

Symmetric Functions and q -Products with an Application to Theoretical Physics

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Abstract

The main purpose of this paper is to show that the Poincaré q -polynomials admit a representation in terms of the symmetric functions and the Patterson-Selberg (or Ruelle-type) spectral functions. We have shown that the q -series elliptic genera can be expressed in terms of q -analogs of the classical special functions, specially the equivalence between the spectral Patterson-Selberg and the Ruelle functions. The main result of this manuscript is to show that this representation can be used in theoretical physics and we analyze them in terms of the Patterson-Selberg spectral function $\mathcal{R}(s)$.

Keywords

Poincaré q -Polynomials, Patterson-Selberg Spectral Functions, Ruelle Spectral Functions

1. Introduction

The main purpose of this paper is to present the explicit formulas of q -polynomials which admit a presentation in terms of the Patterson-Selberg (or Ruelle-type) spectral functions. It has been shown that equivariant methods also can be used to define and analyze (twisted) K-theory [1] [2] [3]. Applying techniques from the equivariant K-theory, one can construct a Chern character and K-theory Euler class. This is manifested by the fact that the q -series elliptic genera can be expressed in terms of q -analogs of the classical special functions, in particular the spectral Patterson-Selberg and Ruelle functions $\mathcal{R}(s)$ [4]. In the present paper we would like to call attention to this connection by discussing some mathematical aspects associated with it.

One of the aims of the present paper is the application of the analytic Patter-

son-Selberg function $\mathcal{R}(s)$ and the symmetric functions (S-functions) in Analysis and Theoretical Physics. $\mathcal{R}(s)$ and S-functions not only provide beautiful formulas in analysis but also promote interesting results in theoretical physics. We mention below some results which will be explored along this manuscript.

Symmetric functions. Symmetric functions are directly connected with the Quantum Hall effect.

Finite-dimensional algebras. These algebras have wide variety of applications in Physics in the so-called quantum group which can be regarded as a deformation, depending on a parameter q , of the universal enveloping algebra of a semi-simple Lie algebra. Therefore, this is not finite-dimensional algebras but they are finitely generated. Their representation theory for q not a root of unity was found to be similar to the corresponding semi-simple Lie algebra.

Quantum groups. Quantum groups are an example of quasi-triangular Hopf algebras and, as such, for each quantum group there exists a universal R-matrix which intertwines with the action of the coproduct. By considering q -deformations of simple Lie superalgebras, one comes across to the concept of quantum supergroups.

Infinite-dimensional algebras. Of central importance in many diverse areas, Mathematics and Theoretical Physics are a particular class of infinite-dimensional algebras called (affine) Kac-Moody algebras. Unlike the finite-dimensional case, where the simple Lie algebras can be realized in terms of a finite number of fermionic/bosonic modes, simple Kac-Moody algebras have various vertex operator realizations. As an example, in terms of a finite number of bosonic free fields, the modes of which generate a Heisenberg algebra.

This paper is organized as follows. In Section 2 we consider the twisted symmetric products, the Euler characteristic $\chi(X)$. Then in Section 3 we analyse the Hirzebruch $\chi(X^n)$ genera and calculate it in the orbifold cycle case. In Section 4 we calculate the orbifold symmetric product. In Section 5 we present the so-called Patterson-Selberg functions $\mathcal{R}(s)$ and discuss their applications. Finally, we summarize our key results in the Conclusion.

2. The Twisted Symmetric Product

Let G be a finite group and X be a G -space, on which a normal subgroup A acts trivially. The G -equivariant K-theory of X can be decomposed as a direct sum of twisted equivariant K-theories of X parametrized by the orbits of the conjugation action of G on the irreducible representations of A [5] (theorems 3.2 and 3.4).

