Applied Mathematics, 2020, 11, 991-999
https://www.scirp.org/journal/am
ISSN Online: 2152-7393
ISSN Print: 2152-7385

Boundary Exact Controllability of the Heat Equation in 1D by Strategic Actuators and a Linear Surjective Compact Operator

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How to cite this paper: Seck, C., Ane, L. and Sène, A. (2020) Boundary Exact Controllability of the Heat Equation in 1D by Strategic Actuators and a Linear Surjective Compact Operator. *Applied Mathematics*, **11**, 991-999.
https://doi.org/10.4236/am.2020.1110065

Received: September 1, 2020
Accepted: October 16, 2020
Published: October 19, 2020

Abstract

In this paper we show a boundary result of controllability by a new approach using a linear, continuous and surjective operator built from the solution of the heat system. And, subsequently, the border exact controllability of the 1D heat equation through a compactness criterion and the use of strategic zone actuators were established.

Keywords

Control, Controllability, Estimations, Strategic Actuator

1. Introduction

These last years, the exact controllability of distributed systems has been significantly enhanced by J. L. Lions [1] [2] with the development of the Hilbert Uniqueness Methods (HUM). It is based essentially on the uniqueness properties of the homogeneous equation by a particular choice of controls, the construction of a Hilbert space and a continuous linear application of this Hilbert space in its dual which is, in fact, an isomorphism that establishes exact controllability.

For hyperbolic problems, this method yielded important results (Lions [3]); although when the controls are small support (Niane [4], Seck [5] [6]), it seems not very effective, likewise when for technical reasons the multiplier method does not give satisfactory results.

As for the parabolic equations, there are the results of Imanuvilov-Fursikov [7] and G. Lebeau-L. Robbiano [8] who proved with different methods but **very**

technical and long, the exact control of the Heat equation.

Also, the harmonic method is inoperative also for this kind of equations.

In this work, to circumvent certain constraints related to estimates in G. Lebeau's work, we show that a new method which solves some of these difficulties. It is based Seck's work; on criteria of surjectivity of a continuous linear operator of a Hilbert space in another construct directly from the problem of exact border controllability.

The criteria are of two types:

1) A surjectivity criterion that is a consequence of the properties of uniqueness (J. L. Lions);

2) A criterion of compactness that derives from the parabolic nature of the operator or the regularity of the control;

In both cases, these criteria are easier to verify than those of the Lions HUM method.

This method which we call **Boundary Exact Controllability by Surjectivity and Compactness** opens wide perspectives to the theory of the exact controllability in general, as well as to the theory of the exact controllability by actuators strategic zones and allows for the parabolic equations, from Schrödinger, plates, linearized Navier-Stockes to solve many questions thus opening up many perspectives.

2. Characterization of Exact Controllability

Indeed, we have the following result of functional analysis (see J. L. Lions and Ramdani [9]) which will allow us to characterize the exact controllability of the heat equation.

For proof see also jeups 2012 Karim-Ramdani or J. L. Lions.

2.1. Exact Controllability Reminders

Let $I =]0,1[$ an open interval of $\mathbb{R}$. We put A to the operator defined by:

$$D(A)=\left\{u \in H_0^1(I) / -\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \in L^2(I)\right\}; \forall u \in D(A), Au = -\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \tag{1}$$

Lemma 1. Let E and H two Hilbert spaces and $\mathcal{L} \in \mathcal{L}(E,H)$. Then, $\mathcal{L}$ is surjective if and only if its adjoint $\mathcal{L}^*$ is bounded below, *i.e.* there exists a constant $C > 0$ such that

$$\left\|\mathcal{L}^*(\beta)\right\|_H \geq C\|\beta\|_E, \quad \forall \beta \in H. \tag{2}$$

Lemma 2. The following assertions are equivalent:

1) The system (12) below is exactly controllable for $T > 0$.

