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Generalized Hermite-Hadamard Type Inequalities Related to Katugampola Fractional Integrals

Hao Wang

Department of Mathematics, College of Science, Hunan City University, Yiyang, China
Email: haowangctgu@163.com

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Abstract

In this paper, we have established a new identity related to Katugampola fractional integrals which generalize the results given by Topul *et al.* and Sarikaya and Budak. To obtain our main results, we assume that the absolute value of the derivative of the considered function φ' is p-convex. We derive several parameterized generalized Hermite-Hadamard inequalities by using the obtained equation. More new inequalities can be presented by taking special parameter values for λ, μ and p. Also, we provide two examples to illustrate our results.

Subject Areas

Mathematical Analysis, Numerical Mathematics

Keywords

p-Convex Mappings, Katugampola Fractional Integrals, Hermite-Hadamard's Inequalities

1. Introduction

Recently, İşcan [1] presented the following concept of p-convex mappings, which is a generalization of ordinary convexity and harmonically convexity.

Definition 1.1 [1] *The mapping* $\pi: I \subset (0,\infty) \to \mathbb{R}$ *is named a p-convex mapping, where* $p \in \mathbb{R}\setminus\{0\}$. *If for all* $s \in [0,1]$ *and* $x, y \in I$, *we have*

$$\pi\left(\left[sx^p + (1-s)y^p\right]^{\frac{1}{p}}\right) \le s\pi(x) + (1-s)\pi(y). \tag{1.1}$$

Many researchers have worked in the properties and inequalities for p-convex functions. For example, Zhang and Wang [2] introduced some properties for p-convexity. Kunt and İşcan [3] established several Hermite-Hadamard-Fejér inequalities involving p-convex mapping. Dragomir *et al.* [4] gave some integral inequalities for differentiable p-convex mappings. Mehreen and Anwar [5] presented several Hermite-Hadamard type inequalities related to exponentially p-convex functions. For more results associated with p-convex functions see references in [6] [7].

In [8], Katugampola introduced a class of fractional integral operator, which generalizes Riemann-Liouville and Hadamard fractional integrals simultaneously.

Definition 1.2 [8] *Let* $[a,b]\subset\mathbb{R}$ *be a finite interval. The left-side and right-side Katugampola fractional integrals of order* $\mu\ge 0$ *of* $\pi\in\chi_c^\sigma(a,b)(c\in\mathbb{R},1\le\sigma\le\infty)$ *are defined respectively by*

$$ {}^pI_{a^+}^{\mu}\pi(x)=\frac{p^{1-\mu}}{\Gamma(\mu)}\int_a^x\frac{s^{p-1}}{\left(x^p-s^p\right)^{1-\mu}}\pi(s)\mathrm{d}s \tag{1.2}$$

and

$$ {}^pI_{b^-}^{\mu}\pi(x)=\frac{p^{1-\mu}}{\Gamma(\mu)}\int_x^b\frac{s^{p-1}}{\left(s^p-x^p\right)^{1-\mu}}\pi(s)\mathrm{d}s, \tag{1.3}$$

where $a<x<b$, $p>0$ *and* $\Gamma(\mu)$ *is the Gamma function and its definition is* $\Gamma(\mu)=\int_0^\infty \mathrm{e}^{-s}s^{\mu-1}\mathrm{d}s$, *if the integrals exist.*

Theorem 1.1 [8] *Let* $\mu>0$ *and* $p>0$. *Then, for* $x<b$, *we have*

$$\lim_{p\to 1}{}^pI_{b^-}^{\mu}\pi(x)=J_{b^-}^{\mu}\pi(x)\text{ and }\lim_{p\to 0^+}{}^pI_{b^-}^{\mu}\pi(x)=\mathcal{H}_{b^-}^{\mu}\pi(x), \tag{1.4}$$

and for $x>a$, *we have*

$$\lim_{p\to 1}{}^pI_{a^+}^{\mu}\pi(x)=J_{a^+}^{\mu}\pi(x)\text{ and }\lim_{p\to 0^+}{}^pI_{a^+}^{\mu}\pi(x)=\mathcal{H}_{a^+}^{\mu}\pi(x), \tag{1.5}$$

where the symbol $J_{a^+}^{\mu}\pi$ *and* $J_{b^-}^{\mu}\pi$ *denote respectively the left-sided and right-sided Riemann-Liouville fractional integrals of the order* $\mu\in\mathbb{R}^+$ *defined by*

$$J_{a^+}^{\mu}\pi(x)=\frac{1}{\Gamma(\mu)}\int_a^x(x-s)^{\mu-1}\pi(s)\mathrm{d}s,\ a<x \tag{1.6}$$

and

$$J_{b^-}^{\mu}\pi(x)=\frac{1}{\Gamma(\mu)}\int_x^b(s-x)^{\mu-1}\pi(s)\mathrm{d}s,\ x<b. \tag{1.7}$$

And the symbol $\mathcal{H}_{a^+}^{\mu}$ *and* $\mathcal{H}_{b^-}^{\mu}\pi$ *denote respectively the left-sided and right-sided Hadamard fractional integrals of order* $\mu\in\mathbb{R}^+$ *defined as*

$$\mathcal{H}_{a^+}^{\mu}\pi(x)=\frac{1}{\Gamma(\mu)}\int_a^x\left(\ln\left(\frac{x}{s}\right)\right)^{\mu-1}\pi(s)\frac{\mathrm{d}s}{s},\ a<x \tag{1.8}$$

and

$$\mathcal{H}_{b^-}^{\mu}\pi(x)=\frac{1}{\Gamma(\mu)}\int_x^b\left(\ln\left(\frac{x}{s}\right)\right)^{\mu-1}\pi(s)\frac{\mathrm{d}s}{s},\ \ x<b. \tag{1.9}$$

Theory of Katugampola fractional integral operators attract widely attention for many authors, some new generalizations, extensions and variations of classically integral inequalities via Katugampola fractional integrals have been established in the literature. For example, Chen and Katugampola [9] obtained Hermite-Hadamard and Hermite-Hadamard-Fejér type inequalities in connection with Katugampola fractional integrals and convex mappings. Delavar and Dragomir [10] studied Katugampola fractional integrals Hermite-Hadamard's mid-point inequalities via Lipschitzian mappings and convex mappings. Toplu *et al.* [11] established the Hermite-Hadamard inequality for *p*-convex mappings via Katugampola fractional integrals. Kermausuor *et al.* [12] introduced some new Katugampola fractional integrals Hermite-Hadamard type inequalities through strongly η-convex mappings. For more information related to Katugampola fractional integral operators, we refer an interested reader to [13]-[17].