Now we turn to the symmetric group and denote by $G = \mathfrak{S}_n$ the symmetric group on n symbols. As before, we denote the non-trivial class of G by α . Using the decomposition formula (theorem 3.4 of [5]), one can calculate $K_{\mathfrak{S}_n}^*(X^n, \alpha)$, where the group acts on the n -fold product of a manifold X by permutation of coordinates. At last, formula for the twisted symmetric products takes the form

$$\begin{aligned}
& \sum q^n \chi \left(K_{\mathbb{S}_n}^* (X^n, \alpha) \otimes \mathbb{C} \right) \\
&= \prod_{n>0} (1 - q^{2n-1})^{-\chi(X)} + \prod_{n>0} (1 + q^{2n-1})^{\chi(X)} \\
&\quad \times \left[1 + \frac{1}{2} \prod_{n>0} (1 + q^{2n})^{\chi(X)} - \frac{1}{2} \prod_{n>0} (1 - q^{2n})^{\chi(X)} \right].
\end{aligned} \tag{1}$$

In Equation (1), $\chi(X)$ is the Euler characteristic of X and $(*)$ implies the conjugacy of the K function. We can easily rewrite Equation (1) in terms of spectral functions $\mathcal{R}(s)$ and we will present it in section 5.

It is known that if X is any compact, connected manifold, then $\chi(X) = Td(X)$. By means of Atiyah-Singer formula one can calculate index of an elliptic operator and express it (*i.e.* express the signature of a manifold) in terms of Pontrjagin classes. As before, we denote the non-trivial class of G by α .

3. The Analysis of the Hirzebruch Genera

Let M be a smooth manifold and G is a finite group of diffeomorphisms. Note that there are two kinds of interesting cohomology theories for the orbifold M/G .

- The orbifold de Rham cohomology $H^*(M/G)$. We easily have isomorphism

$$H^*(M/G) \cong H^*(M)^G \tag{2}$$

- Next cohomology is the delocalized equivariant cohomology

$$H^*(M, G) = \left(\bigoplus_{g \in G} H^*(M^g) \right)^G = \bigoplus_{[g] \in G_*} (M^g)^{Z_g} = \bigoplus_{[g] \in G_*} H^*(M^g/Z_g) \tag{3}$$

Equations (3) have been introduced for the study of the equivariant K-theory. In this equations G_* denotes the set of conjugacy classes of G , Z_g denotes the centralizer of g . For this cohomology theory, one can define an Euler number, denoted by $\chi(M/G)$ and $\chi(M, G)$ respectively.

Definition 3.1

$$\chi(M, G) = 1/|G| \sum_{(g, h)} \chi(M^{\langle g, h \rangle}) \tag{4}$$

Here $\langle g, h \rangle$ is the group generated by g and h , the sum is taken over commuting pairs $(g, h) \in G \times G$

Note that $\chi(M, G)$ as defined in (2) is the Euler number of $H^*(M, G)$ or equivalently $K_G^*(M)$. Therefore, one can easily show that

$$\chi(M, G) = \sum_{(g, h)} 1/|Z_g| \chi(M^g/Z_g), \tag{5}$$

it is a classical analogue of the Lefschitz formula. Indeed:

$$\chi(M/G) = 1/|G| \sum_{g \in G} \chi(M^g) = \sum_{[g] \in G_*} 1/|Z_g| \chi(M^g). \tag{6}$$

Interesting classes of examples are provided by the symmetric products $X^{(n)} = X^n/S_n$. The following formulas have been proved by various authors:

$$\sum_{n \geq 0} \chi(X^{(n)}) q^n = 1/(1-q)^{\chi(X)} \quad (7)$$

$$\sum_{n \geq 0} \chi(X^n, S_n) q^n = \prod_{l \geq 1} 1/(1-q^l)^{\chi(X)} \quad (8)$$

Again, we will present in Section 5, Equation (18), the result

$$\sum_{n \geq 0} \chi(X^{(n)}) q^n = 1/\mathcal{R}(s = 1/2\pi i\tau \left(2 - i\rho(\tau) - \frac{1}{2\pi i\tau} \right))^{\chi(X)} \quad (9)$$

Analysis for cycles. Let $x_j, j = 1, \dots, d$ are the Chern roots of X ($d = \dim X$).