2) The operator $\mathcal{L}_T^*$ is inferiorly bounded, *i.e.* there exists $C_T > 0$ such that

$$\left\|\mathcal{L}_T^*\right\|_{L^2(]0,T[,I)} = \left(\int_0^T \left\|\mathcal{L}_T^* \mathrm{e}^{sA} z_0\right\|^2 \mathrm{d}s\right)^{\frac{1}{2}} \geq C_T \left\|z_0\right\|, \quad \forall z_0 \in I \tag{3}$$

See Lions, El. Jai [10] [11] or also Ramdani-Karim jeups 2012.

Definition 1. Let $\mathcal{L}$ the operator defined below. We will say that the system defined by (12) that it is **exactly controllable in time T** if and only if $\mathcal{L}$ is surjective.

2.2. Preliminary Results of Controllability

Definition 2. An integrable square function $\mu : I \subset \Omega \to \overline{\mathbb{R}}$ is called strategic if it satisfies, for all $\psi_0 \in L^2(I)$, the solution ψ^+ of heat equation

$$\begin{cases} \psi^{+\prime}(t,x) - \Delta\psi^+(t,x) = 0 \text{ in } Q_T =]0,+\infty[\times I \\ \gamma\psi^+(t,x) = 0 \text{ in } \Sigma_T =]0,+\infty[\times \partial I \\ \psi^+(0) = \psi_0 \text{ in } I \end{cases} \tag{4}$$

satisfies:

$$\forall t > 0, \int_I \mu(x)\psi^+(t,x)\,\mathrm{d}x = 0 \text{ then } \psi_0 = 0. \tag{5}$$

Remark.

1) It suffices that the relation (5) be checked over an interval $]0,T[$ so that it is true over $]0,+\infty[$, because of the analycity of $t \mapsto \int_I \mu(x)\psi^+(t,x)\,\mathrm{d}x$ on $\mathbb{R}_+^*$, Brezis [12].

2) Here Ω is an open bounded of $\mathbb{R}^2$, of regular border; $L^2(I)$ is, a priori, the state space and T defines the time horizon considered for the exact controllability of the system (4).

Proposition 3. There are strategic actuators with support contained in any interval $]a,b[$ such that:

$$0 < a < b < 1$$

Proof. We can first notice that μ is strategic if and only if: $\forall k \in \mathbb{N}^*, \mu_k \neq 0$.

Let $a,b \in]0,1[$ such that $a < b$ and assume that: $\mu = \chi_{]a,b[}$.

So, we have

$$\begin{aligned} \mu_k &= \int_0^1 \chi_{]a,b[}(x)\sqrt{2}\sin(k\pi x)\,\mathrm{d}x \\ &= -\frac{\sqrt{2}}{k\pi}\left[\cos(k\pi b) - \cos(k\pi a)\right] \\ &= -\frac{\sqrt{2}}{k\pi}2\sin\left(\frac{k\pi(b-a)}{2}\right)\sin\left(\frac{k\pi(b+a)}{2}\right) \end{aligned}$$

We have $\mu_k = 0$ if and only if

$$\begin{cases} \dfrac{k\pi(b-a)}{2} = l\pi, \quad l \in \mathbb{Z} \\ \text{or} \\ \dfrac{k\pi(b+a)}{2} = r\pi, \quad r \in \mathbb{Z} \end{cases} \tag{6}$$

Therefore, for $\mu_k \neq 0$ it is enough that:

$$b - a \notin \mathbb{Q} \text{ and } b + a \notin \mathbb{Q}.$$

So, if we take $a \in \mathbb{Q}$ and $b = a + r$ where $r \notin \mathbb{Q}$ then $\mu = \chi_{]a,b[}$ is strategic.

Remark. Of course, other strategic actuators can be built without difficulty, see also El. Jai.