In [18], Hu *et al.* established the following identity for right Katugampola fractional integrals to derive several parameterized integral inequalities.

Theorem 1.2 *Let* $p,\mu>0$ *and* $\pi:[a,b]\subseteq(0,\infty)\to\mathbb{R}$ *be a differentiable mapping on* (a,b) *such that* $\pi'\in L^1([a,b])$, *where* $0<a<b$. *Then for all* $m,n\in\mathbb{R}$, *the following identity holds*:

$$\begin{aligned}&m\pi(a)+(n-m)\pi\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)+(1-n)\pi(b)-\frac{p^{\alpha}\Gamma(\alpha+1)}{b^p-a^p}\,{}^{p}I_{b^-}^{\mu}\pi(a)\\&=\frac{b^p-a^p}{p}\int_0^1 z(s)\left(sb^p+(1-s)a^p\right)\pi'\left(\sqrt[p]{sb^p+(1-s)a^p}\right)\mathrm{d}s,\end{aligned} \tag{1.10}$$

where

$$z(s)=\begin{cases}s^{\mu}-m, & s\in\left[0,\frac{1}{2}\right),\\ s^{\mu}-n, & s\in\left[\frac{1}{2},1\right].\end{cases} \tag{1.11}$$

These studies motivated us to establish some trapezium-type inequalities involving Katugampola fractional integrals for the mappings whose first derivative absolute values are *p*-convex. We emphasize that our main results generalize the ones obtained by Sarikaya and Budak [19]. Also, we present two examples to support our results.

2. New Lemma

Before stating the results, we define some notations as follows:

$$m:=\left(\lambda a^p+(1-\lambda)b^p\right)^{\frac{1}{p}}, \tag{2.1}$$

$$n:=\left(\lambda b^p+(1-\lambda)a^p\right)^{\frac{1}{p}} \tag{2.2}$$

and

$$\Phi_{\pi}(\mu,p,\lambda,a,b)$$
$$:=-\frac{p\left[\pi(n)+\pi(m)\right]}{(1-2\lambda)\left(b^{p}-a^{p}\right)}+\frac{\Gamma(1+\mu)p^{\mu+1}}{(1-2\lambda)^{\mu+1}\left(b^{p}-a^{p}\right)^{\mu+1}}\left[{}^{p}I_{n^{+}}^{\mu}\pi(m)+{}^{p}I_{m^{-}}^{\mu}\pi(n)\right]. \tag{2.3}$$

Lemma 2.1 *Assume that* $\mu,p>0$ *and* $\pi:\left[a^{p},b^{p}\right]\subseteq(0,\infty)\to\mathbb{R}$ *be a differentiable mapping on* $\left(a^{p},b^{p}\right)$ *with* $0<a<b$ *satisfying* $\pi'\in L^{1}\left(\left[a^{p},b^{p}\right]\right)$. *Then the following identity exists:*

$$\Phi_{\pi}(\mu,p,\lambda,a,b)$$
$$=\int_{0}^{1}\left[(1-s)^{\mu}-s^{\mu}\right]\left(sm^{p}+(1-s)n^{p}\right)^{\frac{1}{p}-1}\pi'\left(\left[sm^{p}+(1-s)n^{p}\right]^{\frac{1}{p}}\right)\mathrm{d}s, \tag{2.4}$$

where $\lambda\in(0,1]\setminus\frac{1}{2}$.

Proof. It suffices to note that:

$$\xi=\int_{0}^{1}\left[(1-s)^{\mu}-s^{\mu}\right]\left(sm^{p}+(1-s)n^{p}\right)^{\frac{1}{p}-1}\pi'\left(\left[sm^{p}+(1-s)n^{p}\right]^{\frac{1}{p}}\right)\mathrm{d}s$$
$$=\left[\int_{0}^{1}(1-s)^{\mu}\left(sm^{p}+(1-s)n^{p}\right)^{\frac{1}{p}-1}\pi'\left(\left[sm^{p}+(1-s)n^{p}\right]^{\frac{1}{p}}\right)\mathrm{d}s\right]$$
$$+\left[-\int_{0}^{1}s^{\mu}\left(sm^{p}+(1-s)n^{p}\right)^{\frac{1}{p}-1}\pi'\left(\left[sm^{p}+(1-s)n^{p}\right]^{\frac{1}{p}}\right)\mathrm{d}s\right] \tag{2.5}$$
$$:=\xi_{1}+\xi_{2}.$$

Integrating by parts, we obtain

$$\xi_{1}=\frac{p}{(1-2\lambda)\left(b^{p}-a^{p}\right)}\left[\int_{0}^{1}(1-s)^{\mu}\,\mathrm{d}\left(\pi\left(\left[sm^{p}+(1-s)n^{p}\right]^{\frac{1}{p}}\right)\right)\right]$$
$$=\frac{p}{(1-2\lambda)\left(b^{p}-a^{p}\right)}\left[(1-s)^{\mu}\pi\left(\left[sm^{p}+(1-s)n^{p}\right]^{\frac{1}{p}}\right)\Big|_{0}^{1}\right.$$
$$\left.+\mu\int_{0}^{1}(1-s)^{\mu-1}\pi\left(\left[sm^{p}+(1-s)n^{p}\right]^{\frac{1}{p}}\right)\mathrm{d}s\right] \tag{2.6}$$
$$=-\frac{p\pi(n)}{(1-2\lambda)\left(b^{p}-a^{p}\right)}$$
$$+\frac{\mu p}{(1-2\lambda)\left(b^{p}-a^{p}\right)}\int_{0}^{1}(1-s)^{\mu-1}\pi\left(\left[sm^{p}+(1-s)n^{p}\right]^{\frac{1}{p}}\right)\mathrm{d}s.$$