If $\Delta_n(X) = (x, \dots, x) \in X^n$ then notice $(X^n)^{\sigma_n} = \Delta_n(X)$, $X^n|_{\Delta_n(X)}$ is isomorphic to the direct sum of n copies of X and σ_n acts on it by cycling the factors j . The eigenvalues of n cycles on $(\mathbb{C})^n$ are $e^{(2k\pi\sqrt{1/n})}$ for $k = 0, \dots, n-1$ each with multiplicity 1.

$\prod_{k=1}^{n-1} [1 - e^{-x_j - 2k\pi(-1/n)}] = \left(\prod_{k=1}^{\infty} - \prod_{k=n}^{\infty} \right) (1 - e^{-x_j - 2k\pi(-1/n)})$. Then with Appendix, Equation (18), we obtain

$$\prod_{k=1}^{\infty} (1 - e^{-kx_j}) = \mathcal{R}(s = (1/i\tau - x_j - 1/n)(1 - i\rho(\tau))) \quad (10)$$

$$\prod_{k=n}^{\infty} (1 - e^{-kx_j}) = \mathcal{R}(s = (n/i\tau - x_j - 1/n)(1 - i\rho(\tau))) \quad (11)$$

4. The Orbifold Symmetric Product

We will analyze the Poincaré polynomial of *orbispace*

$[X^n/\mathfrak{S}_n] := [X \times \dots \times X/\mathfrak{S}_n]$ [6], whose category objects are n -tuples

(x_1, \dots, x_n) of points in X . In addition, the arrows are elements of the form $(x_1, \dots, x_n; \sigma)$, where $\sigma \in \mathfrak{S}_n$. $(x_1, \dots, x_n; \sigma)$ has its source as (x_1, \dots, x_n) , while its target as $(x_{\sigma(1)}, \dots, x_{\sigma(n)})$.¹

The Poincaré q -polynomials. Let X be a topological space, denoted by $P(X, y)$ its Poincaré polynomial, $P(X, y) = \sum_j b^j(X) y^j$. Here $b^j(X)$ is the j -th Betti number of X . The following formula has been proved by Macdonald [7]

$$\sum_{n=0}^{\infty} P(X^n/\mathfrak{S}_n, y) q^n = \frac{\prod_j (1 + Qy^{2j+1})^{b^{2j+1}(X)}}{\prod_j (1 - Qy^{2j})^{b^{2j}(X)}}. \quad (12)$$

For $y = -1$, Equation (12) reduces to the formula for the Euler characteristic of the symmetric product $\sum_{n=0}^{\infty} \chi(X^n/\mathfrak{S}_n) Q^n = (1-Q)^{-\chi(X)}$, which is valid for

¹The mentioned category is a groupoid for the inverse of $(x_1, \dots, x_n; \sigma)$, which is $(x_1, \dots, x_n; \sigma^{-1})$. This is the reason why we can consider $[X^n/\mathfrak{S}_n]$ as an orbispace.

the topological space X whose cohomology $H^j(X; \text{real})$ is finitely generated for each $j \geq 0$. Using the \mathfrak{S}_n -equivariant K-theory of X^n , one can arrive at the following formula for generation functions [6]:

$\sum_{n=0}^{\infty} \chi_{\mathfrak{S}_n}(X^n/\mathfrak{S}_n) q^n = \prod_{j>0} (1 - Q^j)^{-\chi(X)}$. In terms of the spectral functions $\mathcal{R}(s)$, Equation (12) takes the form

$$\sum_{n=0}^{\infty} P(X^n/\mathfrak{S}_n, y) q^n = \frac{\mathcal{R}\left(s = (1 - i(\log(y))(2j+1)/(2\pi\tau))(1 - i\rho(\tau)) + i\sigma(\tau)\right)^{b_{2j+1}(X)}}{\mathcal{R}\left(s = (1 - i(\log(y))(j/\pi\tau))(1 - i\rho(\tau))\right)^{b_{2j}(X)}}. \quad (13)$$

The Poincaré orbifold polynomial can be defined as follows [6]

$$P_{orb}([X/G], y) := \sum y^j \text{rank } H_{orb}^j([X/G]; \text{real}) \equiv \sum y^j b_{orb}^j([X/G]), \quad (14)$$

where b_{orb}^j is the j -th orbifold Betti number.