We define the Hilbert spaces that follow:

$$\forall T > 0,\ G_T = \left\{ \mu = \sum_{k=1}^{+\infty} \mu_k w_k \Big/ \sum_{k=1}^{+\infty} \mu_k^2 e^{2\lambda_k T} < +\infty \right\} \tag{7}$$

We equip G_T with the following scalar product $(,)_{G_T}$:

$$(x, y)_{G_T} = \sum_{k=1}^{\infty} x_k y_k e^{2\lambda_k T}$$

and, of the norm $\|.\|_{G_T}$.

We know that G_T is a Hilbert space; its dual is defined by:

$$\forall T > 0,\ G_T^* = \left\{ \mu = \sum_{k=1}^{+\infty} \mu_k w_k \Big/ \sum_{k=1}^{+\infty} \mu_k^2 e^{-2\lambda_k T} < +\infty \right\} \tag{8}$$

We equip G_T^* with the scalar product

$$(x, y)_{G_T^*} = \sum_{k=1}^{\infty} x_k y_k e^{-2\lambda_k T}$$

and, of the norm $\|.\|_{G_T^*}$.

We define the duality hook for $x \in G_T^*, y \in G_T$ by

$$\langle x, y \rangle_{G_T^* G_T} = \sum_{k=1}^{\infty} x_k y_k \tag{9}$$

Let $\psi_0 \in G_T^*$, we notice ψ^+ the solution of the following heat equation

$$\begin{cases} \psi^{+'}(t,x) - \Delta\psi^+(t,x) = 0 & \text{in }]0,\infty[\times I \\ \psi^+(t,0) = \psi^+(t,1) = 0 & \text{in }]0,\infty[\\ \psi^+(0,x) = \psi_0(x) & \text{in } I \end{cases} \tag{10}$$

Let $\psi_0 \in G_T$, we notice ψ^- the solution of

$$\begin{cases} \psi^{-'}(t,x) - \Delta\psi^-(t,x) = 0 & \text{in }]0,\infty[\times I \\ \psi^-(t,0) = \psi^-(t,1) = 0 & \text{in }]0,\infty[\\ \psi^-(0,x) = \psi_0(x) & \text{in } I \end{cases} \tag{11}$$

3. Main Result of Boundary Exact Control

In the following, we want to establish an exact controllability result by the construction of a particular linear, continuous and surjective operator. Indeed, we want to solve the following problem:

for all y_0 in a space to be determined after, find $\beta \in L^2(]0,T[)$ such that if y is a solution of the homogeneous heat equation:

$$\begin{cases} y' - \Delta y = 0 & \text{in } Q_T =]0,+\infty[\times I \\ y(t,0) = 0,\ y(t,1) = \beta(t) \\ y(0) = y_0 & \text{in } I \end{cases} \tag{12}$$

then $y(T) = 0$.

Thus, considering that $y(t,1)=\beta(t)$ and either $\mu(x)=x$, we have

$$\begin{cases} -\dfrac{d^2}{dx^2}\mu(x)=0 \\ \mu(0)=0,\ \ \mu(1)=1 \end{cases} \tag{13}$$

and $\mu \in L^2(I)$ so we get $\gamma y(t)=\beta(t)\mu$ on $]0,T[\times \partial I$.

One can formally also see how the operator L can be constructed.

Multiply the Equation (12) by ψ^- solution of the Equation (11), we have:

$$\int_\Omega y(T)\psi^-(T)\,dx-\langle y_0,\psi_0\rangle+\int_0^T\int_{\partial I}\beta(t)\mu(\sigma)\frac{\partial\psi^-(t,\sigma)}{\partial\nu}\,d\sigma dt=0 \tag{14}$$

Let

$$\langle y_0,\psi_0\rangle=\int_0^T\int_{\partial I}\beta(t)\mu(\sigma)\frac{\partial\psi^-(t,\sigma)}{\partial\nu}\,d\sigma dt \tag{15}$$

In order for the second member to make sense, it must be assumed that: $\dfrac{\partial\psi^-(t,\sigma)}{\partial\nu}\in L^2(]0,T[,\partial I)$. It suffices to assume that $\psi^-\in L^2(]0,T[,D(A))$.