Using the change of variable $x^{p}=s\left(\lambda a^{p}+(1-\lambda)b^{p}\right)+(1-s)\left(\lambda b^{p}+(1-\lambda)a^{p}\right)$ for $s\in[0,1]$, then the Equality (2.6) can be written as

$$\xi_{1}=-\frac{p\pi(n)}{(1-2\lambda)\left(b^{p}-a^{p}\right)}+\frac{\mu p^{2}}{(1-2\lambda)^{\mu+1}\left(b^{p}-a^{p}\right)^{\mu+1}}\int_{n}^{m}\frac{x^{p-1}}{\left(m^{p}-x^{p}\right)^{1-\mu}}\pi(x)\mathrm{d}x$$
$$=-\frac{p\pi(n)}{(1-2\lambda)\left(b^{p}-a^{p}\right)}+\frac{\Gamma(1+\mu)p^{\mu+1}}{(1-2\lambda)^{\mu+1}\left(b^{p}-a^{p}\right)^{\mu+1}}\,{}^{p}I_{n^{+}}^{\mu}\pi(m), \tag{2.7}$$

and similarly, we have

$$\xi_2 = -\frac{p}{(1-2\lambda)(b^p-a^p)}\left[\int_0^1 s^\mu \mathrm{d}\left(\pi\left(\left[sm^p+(1-s)n^p\right]^{\frac{1}{p}}\right)\right)\right.$$

$$= -\frac{p}{(1-2\lambda)(b^p-a^p)}\left[s^\mu \pi\left(\left[sm^p+(1-s)n^p\right]^{\frac{1}{p}}\right)\Big|_0^1\right.$$

$$\left.+\mu\int_0^1 s^{\mu-1}\pi\left(\left[sm^p+(1-s)n^p\right]^{\frac{1}{p}}\right)\mathrm{d}s\right] \tag{2.8}$$

$$= -\frac{p\pi(m)}{(1-2\lambda)(b^p-a^p)}+\frac{\mu p^2}{(1-2\lambda)^{\mu+1}(b^p-a^p)^{\mu+1}}\int_n^m \frac{x^{p-1}}{(x^p-n^p)^{1-\mu}}\pi(x)\mathrm{d}x$$

$$= -\frac{p\pi(m)}{(1-2\lambda)(b^p-a^p)}+\frac{\Gamma(1+\mu)p^{\mu+1}}{(1-2\lambda)^{\mu+1}(b^p-a^p)^{\mu+1}}\,{}^pI_{m^-}^{\mu}\pi(n).$$

Adding the Equality (2.7) and Equality (2.8) together, we get

$$\xi = \xi_1+\xi_2$$

$$= -\frac{p[\pi(n)+\pi(m)]}{(1-2\lambda)(b^p-a^p)}+\frac{\Gamma(1+\mu)p^{\mu+1}}{(1-2\lambda)^{\mu+1}(b^p-a^p)^{\mu+1}}\left[{}^pI_{n^+}^{\mu}\pi(m)+{}^pI_{m^-}^{\mu}\pi(n)\right]. \tag{2.9}$$

This completes the proof.

Remark 2.1 *Choosing* $p=1$ *in Lemma* 2.1, *we have Lemma* 2.1 *presented by Sarikaya and Budak in* [19].

Remark 2.2 *Taking* $\lambda=0$ *in Lemma* 2.1, *we have*

$$-\frac{p[\pi(a)+\pi(b)]}{b^p-a^p}+\frac{\Gamma(1+\mu)p^{\mu+1}}{(b^p-a^p)^{\mu+1}}\left[{}^pI_{a^+}^{\mu}\pi(b)+{}^pI_{b^-}^{\mu}\pi(a)\right]$$
$$= \int_0^1\left[(1-s)^\mu-s^\mu\right]\left(sb^p+(1-s)a^p\right)^{\frac{1}{p}-1}\pi'\left(\left[sb^p+(1-s)a^p\right]^{\frac{1}{p}}\right)\mathrm{d}s. \tag{2.10}$$

Similarly, putting $\lambda=1$ *in Lemma* 2.1, *we obtain*

$$\frac{p[\pi(a)+\pi(b)]}{b^p-a^p}+\frac{\Gamma(1+\mu)p^{\mu+1}}{(b^p-a^p)^{\mu+1}}\left[{}^pI_{a^-}^{\mu}\pi(b)+{}^pI_{b^+}^{\mu}\pi(a)\right]$$
$$= \int_0^1\left[(1-s)^\mu-s^\mu\right]\left(sa^p+(1-s)b^p\right)^{\frac{1}{p}-1}\pi'\left(\left[sa^p+(1-s)b^p\right]^{\frac{1}{p}}\right)\mathrm{d}s, \tag{2.11}$$

which is proved by Toplu et al. in [11].

3. Main Results

We now present some katugampola fractional integrals inequalities with multiple parameters related to p-convex mappings.

Theorem 3.1 *Let* $\pi:\left[a^p,b^p\right]\subset(0,\infty)\to R$ *be a differentiable mapping on* $\left(a^p,b^p\right)$ *with* $0<a<b$ *such that* $\pi'\in L^1\left(\left[a^p,b^p\right]\right)$. *If* $|\pi'|^q$ *is p-convex on* $\left[a^p,b^p\right]$ *for* $p>0$, $q\ge 1$, *then the following inequality for katugampola fractional integrals holds*:

$$\left|\Phi_{\pi}\left(\mu, p, \lambda, a, b\right)\right| \leq \eta^{1-p}\left[\frac{2}{\mu+1}\left(1-\frac{1}{2^{\mu}}\right)\right]\left[\frac{\left|\pi'(m)\right|^{q}+\left|\pi'(n)\right|^{q}}{2}\right]^{\frac{1}{q}}, \tag{3.1}$$

where $\mu > 0$, $\lambda \in (0,1] \setminus \frac{1}{2}$ *and*

$$\eta = \begin{cases} a, & 1 \leq p < \infty, \\ b, & 0 < p < 1. \end{cases} \tag{3.2}$$