Remark 4.1 For any arbitrary (fixed) positive integers m_1, \dots, m_r the following equality holds

$$\sum_{n_1, \dots, n_r=0}^{\infty} F(n_1, \dots, n_r) z_1^{n_1}, \dots, z_r^{n_r} = \sum_{N=0}^{\infty} \sum_{\substack{n_1 m_1 + \dots + n_r m_r = N \\ n_1, \dots, n_r \geq 0}} F(n_1, \dots, n_r) z_1^{n_1}, \dots, z_r^{n_r}. \quad (15)$$

For the symmetric product, viewed as an orbifold groupoid $[X^n/\mathfrak{S}_n]$, it follows that

$$H_{orb}^*([X^n/\mathfrak{S}_n]; \text{real}) \equiv \bigoplus_{\sum j n_j = n} \bigotimes_j H^*(X^{n_j}; \text{real})^{\mathfrak{S}_{n_j}}. \quad (16)$$

Using Equation (15) for calculating the orbifold Poincaré polynomial, we obtain

$$\begin{aligned} & \sum_{n=0}^{\infty} q^n P_{orb}([X^n/\mathfrak{S}_n], y) \\ &= \sum_{n=0}^{\infty} q^n \left(\sum_{\sum j n_j = n} \prod_j P(X^{n_j}/\mathfrak{S}_{n_j}, y) \right) \\ &= \prod_{j>0} \prod_{n \geq 0} \frac{(1 + q^n y^{2j+1})^{b_{2j+1}(X)}}{(1 - q^n y^{2j})^{b_{2j}(X)}} \\ &= \prod_{j>0} \frac{\left[\mathcal{R}\left(s = -i(2j+1)\log(y)/(2\pi\tau)(1 - i\rho(\tau)) + i\sigma(\tau)\right) \right]^{b_{2j+1}(X)}}{\left[\mathcal{R}\left(s = -ij\log(y)/(\pi\tau)(1 - i\rho(\tau))\right) \right]^{b_{2j}(X)}}. \end{aligned} \quad (17)$$

For these formulas the cohomology of X must be finitely generated at each n . The first line in (17) is similar to the equality (2.32) of [6]. As before $b^j(X)$ is the j -th Betti number of X .

5. The Ruelle-Type Spectral Functions

Let us introduce the Ruelle-type spectral function of a hyperbolic geometry $\mathcal{R}(s)$ [4]. The function $\mathcal{R}(s)$ is an alternating product of more complicated

factors, each one being the so-called Patterson-Selberg function,

$$\prod_{n=\ell}^{\infty} (1 - q^{an+\varepsilon}) = \mathcal{R}(s = (a\ell + \varepsilon)(1 - i\rho(\tau)) + 1 - a), \quad (18)$$

$$\prod_{n=\ell}^{\infty} (1 + q^{an+\varepsilon}) = \mathcal{R}(s = (a\ell + \varepsilon)(1 - i\rho(\tau)) + 1 - a + i\sigma(\tau)), \quad (19)$$

where $q \equiv e^{2\pi i\tau}$, $\rho(\tau) = \text{Re } \tau / \text{Im } \tau$, $\sigma(\tau) = (2 \text{Im } \tau)^{-1}$, a is a real number, $\varepsilon, b \in \mathbb{C}$, $\ell \in \mathbb{Z}_+$. The readers can find explicit formula for $\mathcal{R}(s)$ and its analytic properties in the reference [4].