We know that:

$$\sum_{k=1}^{\infty}\int_0^T\lambda_k^2\psi_{0k}^2 e^{2\lambda_k t}dt := \sum_{k=1}^{\infty}\frac{1}{2}\lambda_k\psi_{0k}^2\left[e^{2\lambda_k T}-1\right],$$

Therefore, we deduce that

$$\sum_{k=1}^{\infty}\int_0^T\lambda_k^2\psi_{0k}^2 e^{2\lambda_k t}dt \le \sum_{k=1}^{\infty}\frac{1}{2}\lambda_k\psi_{0k}^2 e^{2\lambda_k T}<+\infty.$$

Remember that we had defined the following spaces:

$$G_T=\left\{\mu=\sum_{k=1}^{\infty}\mu_k w_k \middle/ \sum_{k=1}^{\infty}\lambda_k\mu_k^2 e^{2\lambda_k T}<+\infty\right\}$$

and we equip it with the following scalar product:

$$(x,y)_{G_T}=\sum_{k=1}^{+\infty}\lambda_k x_k y_k e^{2\lambda_k T}$$

and the natural norm $\|.\|_{G_T}$ of dual G_T^* defined by

$$G_T^*=\left\{\mu=\sum_{k=1}^{\infty}\mu_k w_k \middle/ \sum_{k=1}^{\infty}\lambda_k\mu_k^2 e^{-2\lambda_k T}<+\infty\right\}$$

also provided with the scalar product $(,)_{G_T^*}$ and its natural norm.

With the previous notations, the formula (15) is written

$$\langle y_0,\psi_0\rangle_{G_T^*,G_T}=\int_0^T\int_{\partial I}\beta(t)\mu(\sigma)\frac{\partial\psi^-(t,\sigma)}{\partial\nu}\,d\sigma dt \tag{16}$$

$$=\int_0^T\beta(t)\int_{\partial I}\mu(\sigma)\frac{\partial\psi^-(t,\sigma)}{\partial\nu}\,d\sigma dt \tag{17}$$

By integrating by part by Green, we obtain

$$\langle y_0,\psi_0\rangle_{G_T^*,G_T}=\int_0^T\beta(t)\int_I\Delta\psi^-\mu\,dx dt \tag{18}$$

$$= -\int_0^T \beta(t) \int_I \psi^{-\prime} \mu \mathrm{d}x\mathrm{d}t \tag{19}$$

$$= -\int_0^T \beta(t) \left\langle \mu^{-\prime}, \psi_0 \right\rangle_{G_T^*, G_T} \mathrm{d}t \tag{20}$$

$$= -\left\langle \int_0^T \beta(t) \mu^{-\prime} \mathrm{d}t, \psi_0 \right\rangle_{G_T^*, G_T}. \tag{21}$$

where

$$y_0 = -\int_0^T \beta(t) \mu^{-\prime} \mathrm{d}t \ \text{ in } G_T^* \tag{22}$$

which allows us to define the operator

$$L_1(\beta) = -\int_0^T \beta(t) \mu^{-\prime} \mathrm{d}t \ \text{ in } G_T^* \tag{23}$$

Remark. We can notice that: $x = y$ in G_T^* equals

$$(-\Delta)^{-1} x^+(T) = (-\Delta)^{-1} y^+(T) \ \text{ in } L^2(I) \tag{24}$$

We thus define the operator L by

$$L(\beta) = \int_0^T \beta(t) (-\Delta)^{-1} \mu^{+\prime}(T-t) \mathrm{d}t \tag{25}$$

Using the relationship

$$\mu^{+\prime}(T-t) - \Delta\mu^+(T-t) = 0 \ \text{ in }]0,T[\times I \tag{26}$$

Remark. The Lemma 1 and Lemma 2, responds to another philosophy than the usual one whose main hypothesis is the coercivity that assumes the verification of an estimate difficult to establish in explicit spaces.