Proof. Case 1: $p \geq 1$. By means of Lemma 2.1, one has

$$\begin{aligned} &\left|\Phi_{\pi}\left(\mu, p, \lambda, a, b\right)\right| \\ &\leq \int_0^1 \left|(1-s)^{\mu} - s^{\mu}\right| \left|\left(sm^p + (1-s)n^p\right)^{\frac{1}{p}-1}\right| \left|\pi'\left(\left[sm^p + (1-s)n^p\right]^{\frac{1}{p}}\right)\right| \mathrm{d}s. \end{aligned} \tag{3.3}$$

For all $0 \leq s \leq 1$, applying the fact that

$$\left(s\left(\lambda a^p + (1-\lambda)b^p\right) + (1-s)\left(\lambda b^p + (1-\lambda)a^p\right)\right)^{\frac{1}{p}-1} \leq \left(a^p\right)^{\frac{1}{p}-1} = a^{1-p}, \tag{3.4}$$

we obtain

$$\left|\Phi_{\pi}\left(\mu, p, \lambda, a, b\right)\right| \leq a^{1-p}\left[\int_0^1 \left|(1-s)^{\mu} - s^{\mu}\right| \left|\pi'\left(\left[sm^p + (1-s)n^p\right]^{\frac{1}{p}}\right)\right| \mathrm{d}s\right]. \tag{3.5}$$

Using the Hölder inequality for inequality (3.5), we deduce

$$\begin{aligned} &\left|\Phi_{\pi}\left(\mu, p, \lambda, a, b\right)\right| \\ &\leq a^{1-p}\left[\int_0^1 \left|(1-s)^{\mu} - s^{\mu}\right| \mathrm{d}s\right]^{1-\frac{1}{q}} \left[\int_0^1 \left|(1-s)^{\mu} - s^{\mu}\right| \left|\pi'\left(\left[sm^p + (1-s)n^p\right]^{\frac{1}{p}}\right)\right|^q \mathrm{d}s\right]^{\frac{1}{q}} \\ &= a^{p-1}\left(\int_0^{\frac{1}{2}} \left[(1-s)^{\mu} - s^{\mu}\right] \mathrm{d}s + \int_{\frac{1}{2}}^1 \left[s^{\mu} - (1-s)^{\mu}\right] \mathrm{d}s\right)^{1-\frac{1}{q}} \\ &\cdot \left[\int_0^1 \left|(1-s)^{\mu} - s^{\mu}\right| \left|\pi'\left(\left[sm^p + (1-s)n^p\right]^{\frac{1}{p}}\right)\right|^q \mathrm{d}s\right]^{\frac{1}{q}}. \end{aligned} \tag{3.6}$$

Since $\left|\pi'\right|^q$ is p-convex, we get

$$\begin{aligned} &\left|\Phi_{\pi}\left(\mu, p, \lambda, a, b\right)\right| \\ &\leq a^{p-1}\left[\frac{2}{\mu+1}\left(1-\frac{1}{2^{\mu}}\right)\right]^{1-\frac{1}{q}} \left[\int_0^1 \left|(1-s)^{\mu} - s^{\mu}\right| \left[s\left|\pi'(m)\right|^q + (1-s)\left|\pi'(n)\right|^q\right] \mathrm{d}s\right]^{\frac{1}{q}} \\ &= a^{p-1}\left[\frac{2}{\mu+1}\left(1-\frac{1}{2^{\mu}}\right)\right]^{1-\frac{1}{q}} \left[\frac{1}{\mu+1}\left(1-\frac{1}{2^{\mu}}\right)\right]^{\frac{1}{q}} \left[\left|\pi'(m)\right|^q + \left|\pi'(n)\right|^q\right]^{\frac{1}{q}} \\ &= a^{p-1}\left[\frac{2}{\mu+1}\left(1-\frac{1}{2^{\mu}}\right)\right]\left[\frac{\left|\pi'(m)\right|^q + \left|\pi'(n)\right|^q}{2}\right]^{\frac{1}{q}}. \end{aligned} \tag{3.7}$$

Case 2: $0 < p < 1$. For all $0 \le s \le 1$, we deduce that

$$\left(s\left(\lambda a^p + (1-\lambda)b^p\right) + (1-s)\left(\lambda b^p + (1-\lambda)a^p\right)\right)^{\frac{1}{p}-1} \le \left(b^p\right)^{\frac{1}{p}-1} = b^{1-p}. \quad (3.8)$$

Using above process with relation inequality (3.8), we obtain

$$\begin{aligned}
&\left|\Phi_\pi(\mu, p, \lambda, a, b)\right| \\
&\le b^{p-1}\left[\frac{2}{\mu+1}\left(1-\frac{1}{2^\mu}\right)\right]^{1-\frac{1}{q}}\left[\int_0^1\left|(1-s)^\mu - s^\mu\right|\left[s\left|\pi'(m)\right|^q + (1-s)\left|\pi'(n)\right|^q\right]\mathrm{d}s\right]^{\frac{1}{q}} \\
&= b^{p-1}\left[\frac{2}{\mu+1}\left(1-\frac{1}{2^\mu}\right)\right]^{1-\frac{1}{q}}\left[\frac{1}{\mu+1}\left(1-\frac{1}{2^\mu}\right)\right]^{\frac{1}{q}}\left[\left|\pi'(m)\right|^q + \left|\pi'(n)\right|^q\right]^{\frac{1}{q}} \\
&= b^{p-1}\left[\frac{2}{\mu+1}\left(1-\frac{1}{2^\mu}\right)\right]\cdot\left[\frac{\left|\pi'(m)\right|^q + \left|\pi'(n)\right|^q}{2}\right]^{\frac{1}{q}}.
\end{aligned} \quad (3.9)$$

Thus, the proof is completed.

