-(m-2)}{p(m-1)}+p\delta-|p\delta+1|\varepsilon\right)\int_M \eta^2\left|\nabla|\omega|^p\right|^2\\ &\leq -p\int_M\left(BiRic\left(\frac{X}{|X|},N\right)-\varphi(H,S)\right)\eta^2|\omega|^{2p}\\ &\quad +\left(p\delta+\frac{|p\delta+1|}{\varepsilon}\right)\int_M|\nabla\eta|^2|\omega|^{2p}.\end{aligned} \tag{55}$$

Thus

$$\mathbf{A}\int_M\eta^2\left|\nabla|\omega|^p\right|^2+\mathbf{B}\int_M\eta^2|\omega|^{2p}\leq\mathbf{C}\int_M|\nabla\eta|^2|\omega|^{2p} \tag{56}$$

Set

$$\begin{aligned}\mathbf{A}&=\frac{2p(m-1)-(m-2)}{p(m-1)}+p\delta-|p\delta+1|\varepsilon,\\ \mathbf{B}&=p\left(BiRic\left(\frac{X}{|X|},N\right)-\varphi(H,S)\right)\\ \mathbf{C}&=p\delta+\frac{|p\delta+1|}{\varepsilon}.\end{aligned} \tag{57}$$

Let B_r be a geodesic ball of radius $r>0$ on M^m centered at the point *p*. Choose a cut-off function η satisfying

$$\begin{cases} \eta = 0 \quad \text{in } M \setminus B_{2r}, \\ \eta = 1 \quad \text{in } B_r, \\ |\nabla \eta| \le \dfrac{2}{r} \quad \text{in } B_{2r} \setminus B_r. \end{cases} \tag{58}$$

Let $0 \le \eta \le 1$. Using (56) with (58), we obtain

$$\mathbf{A}\int_{B_r} \left|\nabla |\omega|^p\right|^2 \le \mathbf{C}\left(\frac{4}{r^2}\right)\int_{B_{2r}\setminus B_r} |\omega|^{2p}. \tag{59}$$

Taking $r \to \infty$, we get $\nabla|\omega| = 0$, and $|\omega| = |X|$ is constant. Hence,

$$|\nabla \omega|^2 = \frac{m}{m-1}\left|\nabla|\omega|\right|^2 = 0, \quad BiRic\left(\frac{X}{|X|}, N\right) = \varphi(H,S). \tag{60}$$

By (60) we obtain

$$Ric(\omega,\omega) + \delta\left(Ric(N,N) + S\right) = 0. \tag{61}$$

Moreover, since $\nabla|\omega| = 0$, and $|\omega| = |X|$ is constant, the Bochner formula implies

$$Ric(X,X) = 0. \tag{62}$$

Thus, by (62) we can deduce

$$Ric(N,N) + S = 0. \tag{63}$$

Therefore, for any unite tangent vector Y, it follows from (31) and (63) that

$$\begin{aligned} Ric(Y,Y) &\ge BiRic(Y,N) - \delta\left(Ric(N,N) + S\right) - \varphi(H,S) \\ &= BiRic(Y,N) - \varphi(H,S) \ge 0. \end{aligned} \tag{64}$$

Thus, using (32) with (64) we get

$$\begin{aligned} &BiRic(Y,N) \\ &\ge \left(\frac{m-1}{m} - \delta\right)S - \frac{1}{m^2}\left\{2(m-1)H^2 - (m-2)H\sqrt{(m-1)\left(mS - H^2\right)}\right\}. \end{aligned} \tag{65}$$

Assume that M^m is a minimal stable hypersurface immersed in space form $\mathbb{N}_c^{m+1}$. Hence $H = 0$, and this implies

$$BiRic(Y,N) \ge \left(\frac{m-1}{m} - \delta\right)S. \tag{66}$$

Then there is no nontrivial L^p p-harmonic 1-forms on M^m. Hence we get the prove as assumption in theorem.

Corollary 5.2. Let $x : M^m \to \mathbb{N}_c^{m+1}$, $m \ge 3$, be a complete noncompact orientable δ-stable minimal hypersurface M^m immersed in space form $\mathbb{N}_c^{m+1}$ with nonnegative BiRic curvature bounded from below. If $BiRic - \varphi(H,S) \ge 0$ for any positive number δ satisfy

$$\delta \le \frac{m-1}{m}.$$

Then there is no nontrivial L^p p-harmonic 1-form on M^m.

Corollary 5.3. Let $x: M^m \to \mathbb{N}_c^{m+1}$, $m \geq 3$, be a complete noncompact orientable δ-stable hypersurface M^m immersed in space form $\mathbb{N}_c^{m+1}$. If $BiRic = \varphi(H,S) = 0$, then one of the following conditions holds

1) M is minimal and S is totally geodesic.

2) M is minimal and $\delta = \frac{m-1}{m}$.

Then there is no nontrivial L^p p-harmonic 1-form on M^m.

Theorem 5.4. Let $x: M^m \to \mathbb{N}_c^{m+1}$, $m \geq 3$, be a complete noncompact orientable δ-stable minimal hypersurface M^m immersed in space form $\mathbb{N}_c^{m+1}$ with nonnegative BiRic curvature bounded from below. If M^m satisfy

$$\lambda_1(M) > \frac{BiRic\left(\frac{X}{|X|}, N\right) - \varphi(H,S)}{\delta}.$$

Then there is no nontrivial L^p p-harmonic 1-form on M^m.