So, we have the following boundary controllability result

Theorem 4 (Main result). For all $y_0 \in G_T^*$, it exists $\beta \in L^2(]0,T[)$ such that if y is the solution of

$$\begin{cases} y' - \Delta y = 0 \ \text{ in } Q_T =]0,+\infty[\times I \\ y(t,0) = 0, \ y(t,1) = \beta(t) \\ y(0) = y_0 \ \text{ in } I \end{cases} \tag{27}$$

then $y(T) = 0$.

Proof. **First step**:

By construction, $L \in \mathcal{L}(L^2(]0,T[), L^2(I))$. Indeed, the operator L is exactly defined by a formula (26); So just show that μ is strategic.

We have $\mu \in L^2(I)$, after that

$$\begin{aligned} \mu_k &= \sqrt{2}\int_0^1 x \sin(k\pi x) \mathrm{d}x, \ \forall k \geq 1 \\ &= -\frac{\sqrt{2}}{k\pi}(-1)^k + \frac{\sqrt{2}}{k\pi} \sin(k\pi x)\Big|_0^1 \\ &= -\sqrt{2}\frac{(-1)^k}{k\pi} \quad \forall k \geq 1 \end{aligned}$$

So μ is strategic: not degenerated.

Second step:

Let's show that $L \in \mathcal{L}(L^2(]0,T[), L^2(I))$ is surjective?

We know that the operator L is defined by:

$$L(\beta)=\int_0^T \beta(t)(-\Delta)^{-1}\mu^{+\prime}(T-t)\mathrm{d}t \tag{28}$$

$$=\int_0^T \beta(t)\sum_{k=1}^{+\infty}\mu_k \mathrm{e}^{-\lambda_k(T-t)}w_k \mathrm{d}t \tag{29}$$

$$=\int_0^T \sum_{k=1}^{\infty}\beta(t)\mu_k \mathrm{e}^{-\lambda_k(T-t)}w_k \mathrm{d}t \tag{30}$$

and his dual is

$$L^*(\beta)=\int_0^T \sum_{k=1}^{\infty}\beta(t)\mu_k \mathrm{e}^{-\lambda_k(t-T)}w_k \mathrm{d}t \tag{31}$$

$$=\int_0^T \sum_{k=1}^{\infty}\beta(t)\mu_k \mathrm{e}^{\lambda_k(T-t)}w_k \mathrm{d}t \tag{32}$$

So

$$\left\|L^*(\beta)\right\|_{G_T^*}^2 \geq \int_0^T \sum_{k=1}^{\infty}\left\|\beta(t)\right\|^2\left\|\mu_k\right\|^2 \mathrm{e}^{-2\lambda_k(T-t)}\left\|w_k\right\|^2 \mathrm{d}t \tag{33}$$

$$\geq \left\|\beta\right\|_{L^2(]0,T[)}^2\left\|\mu_k\right\|^2 \mathrm{e}^{-2\lambda_k(T-t)}\left\|w_k\right\|^2 \tag{34}$$

$$\geq K\cdot\left\|\beta\right\|_{L^2(]0,T[)}^2 \tag{35}$$

where K a constant defined by $K=\min_{\{k\geq 1\}}\left\|\mu_k\right\|^2 \mathrm{e}^{-2\lambda_k(T-t)}\left\|w_k\right\|^2$.

By the Lemma 1 and Lemma 2, we can deduce that L is surjective.

Third step: conclusion.

We know that μ is not degenerate, the operator L is surjective and, in addition, the operator LL^* is compact see also Seck.