Corollary 3.1 *In Theorem* 3.1, *if we take* $q = 1$, *then we have*

$$\left|\Phi_\pi(\mu, p, \lambda, a, b)\right| \le \eta^{1-p}\left[\frac{1}{\mu+1}\left(1-\frac{1}{2^\mu}\right)\right]\left[\left|\pi'(m)\right| + \left|\pi'(n)\right|\right], \quad (3.10)$$

where η *is define by* (3.2).

Corollary 3.2 *In Theorem* 3.1, *if we choose* $p = 1$, *then we obtain Theorem* 2.3 *in* [19].

Corollary 3.3 *In Theorem* 3.1, *if we put* $\lambda = 0$, *then we get*

$$\begin{aligned}
&-\frac{p\left[\pi(a)+\pi(b)\right]}{b^p - a^p} + \frac{\Gamma(1+\mu)p^{\mu+1}}{\left(b^p - a^p\right)^{\mu+1}}\left[{}^pI^\mu_{a^+}\pi(b) + {}^pI^\mu_{b^-}\pi(a)\right] \\
&\le \eta^{p-1}\left[\frac{2}{\mu+1}\left(1-\frac{1}{2^\mu}\right)\right]\left[\frac{\left|\pi'(a)\right|^q + \left|\pi'(b)\right|^q}{2}\right]^{\frac{1}{q}},
\end{aligned} \quad (3.11)$$

where η *is defined by* (3.2).

Now, we prepare to introduce the second theorem as follows.

Theorem 3.2 *Let* $p, \mu > 0$ *and* $\pi : \left[a^p, b^p\right] \subset (0, \infty) \to \mathbb{R}$ *be a differentiable mapping on* $\left(a^p, b^p\right)$ *with* $0 < a < b$ *such that* $\pi' \in L^1([a, b])$. *If* $\left|\pi'\right|^q$ *is p-convex on* $\left[a^p, b^p\right]$ *for* $q > 1$ *and give constant* $r > 1$ *such that* $\frac{1}{r} + \frac{1}{q} = 1$, *then the following inequality holds*:

$$\left|\Phi_\pi(\mu, p, \lambda, a, b)\right| \le \eta^{1-p}\left[\frac{2}{r\mu+1}\left(1-\frac{1}{2^{\mu r}}\right)\right]^{\frac{1}{r}}\left[\frac{\left|\pi'(m)\right|^q + \left|\pi'(n)\right|^q}{2}\right]^{\frac{1}{q}}. \quad (3.12)$$

where η *is defined by* (3.2) *and* $\lambda \in (0,1] \setminus \frac{1}{2}$.

Proof. Case 1: $p \ge 1$. Continuing from inequality (3.5), and using Hölder inequality, we obtain

$$\begin{aligned}
&\left|\Phi_\pi(\mu,p,\lambda,a,b)\right| \\
&\le a^{1-p}\left[\int_0^1\left|(1-s)^\mu-s^\mu\right|\left|\pi'\left(\left[sm^p+(1-s)n^p\right]^{\frac{1}{p}}\right)\right|\mathrm{d}s\right] \\
&\le a^{1-p}\left[\int_0^1\left|(1-s)^\mu-s^\mu\right|^r\mathrm{d}s\right]^{\frac{1}{r}}\left[\int_0^1\left|\pi'\left(\left[sm^p+(1-s)n^p\right]^{\frac{1}{p}}\right)\right|^q\mathrm{d}s\right]^{\frac{1}{q}} \qquad (3.13)\\
&= a^{1-p}\left(\int_0^{\frac{1}{2}}\left[(1-s)^\mu-s^\mu\right]^r\mathrm{d}s+\int_{\frac{1}{2}}^1\left[s^\mu-(1-s)^\mu\right]^r\mathrm{d}s\right)^{\frac{1}{r}} \\
&\quad\cdot\left[\int_0^1\left|\pi'\left(\left[sm^p+(1-s)n^p\right]^{\frac{1}{p}}\right)\right|^q\mathrm{d}s\right]^{\frac{1}{q}}.
\end{aligned}$$

Since $|\pi'|^q$ is *p*-convex, we get

$$\begin{aligned}
&\int_0^1\left|\pi'\left(\left[sm^p+(1-s)n^p\right]^{\frac{1}{p}}\right)\right|^q\mathrm{d}s \\
&\le\int_0^1\left[s\left|\pi'(m)\right|^q+(1-s)\left|\pi'(n)\right|^q\right]\mathrm{d}s \qquad (3.14)\\
&=\frac{\left|\pi'(m)\right|^q+\left|\pi'(n)\right|^q}{2}.
\end{aligned}$$

By calculation, we have

$$\left(\int_0^{\frac{1}{2}}\left[(1-s)^\mu-s^\mu\right]^r\mathrm{d}s+\int_{\frac{1}{2}}^1\left[s^\mu-(1-s)^\mu\right]^r\mathrm{d}s\right)^{\frac{1}{r}}=\left[\frac{2}{r\mu+1}\left(1-\frac{1}{2^{\mu r}}\right)\right]^{\frac{1}{r}}. \qquad (3.15)$$

Using inequality (3.14) and (3.15) in inequality (3.13), we deduce

$$\left|\Phi_\pi(\mu,p,\lambda,a,b)\right|\le a^{1-p}\left[\frac{2}{r\mu+1}\left(1-\frac{1}{2^{\mu r}}\right)\right]^{\frac{1}{r}}\left[\frac{\left|\pi'(m)\right|^q+\left|\pi'(n)\right|^q}{2}\right]^{\frac{1}{q}}. \qquad (3.16)$$

Case 2: $0<p<1$. By utilizing the above process with relation inequality (3.8), we obtain

$$\begin{aligned}
&\left|\Phi_\pi(\mu,p,\lambda,a,b)\right| \\
&\le b^{1-p}\left[\int_0^1\left|(1-s)^\mu-s^\mu\right|^r\mathrm{d}s\right]^{\frac{1}{r}}\left[\int_0^1\left|\pi'\left(\left[sm^p+(1-s)n^p\right]^{\frac{1}{p}}\right)\right|^q\mathrm{d}s\right]^{\frac{1}{q}} \qquad (3.17)\\
&\le b^{1-p}\left[\frac{2}{r\mu+1}\left(1-\frac{1}{2^{\mu r}}\right)\right]^{\frac{1}{r}}\left[\frac{\left|\pi'(m)\right|^q+\left|\pi'(n)\right|^q}{2}\right]^{\frac{1}{q}}.
\end{aligned}$$

Thus, the proof is completed.