The Bailey's Transform and Infinite Hierarchy of the Bailey's Chain

Let us begin with notations: $(a; q)_{\infty} := \prod_{m=0}^{\infty} (1 - aq^m)$, and $(a; q)_n := (a; q)_{\infty} / (aq^n; q)_{\infty}$. Bailey made the following observation [8] (see also [9]), which is known as *Bailey's transform*.

Theorem 5.1 (lectures [9], Theorem 3.1) *Let for a suitable convergence conditions,*

$$\beta_n = \sum_{r=0}^n \alpha_r u_{n-r} v_{n+r}, \quad (20)$$

$$\gamma_n = \sum_{r=n}^{\infty} \delta_r u_{r-n} v_{r+n}, \quad (21)$$

then

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n. \quad (22)$$

Proof.

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \sum_{r=n}^{\infty} \alpha_n \delta_r u_{r-n} v_{r+n} = \sum_{r=0}^{\infty} \sum_{n=0}^r \alpha_n \delta_r u_{r-n} v_{r+n} = \sum_{r=0}^{\infty} \delta_r \beta_r. \quad (23)$$

Important application of the Bailey's transform and Bailey's Lemma can be found in [8], Section 4

Let for $n \geq 0$

$$\beta_n = \sum_{r=0}^n \frac{\alpha_r}{(q; q)_{n-r} (aq; q)_{n+r}}, \quad (24)$$

then also

$$\beta'_n = \sum_{r=0}^n \frac{\alpha'_r}{(q; q)_{n-r} (aq; q)_{n+r}}. \quad (25)$$

In addition

$$\alpha'_r = \alpha_r \frac{(\rho_1; q)_r (\rho_2; q)_r (aq/\rho_1 \rho_2)^r}{(aq/\rho_1; q)_r (aq/\rho_2; q)_r}, \quad (26)$$

$$\beta'_n = \sum_{r \geq 0} \beta_r \frac{(\rho_1; q)_r (\rho_2; q)_r (aq/\rho_1 \rho_2; q)_{n-r} (aq/\rho_1 \rho_2)^j}{(q; q)_{n-j} (aq/\rho_1; q)_n (aq/\rho_2; q)_n}. \quad (27)$$

We apply Bailey's transform

$$u_{n-r} := \prod_{m=0}^{n-r} (1 - q^{m+1}) \equiv (q; q)_{n-r} = \frac{(q; q)_{\infty}}{(qq^{n-r}; q)_{\infty}} = \prod_{m=0}^{\infty} \frac{1 - q^{m+1}}{1 - q^{m+n-r+1}} \quad (28)$$

$$= \frac{\mathcal{R}(s=1-i\rho(\tau))}{\mathcal{R}(s=(n-r+1)(1-i\rho(\tau)))}, \quad (29)$$

$$v_{n+r} := \prod_{m=0}^{n+r} (1-aq^{m+1}) \equiv (aq; q)_{n+r} = \frac{(q; q)_{\infty}}{(aq^{n+r}; q)_{\infty}} = \prod_{m=0}^{\infty} \frac{1-q^{m+1}}{1-aq^{m+n+r+1}} \quad (30)$$

$$= \frac{\mathcal{R}(s=1-i\rho(\tau))}{\mathcal{R}(s=(n+r+\xi+1)(1-i\rho(\tau)))}, \quad (31)$$

where $\xi = \log a / 2\pi i \tau$.