Let now $y_0 \in G_T^*$, it exists $\beta \in L^2(\Omega)$ such that

$$(-\Delta)^{-1}y^+(T)=-\int_0^T \beta(t)\mu^+(T-t)\mathrm{d}t$$

so

$$y^+(T)=-\int_0^T \beta(t)(-\Delta)\mu^+(T-t)\mathrm{d}t$$

Let

$$y^+(T)=-\int_0^T \beta(t)\mu^{+\prime}(T-t)\mathrm{d}t \text{ so } y_0=-\int_0^T \beta(t)\mu^{-\prime}(t)\mathrm{d}t$$

Let $\psi_0 \in G_T$, we have

$$\langle y_0,\psi_0\rangle_{G_T^*,G_T}=\sum_{k=1}^{+\infty}\int_0^T \beta(t)\mu_k \mathrm{e}^{\lambda_k t}\lambda_k\psi_{0k}\mathrm{d}t$$

As

$$\sum_{k=1}^{+\infty}\int_0^T\left|\beta(t)\right|\left|\mu_k\right|\left|\lambda_k\right|\psi_{0k}\mathrm{e}^{\lambda_k t}\mathrm{d}t$$

$$\leq \frac{1}{\sqrt{2}}\sum_{k=1}^{\infty}\left\|\beta\right\|_{L^2(]0,T[)}\left|\mu_k\right|\lambda_k\left|\psi_{0k}\right|\mathrm{e}^{\lambda_k T}$$

$$\leq \frac{1}{\sqrt{2}}\left\|\beta\right\|_{L^2(]0,T[)}\left\|\mu\right\|_{L^2(]0,T[)}\left\|\psi_0\right\|_{G_T}$$

So,

$$\begin{aligned}\langle y_0,\psi_0\rangle_{G_T^*,G_T} &= -\int_0^T \beta(t)\sum_{k=1}^{+\infty}\mu_k \mathrm{e}^{\lambda_k t}\lambda_k\psi_{0k}\,\mathrm{d}t\\ &= \int_0^T \beta(t)\int_I \mu\psi^{-\prime}(t)\,\mathrm{d}x\mathrm{d}t\\ &= \int_0^T \beta(t)\int_I \mu\Delta\psi^{-}(t)\,\mathrm{d}x\mathrm{d}t\\ &= \int_0^T\int_{\partial I}\frac{\partial\psi^{-}(t,\sigma)}{\partial\nu}\mu\,\mathrm{d}\sigma\mathrm{d}t\end{aligned}$$

Now, multiply the system (27) by $\psi^{-}(t)$ integrating by parts

$$\langle y(T),\psi^{-}(T)\rangle_{H^{-1}(I),H_0^1(I)}-\langle y_0,\psi_0\rangle_{G_T^*,G_T}=-\int_0^T\beta(t)\frac{\partial\psi^{-}(t,1)}{\partial\nu}\mathrm{d}t$$

So

$$\forall\psi_0\in G_T,\langle y(T),\psi^{-}(T)\rangle_{H^{-1}(I),H_0^1(I)}=0 \tag{36}$$

From where $y(T)=0$: which completes the proof.

4. Conclusion and Perspective

The exact controllability results by the HUM (Hilbert Uniqueness Method of Lions) method are not suitable for parabolic type operators. To get around these difficulties, in particular the coercivity hypothesis for the establishment of the inverse inequality, many have used Carlemann inequalities Fursikov-Imanuvilov, Lebeau-Robbiano, Khodja [13], Tusnack [14], ... But the length and heaviness of the calculations in the Carleman inequalities are dissuasive, from where this idea came to us to couple the notion of strategic zone actuators and the subjectivity-compactness of a linear operator which allowed us to have the exact controllability. And this technique opens up many perspectives for linear and semi-linear parabolic systems, of Schrodinger, of plates, ...

Acknowledgements

- The authors thank the referees in advance for their comments and suggestions.
- The authors thank the Dean of FASTEF ex ENS of the University Cheikh Anta Diop in Dakar and his assessor for their financial and moral support.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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