Corollary 3.4 *In Theorem* 3.2, *if we take* $p=1$, *then we have Theorem* 2.6 *in* [19].

Next, we will use the well-known Young's inequality

$$XY \le \frac{X^r}{r} + \frac{Y^q}{q}, \forall X, Y \ge 0, r \cdot q > 1, \frac{1}{r} + \frac{1}{q} = 1 \tag{3.18}$$

in Theorem 3.2 to get the following two corollaries.

Corollary 3.5 *Under all assumptions of Theorem* 3.2, *we deduce*

$$\left|\Phi_\pi(\mu, p, \lambda, a, b)\right| \le \eta^{1-p} \left[\frac{\frac{2}{r\mu+1}\left(1 - \frac{1}{2^{\mu r}}\right)}{r} + \frac{\left|\pi'(m)\right|^q + \left|\pi'(n)\right|^q}{2q} \right], \tag{3.19}$$

where η is defined by (3.2).

Corollary 3.6 *Under the conditions of Theorem* 3.2, *if we take* $\lambda = 0$, *then we have*

$$\begin{aligned} &-\frac{p\left[\pi(a)+\pi(b)\right]}{b^p - a^p} + \frac{\Gamma(1+\mu)p^{\mu+1}}{\left(b^p - a^p\right)^{\mu+1}}\left[{}^pI_{a^+}^{\mu}\pi(b) + {}^pI_{b^-}^{\mu}\pi(a)\right] \\ &\le \eta^{1-p}\left[\frac{\frac{2}{r\mu+1}\left(1 - \frac{1}{2^{\mu r}}\right)}{r} + \frac{\left|\pi'(a)\right|^q + \left|\pi'(b)\right|^q}{2q} \right], \end{aligned} \tag{3.20}$$

where η *is defined by* (3.2).

We will apply the following special functions in the next theorem.

1) The beta function,

$$\beta(x, y) = \int_0^1 s^{x-1}(1-s)^{y-1}\,\mathrm{d}s, \quad x, y > 0. \tag{3.21}$$

2) The hypergeometric function,

$${}_2F_1(a, b; c; z) = \frac{1}{\beta(b, c-b)}\int_0^1 s^{b-1}(1-s)^{c-b-1}(1-zs)^{-a}\,\mathrm{d}s, \ c > b > 0, |z| < 1. \tag{3.22}$$

Theorem 3.3 *Let* $p, \mu > 0$ *and* $\pi : \left[a^p, b^p\right] \subset (0, \infty) \to \mathbb{R}$ *be a differentiable mapping on* $\left(a^p, b^p\right)$ *with* $0 < a < b$ *such that* $\pi' \in L^1([a, b])$. *If* $\left|\pi'\right|^q$ *is p-convex on* $\left[a^p, b^p\right]$ *for* $q > 1$ *and give constant* $r > 1$ *such that* $\frac{1}{r} + \frac{1}{q} = 1$, *then the following inequality exists*:

$$\begin{aligned} &\left|\Phi_\pi(\mu, p, \lambda, a, b)\right| \\ &\le \left[\frac{2}{r\mu+1}\left(1 - \frac{1}{2^{\mu r}}\right)\right]^{\frac{1}{r}} \left\{ \frac{m^{(1-p)q}}{2} \cdot {}_2F_1\left(q - \frac{q}{p}, 1:3; \frac{m^p - n^p}{m^p}\right)\left|\pi'(m)\right|^q \right. \\ &\left. + \frac{m^{(1-p)q}}{2}\left[2 \cdot {}_2F_1\left(q - \frac{q}{p}, 1:2; \frac{m^p - n^p}{m^p}\right) - {}_2F_1\left(q - \frac{q}{p}, 1:3; \frac{m^p - n^p}{m^p}\right)\right]\left|\pi'(n)\right|^q \right\}^{\frac{1}{q}}. \end{aligned} \tag{3.23}$$

Proof. Applying Lemma 2.1, Hölder inequality and the *p*-convexity of $\left|\pi'\right|^q$, one has

$$
\begin{aligned}
&\left|\Phi_{\pi}(\mu,p,\lambda,a,b)\right| \\
&\le \int_0^1 \left|(1-s)^{\mu}-s^{\mu}\right|\left|\left(sm^p+(1-s)n^p\right)^{\frac{1}{p}-1}\right|\left|\pi'\left(\left[sm^p+(1-s)n^p\right]^{\frac{1}{p}}\right)\mathrm{d}s\right| \\
&\le \left[\int_0^1 \left|(1-s)^{\mu}-s^{\mu}\right|^r \mathrm{d}s\right]^{\frac{1}{r}}\left[\int_0^1 \left(sm^p+(1-s)n^p\right)^{\left(\frac{1}{p}-1\right)q}\left|\pi'\left(\left[sm^p+(1-s)n^p\right]^{\frac{1}{p}}\right)\right|^q \mathrm{d}s\right]^{\frac{1}{q}} \\
&\le \left[\frac{2}{r\mu+1}\left(1-\frac{1}{2^{\mu r}}\right)\right]^{\frac{1}{r}}\left[\int_0^1 \left(sm^p+(1-s)n^p\right)^{\left(\frac{1}{p}-1\right)q}\left[s\left|\pi'(m)\right|^q+(1-s)\left|\pi'(n)\right|^q\right]\mathrm{d}s\right]^{\frac{1}{q}} \\
&= \left[\frac{2}{r\mu+1}\left(1-\frac{1}{2^{\mu r}}\right)\right]^{\frac{1}{r}}\left\{\left|\pi'(m)\right|^q \int_0^1 s\left(sm^p+(1-s)n^p\right)^{\left(\frac{1}{p}-1\right)q}\mathrm{d}s\right. \\
&\quad \left.+\left|\pi'(n)\right|^q \int_0^1 (1-s)\left(sm^p+(1-s)n^p\right)^{\left(\frac{1}{p}-1\right)q}\mathrm{d}s\right\}^{\frac{1}{q}}.
\end{aligned}
\tag{3.24}
$$