Proof: From the definition (4.1) and replacing φ by $\eta|\omega|^p$ we get

$$\lambda_1(M)\int_M \eta^2 |\omega|^{2p} \leq \int_M \left|\nabla\left(\eta|\omega|^p\right)\right|^2. \tag{67}$$

Thus,

$$\lambda_1 \int_M \eta^2 |\omega|^{2p} \leq \int_M \eta^2 \left|\nabla|\omega|^p\right|^2 + \int_M |\nabla\eta|^2 |\omega|^{2p} + 2\int_M \eta|\omega|^p \left\langle \nabla\eta, \nabla|\omega|^p \right\rangle. \tag{68}$$

Using Cauchy-Schwartz inequality

$$2\left|\int_M \eta|\omega|^p \left\langle \nabla\eta, \nabla|\omega|^p \right\rangle\right| \leq s\int_M \eta^2 \left|\nabla|\omega|^p\right|^2 + \frac{1}{s}\int_M |\nabla\eta|^2 |\omega|^{2p}, \tag{69}$$

where $s > 0$, using (68) with (69), and multiplying both said by $\mathbf{B}$ we get

$$\mathbf{B}\int_M \eta^2 |\omega|^{2p} \leq \frac{\mathbf{B}(1+s)}{\lambda_1}\int_M \eta^2 \left|\nabla|\omega|^p\right|^2 + \frac{\mathbf{B}\left(1+\frac{1}{s}\right)}{\lambda_1}\int_M |\nabla\eta|^2 |\omega|^{2p}. \tag{70}$$

Compining (56) with (70), we get

$$\mathbf{D}\int_M \eta^2 \left|\nabla|\omega|^p\right|^2 \leq \mathbf{E}\int_M |\nabla\eta|^2 |\omega|^{2p}. \tag{71}$$

Set

$$\mathbf{D} = \mathbf{A} + \frac{\mathbf{B}(1+s)}{\lambda_1}, \quad \mathbf{E} = \mathbf{C} - \frac{\mathbf{B}\left(1+\frac{1}{s}\right)}{\lambda_1}, \tag{72}$$

for some constant $\mathbf{E} > 0$

$$\mathbf{E} = \mathbf{C} - \frac{\mathbf{B}\left(1+\frac{1}{s}\right)}{\lambda_1} > 0. \tag{73}$$

Thus,

$$p\delta + \frac{|p\delta+1|}{\varepsilon} > \frac{p\left(BiRic\left(\frac{X}{|X|}, N\right) - \varphi(H,S)\right)\left(1+\frac{1}{s}\right)}{\lambda_1} \tag{74}$$

Choosing ε and s small enough, we get

$$\lambda_1(M) > \frac{BiRic\left(\frac{X}{|X|}, N\right) - \varphi(H,S)}{\delta}. \tag{75}$$

Now we observe that

$$\varphi(H,S) = \frac{-S}{m} - \frac{2(m-1)H^2}{m^2} + \frac{(m-2)H\sqrt{(m-1)(mS-H^2)}}{m^2} \le \frac{S}{m}. \tag{76}$$

This implies

$$BiRic\left(\frac{X}{|X|}, N\right) = \frac{S}{m} \ge 0. \tag{77}$$

Using (58) with (71), we obtain

$$\mathbf{D}\int_{B_r} \left|\nabla|\omega|^p\right|^2 \le \mathbf{E}\left(\frac{4}{r^2}\right)\int_{B_{2r}\setminus B_r} |\omega|^{2p}. \tag{78}$$

Taking $r \to \infty$, we get $\omega = 0$. Then there are no nontrivial L^p p-harmonic 1-forms on M^m. Hence we get the conclusion.

On the other hand, Dung and Seo [3] proved that

$$\frac{m-1}{m}S - \frac{2(m-1)H^2}{m^2} + \frac{(m-2)H\sqrt{(m-1)(mS-H^2)}}{m^2} \le \frac{\sqrt{m-1}}{2}S.$$

In fact, in [3], Dung showed that

$$\begin{aligned}
&\frac{m-1}{m}S - \frac{2(m-1)H^2}{m^2} + \frac{(m-2)H\sqrt{(m-1)(mS-H^2)}}{m^2} \\
&= \frac{\sqrt{m-1}}{2}S - \frac{\sqrt{m-1}}{2m^2}\left(\frac{(m-2)\sqrt{mS-H^2}}{\sqrt{m-1}+1} - \left(\sqrt{m-1}+1\right)H^2\right)^2 \\
&\le \frac{\sqrt{m-1}}{2}S.
\end{aligned} \tag{79}$$

This implies that $\varphi(H,S) \le \left(\frac{\sqrt{m-1}}{2} - \delta\right)S$. Therefore, Theorem 5.4 implies the following conclusion.

Corollary 5.5. Let $x: M^m \to \mathbb{N}_c^{m+1}$, $m \ge 3$, be a complete noncompact δ-stable minimal hypersurface immersed in space form $\mathbb{N}_c^{m+1}$ with nonnegative BiRic curvature bounded from below. Suppose that one of the following conditions holds. Then there is no nontrivial L^p p-harmonic 1-form on M^m.

1) If $BiRic\left(\frac{X}{|X|}, N\right) = \frac{S}{m} = 0$, then S is totally geodesic.

2) If $BiRic = \left(\sqrt{\frac{m-1}{2}} - \delta\right)S = 0$, then either $\delta = \sqrt{\frac{m-1}{2}}$ or S is totally geodesic.

6. Conclusion

We investigated the space of L^p p-harmonic 1-forms on a complete noncompact orientable δ-stable hypersurfaces that are immersed in space form with nonnegative BiRic curvature. We proved the nonexistence of L^p p-harmonic 1-forms on M^m. Moreover, we obtained some vanishing properties for this class of harmonic 1-forms.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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