Using Equations (29) and (31) we have:

$$\beta_n = \sum_{r=0}^n \alpha_r \frac{\mathcal{R}(s=(n-r+1)(1-\rho(\tau))) \cdot \mathcal{R}(s=(n-r+1)(1-\rho(\tau)))}{\mathcal{R}(s=1-i\rho(\tau)) \cdot \mathcal{R}(s=1-i\rho(\tau))}, \quad (32)$$

$$\beta'_n = \sum_{r=0}^n \alpha'_r \frac{\mathcal{R}(s=(n-r+1)(1-\rho(\tau))) \cdot \mathcal{R}(s=(n-r+1)(1-\rho(\tau)))}{\mathcal{R}(s=1-i\rho(\tau)) \cdot \mathcal{R}(s=1-i\rho(\tau))}. \quad (33)$$

In addition (see [9]),

$$\alpha'_r = \alpha_r \frac{(\rho_1; q)_r (\rho_2; q)_r (aq/\rho_1 \rho_2)^r}{(aq/\rho_1; q)_r (aq/\rho_2; q)_r}, \quad (34)$$

$$\beta'_n = \sum_{r \geq 0} \beta_r \frac{(\rho_1; q)_r (\rho_2; q)_r (aq/\rho_1 \rho_2; q)_{n-r} (aq/\rho_1 \rho_2)^r}{(q; q)_{n-j} (aq/\rho_1; q)_n (aq/\rho_2; q)_n}. \quad (35)$$

Important special cases of Bailey's Lemma and iterated of these results where found in [9]. Note that a pair of sequences (α_n, β_n) is called a *Bailey pair*. Thus if (α_n, β_n) is a Bailey pair, then (α'_n, β'_n) is a new pair which is given by (34) and (35).

Bailey's Chain. Using Bailey initial pair (α'_n, β'_n) we can create a new pair (α''_n, β''_n) by applying Bailey's Lemma. Continuing this process we create a sequence of Baley pairs, Bailey's chain:

$$(\alpha_n, \beta_n) \rightarrow (\alpha'_n, \beta'_n) \rightarrow (\alpha''_n, \beta''_n) \rightarrow (\alpha'''_n, \beta'''_n) \rightarrow \dots \quad (36)$$

Let us assume $(\alpha_n, \beta_n) = (\alpha_n^0, \beta_n^0)$, then this allows to extend the Bailey chain (36) to the left [9]:

$$\dots \rightarrow (\alpha_n^{(-2)}, \beta_n^{(-2)}) \rightarrow (\alpha_n^{(-1)}, \beta_n^{(-1)}) \rightarrow (\alpha_n^0, \beta_n^0) \rightarrow (\alpha'_n, \beta'_n) \rightarrow \dots \quad (37)$$

A Bailey pair is uniquely determined given either sequence α_n or β_n . Let a sequence α_n be given, then β_n sequence can be found. Equation (24) and Equation (25) can be inverted, and the result is [9]:

$$\begin{aligned} \alpha_n &= (1-aq^{2n}) \sum_{k=0}^n \beta_k \frac{(aq; q)_{n+k-1} (-1)^{n-k} q^{C_2^{n-k}}}{(q; q)_{n-k}} \\ &= (1-aq^{2n}) \sum_{k=0}^n \beta_k \frac{\prod_{m=0}^{\infty} (1-aq^{m+n+k}) (-1)^{n-k} q^{C_2^{n-k}}}{\prod_{m=0}^{\infty} (1-q^{m+n-k+1})}. \end{aligned} \quad (38)$$

Finally, we obtain

$$\alpha_n = (1 - aq^{2n}) \sum_{k=0}^n \beta_k \frac{\mathcal{R}(s = (n+k)(1 - i\rho(\tau))) (-1)^{n-k} q^{C_2^{n-k}}}{\mathcal{R}(s = (n-k+1)(1 - \rho(\tau)))}. \quad (39)$$

6. Conclusion

The main purpose of this paper is to present an explicit formula of q -polynomials which admit presentation in terms of the Patterson-Selberg (or Ruelle-type) spectral functions. Equivariant methods also can be used to define and analyze (twisted) K-theory. Applying techniques from the equivariant K-theory, one can construct a Chern character and K-theory Euler class. This is manifested by the fact that the q -series elliptic genera can be expressed in terms of q -analogs of the classical special functions, in particular the spectral Patterson-Selberg and Ruelle functions $\mathcal{R}(s)$. Our main results were presented in Section 5 and they were applied them in some interesting problems in theoretical physics.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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