By calculation, we obtain

$$
\int_0^1 s\left(sm^p+(1-s)n^p\right)^{\left(\frac{1}{p}-1\right)q}\mathrm{d}s=\frac{m^{(1-p)q}}{2}\cdot {}_2F_1\left(q-\frac{q}{p},1;3;\frac{m^p-n^p}{m^p}\right) \tag{3.25}
$$

and

$$
\begin{aligned}
&\int_0^1 (1-s)\left(sm^p+(1-s)n^p\right)^{\left(\frac{1}{p}-1\right)q}\mathrm{d}s \\
&=\frac{m^{(1-p)q}}{2}\left[2\cdot {}_2F_1\left(q-\frac{q}{p},1;2;\frac{m^p-n^p}{m^p}\right)-{}_2F_1\left(q-\frac{q}{p},1;3;\frac{m^p-n^p}{m^p}\right)\right].
\end{aligned}
\tag{3.26}
$$

Utilizing inequality (3.25) and (3.26) in inequality (3.24), we can obtain desired inequality (3.23). The proof is completed.

Corollary 3.7 *Under the conditions of Theorem* 3.3, *if we take* $\lambda=0$, *then we obtain*

$$
\begin{aligned}
&-\frac{p\left[\pi(a)+\pi(b)\right]}{b^p-a^p}+\frac{\Gamma(1+\mu)p^{\mu+1}}{\left(b^p-a^p\right)^{\mu+1}}\left[{}^pI_{a^+}^{\mu}\pi(b)+{}^pI_{b^-}^{\mu}\pi(a)\right] \\
&\le \left[\frac{2}{r\mu+1}\left(1-\frac{1}{2^{\mu r}}\right)\right]^{\frac{1}{r}}\left\{\frac{b^{(1-p)q}}{2}\cdot {}_2F_1\left(q-\frac{q}{p},1:3;\frac{b^p-a^p}{b^p}\right)\left|\pi'(b)\right|^q\right. \\
&\quad \left.+\frac{b^{(1-p)q}}{2}\left[2\cdot {}_2F_1\left(q-\frac{q}{p},1:2;\frac{b^p-n^p}{a^p}\right)-{}_2F_1\left(q-\frac{q}{p},1:3;\frac{b^p-a^p}{b^p}\right)\right]\left|\pi'(a)\right|^q\right\}^{\frac{1}{q}}.
\end{aligned}
\tag{3.27}
$$

4. Examples

In this part, we obtain two examples to illustrate our main results.

Example 4.1 *Let* $\pi(x)=\frac{1}{2}x^2$, $x\in(0,\infty)$, *then* $\left|\pi'(x)\right|=x$ *is p-convex for*

$p<1$. *Choosing* $\lambda=0$, $\mu=1.5$, $p=0.5$, $q=1$, $a=4$ *and* $b=9$, *then all the assumptions in Theorem* 3.1 *are satisfied. We have*

$$\begin{aligned}
&\left|\Phi_{\pi}(\mu,p,\lambda,a,b)\right| \\
&=-\frac{p\left[\pi(n)+\pi(m)\right]}{(1-2\lambda)\left(b^{p}-a^{p}\right)}+\frac{\Gamma(1+\mu)\,p^{\mu+1}}{(1-2\lambda)^{\mu+1}\left(b^{p}-a^{p}\right)^{\mu+1}}\left[{}^{p}I^{\mu}_{n^{+}}\pi(m)+{}^{p}I^{\mu}_{m^{-}}\pi(n)\right] \\
&=\left|-0.5\left[\pi(4)+\pi(9)\right]+\Gamma(2.5)0.5^{2.5}\left[{}^{0.5}I^{1.5}_{4^{+}}\pi(9)+{}^{0.5}I^{1.5}_{9^{-}}\pi(4)\right]\right|\approx 8.9344 \quad (4.1) \\
&\le b^{1-p}\left[\frac{2}{\mu+1}\left(1-\frac{1}{2^{\mu}}\right)\right]\left[\frac{\left|\pi'(m)\right|^{q}+\left|\pi'(n)\right|^{q}}{2}\right]^{\frac{1}{q}} \\
&=9^{(1-0.5)}\frac{2}{1.5+1}\left(1-\frac{1}{2^{1.5}}\right)\left(\frac{\pi'(4)+\pi'(9)}{2}\right)\approx 10.0846.
\end{aligned}$$

This proves that the described result in Theorem 3.1 *is correct.*

Example 4.2 *Let* $\pi(x)=\dfrac{1}{1-p}x^{1-p}$, $x\in(0,\infty)$, *then* $\left|\pi'(x)\right|=x^{-p}$ *is p-convex for* $p>1$. *Taking* $\lambda=0$, $\mu=1.5$, $p=2$, $q=1$, $a=1$ *and* $b=3$, *then all the assumptions in Theorem* 3.1 *are satisfied. We have*

$$\begin{aligned}
&\left|\Phi_{\pi}(\mu,p,\lambda,a,b)\right| \\
&=-\frac{p\left[\pi(n)+\pi(m)\right]}{(1-2\lambda)\left(b^{p}-a^{p}\right)}+\frac{\Gamma(1+\mu)\,p^{\mu+1}}{(1-2\lambda)^{\mu+1}\left(b^{p}-a^{p}\right)^{\mu+1}}\left[{}^{p}I^{\mu}_{n^{+}}\pi(m)+{}^{p}I^{\mu}_{m^{-}}\pi(n)\right] \\
&=\left|-\frac{2\left[\pi(1)+\pi(3)\right]}{8}+\frac{\Gamma(2.5)2^{2.5}}{8^{2.5}}\left[{}^{2}I^{1.5}_{1^{+}}\pi(3)+{}^{2}I^{1.5}_{3^{-}}\pi(1)\right]\right|\approx 0.0672 \quad (4.2) \\
&\le a^{1-p}\left[\frac{2}{\mu+1}\left(1-\frac{1}{2^{\mu}}\right)\right]\left[\frac{\left|\pi'(m)\right|^{q}+\left|\pi'(n)\right|^{q}}{2}\right]^{\frac{1}{q}} \\
&=1^{(1-2)}\frac{2}{1.5+1}\left(1-\frac{1}{2^{1.5}}\right)\left(\frac{\pi'(1)+\pi'(3)}{2}\right)\approx 0.17146
\end{aligned}$$

This proves that the described result in Theorem 3.1 *is correct.*

5. Conclusion

In this paper, we assume that the absolute value of the derivative of the considered function π' is *p*-convex to obtain some inequalities for Katugampola fractional integrals. More new results can be derived by taking special parameter values for λ, μ and *p*. We emphasize that certain results proved in this article generalize the ones obtained by Sarikaya and Budak [19] and Topul *et al.* [11].

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] İşcan, İ. (2016) Ostrowski Type Inequalities for p-Convex Functions. *New Trends in Mathematical Sciences*, **4**, 140-150. https://doi.org/10.20852/ntmsci.2016318838

[2] Zhang, K.S. and Wan, J.P. (2017) p-Convex Functions and Their Properties. *Pure and Applied Mathematics*, **23**, 130-133.

[3] Kunt, M. and İşcan, İ. (2017) Hermite-Hadamard-Fejér Type Inequalities for p-Convex Functions. *Arab Journal of Mathematical Sciences*, **23**, 215-230. https://doi.org/10.1016/j.ajmsc.2016.11.001

[4] Dragomir, S.S., Latif, M.A. and Momoniat, E. (2020) Improved Hermit-Hadamard type Inequalities by Using the p-Convexity of Differentiable Mappings. *Quaestions Mathematicae*, in press. https://doi.org/10.2989/16073606.2020.1768170

[5] Mehreen, N. and Anwar, M. (2019) Hermite-Hadamard Type Inequalities for Exponentially p-Convex Functions and Expenentially s-Convex Functions in the Second Sense with Applications. *Journal of Inequalities and Applications*, **2019**, Article No. 92. https://doi.org/10.1186/s13660-019-2047-1

[6] Latif, M.A. and Hussain, S. (2019) Some Generalized Hermite-Hadamard and Simpson Type Inequalities by Using the p-Convexity of Differentiable Mappings. *Communications Faculty of Sciences University of Ankara Series A*1 *Mathematics and Statistics*, **68**, 1543-1555. https://doi.org/10.31801/cfsuasmas.422991

[7] Salaş, S., Erdaş, Y., Toplu, T. and Set, E. (2019) On Some Generalized Fractional Integral Inequalities for p-Convex Functions. *Fractal and Fractional*, **3**, 29. https://doi.org/10.3390/fractalfract3020029

[8] Katugampola, U.N. (2011) New Approach to a Generalized Fractional Integral. *Applied Mathematics and Computation*, **218**, 860-865. https://doi.org/10.1016/j.amc.2011.03.062

[9] Chen, H. and Katugampola, U.N. (2017) Hermite-Hadamard and Hermite-Hadamard-Fejér Type Inequalities for Generalized Fractional Integrals. *Journal of Mathematical Analysis and Applications*, **446**, 1274-1291. https://doi.org/10.1016/j.jmaa.2016.09.018

[10] Delavar, M.R. and Dragomir, S.S. (2020) Hermite-Hadamard's Mid-Point Type Inequalities for Generalized Fractional Integrals. *RACSAM*, **114**, Article No. 73. https://doi.org/10.1007/s13398-020-00795-6

[11] Toplu, T., Set, E., İşcan, İ. and Maden, S. (2019) Hermit-Hadamard Type Inequalities for p-Convex Functions via Katugampola Fractional Integrals. *Facta Universitatis*, *Series*: *Mathematics and Informatics*, **34**, 149-164.

[12] Kermausuor, S., Nwaeze, E.R. and Tameru, A.M. (2019) New Integral Inequalities via the Katugampola Fractional Integrals for Functions Whose Second Derivatives Are Strongly η-Convex. *Mathematics*, **7**, 1-14. https://doi.org/10.3390/math7020183

[13] Yaldiz, H. and Akdemir, A.O. (2018) Katugamola Fractional Integrals within the Class of Convex Functions. *Turkish Journal of Science*, **3**, 40-50.

[14] Farid, G. Katugampola, U.N. and Usman, M. (2017) Ostrowski Type Fractional Integral Inequalities for s-Godunova-Levin Functions via Katugampola Fractional

Integrals. *Open Journal of Mathematical Science*, **1**, 97-110. https://doi.org/10.30538/oms2017.0010

[15] Gürbüz, M., Taşdan, Y. and Set, E. (2019) Ostrowski Type Inequalities via the Katugamploa Fractional Integrals. *AIMS Mathematics*, **5**, 42-53. https://doi.org/10.3934/math.2020004

[16] Katugamploa, U.N. (2014) New Approach to Generalized Fractional Derivatives. *Bulletin of Mathematical Analysis and Applications*, **6**, 1-15.

[17] Kermausuor, S. (2019) Generalized Ostrowski-Type Inequalities Involving Second Derivatives via the Katugampola Fractional Integrals. *Journal of Nonlinear Sciences and Applications*, **12**, 509-522. https://doi.org/10.22436/jnsa.012.08.02

[18] Hu, G., Lei, H. and Du, T.S. (2020) Some Parameterized Integral Inequalities for *p*-Convex Mappings via the Right Katugampola Fractional Integrals. *AIMS Mathematics*, **5**, 1425-1445. https://doi.org/10.3934/math.2020098

[19] Sarikaya, M.Z. and Budak, H. (2016) Generalized Hermite-Hadamard Type Intergal Inequalities for Fractional Integrals. *Format*, **30**, 1315-1326. https://doi.org/10.2298/FIL1605315S