Engineering, 2020, 12, 652-665
https://www.scirp.org/journal/eng
ISSN Online: 1947-394X
ISSN Print: 1947-3931

Output Feedback Regulation for 1-D Anti-Stable Wave Equation with External System Disturbance

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How to cite this paper: Li, Z.Y. (2020) Output Feedback Regulation for 1-D Anti-Stable Wave Equation with External System Disturbance. *Engineering*, **12**, 652-665.
https://doi.org/10.4236/eng.2020.129046

Received: September 1, 2020
Accepted: September 19, 2020
Published: September 22, 2020

Abstract

This paper has studied the output feedback regulation problem for 1-D anti-stable wave equation with distributed disturbance and a given reference signal generated by a finite-dimensional exosystem. We first design an observer for both exosystem and auxiliary PDE system to recover the state. Then we show the well-posedness of the regulator equations and propose an observer-based feedback control law to regulate the tracking error to zero exponentially and keep all the states bounded.

Keywords

Anti-Stable Wave Equation, Distributed Disturbance, Exosystem, Output Feedback Regulation

1. Introduction

Output feedback regulation is a classical topic of control theory and engineering practice. A feedback regulator is designed for the controlled system, so that the signal to be regulated can track the target reference signal and the system keeps stable. Many practical problems such as aircraft landing, missile tracking and robot control all depend on output regulation. The wave equation with anti-damping term can simulate many engineering problems like pipeline combustion, acoustic instability or stick slip instability during drilling, which is of great significance to study the output regulation of anti-stable wave model.

In the last few years, a quiet great progress has been made both in the output feedback stabilization [1]-[7] and the output feedback regulation [8]-[13]. [13] realizes the output tracking and disturbance rejection of a 1-D anti-stable wave system with general boundary disturbance collocated with control by proportional control, in which the external disturbance of the controlled system is at the same end as the control input. Output regulation of 1-D wave equation with both internal and external uncertainties is considered in [12] and finally achieves exponential tracking. In [9], the tracking problem of coupled wave equations with external disturbance is solved through the backstepping method. F.F. Jin and B.Z. Guo studies the output tracking problem of 1-D anti-stable wave equation with disturbance generated by external system in [11] via the reversible backstepping transformation. In [1] [2], a transport equation is introduced to deal with the anti-damping term on the boundary. Inspired by this, we further study the output feedback regulation problem of anti-stable wave equation by this method.

In this paper, we focus on the output tracking for 1-D anti-stable wave system with in-domain disturbance generated by an exosystem described by

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t) + f(x)d(t), x \in (0,1), t > 0, \\ w_x(0,t) = -qw_t(0,t), t \geq 0, \\ w_x(1,t) = U(t), t \geq 0, \\ w(x,0) = w_0(x), w_t(x,0) = w_1(x), x \in [0,1], \end{cases} \tag{1.1}$$

where $w(0,t)$ is the output to be regulated, the displacement $w(0,t)$ and its derivative $w_t(0,t-\tau), \tau \in [0,1]$ with time-delay are the measured output. $U(t)$ represents the input (control), $q > 0 (\neq 1)$ an unknown constant parameter ($q \neq 1$ is to avoid the real part of the plant eigenvalues tending to positive infinity). $(w_0(x), w_1(x))$ is initial condition, $r(t)$ is a given reference signal, $f(x) \in C[0,1]$ represents the unknown intensity of the distributed external disturbance $d(t)$. $d(t), r(t)$ are generated by following exosystem

$$\begin{cases} \dot{v}(t) = Sv(t), t > 0, \\ d(t) = P_1 v(t), t \geq 0, \\ r(t) = P_2 v(t), t \geq 0, \\ v(0) = v_0 \in C^n, \end{cases} \tag{1.2}$$

where S is a diagonalizable matrix with all eigenvalues on the imaginary axis. For design purpose, we suppose that the initial value v_0 is unknown and so do the state $v(t)$. Rewrite S as (S_1, S_2) and $d(t), r(t)$ can be written as

$$\begin{cases} \dot{v}_1(t) = S_1 v_1(t), t > 0, \\ \dot{v}_2(t) = S_2 v_2(t), t > 0, \\ d(t) = Q_1 v_1(t), t \geq 0, \\ r(t) = Q_2 v_2(t), t \geq 0, \\ v_1(0) = v_{10} \in C^{n_1}, v_2(0) = v_{20} \in C^{n_2}. \end{cases} \tag{1.3}$$

Here $r(t)$ is known but $d(t)$ is unknown due to the uncertainty of v_{10}. We have $n_1 + n_2 = n$ and $v = \begin{pmatrix} v_1^{\mathrm{T}} & v_2^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}}$, and assume that the eigenvalues of matrix S_1 are distinct and $\sum\left(Q_2^{\mathrm{T}}, S_2\right)$ is observable. The objective of this paper is to design an observer-based output feedback regulator for system (1.1) to regulate the tracking error $e(t)=w(0,t)-r(t)$ to zero and simultaneously keep all the states bounded. The advanced nature of our result lies in that the measured output is at the left end and may admit time-delay, which makes the regulation problem of (1.1) challenging.

The rest of the paper is organized as follows. In Section 2, an auxiliary stable system is constructed by introducing a transport equation and a regulator equation, and its observer is derived. We propose the output feedback control law for the auxiliary system and obtain the closed-loop system in Section 3. The main results are presented in Section 4 and Section 5 concludes this paper.

2. State Observer Design

In order to deal with the anti-damping $-qw_t(0,t)$ in system (1.1), we introduce the following transport equation in [1] [2]

$$\begin{cases} \delta_t(x,t)=-\delta_x(x,t), x\in(0,1), t>0, \\ \delta(0,t)=-\dfrac{q+c_0}{1+c_0}w(0,t), t\geq 0, \\ \delta(x,0)=\delta_0(x), x\in[0,1], \end{cases} \tag{2.1}$$

where δ_0 is the initial value, $c_0>0$ a tuning parameter.

In the rest of this paper, we omit the obvious domain $x\in[0,1]$ and $t\in[0,+\infty)$ when there is no confusion.

Let

$$\begin{cases} v_1(t)=v_1(t), v_2(t)=v_2(t), \\ u(x,t)=w(x,t)+\delta(x,t), \end{cases} \tag{2.2}$$

then $u(x,t)$ is governed by

$$\begin{cases} \dot{v}_1(t)=S_1v_1(t), t>0, \\ \dot{v}_2(t)=S_2v_2(t), t>0, \\ u_{tt}(x,t)=u_{xx}(x,t)+f(x)Q_1v_1(t), \\ u_x(0,t)=c_0u_t(0,t), \\ u_x(1,t)=U(t)+\delta_x(1,t), \\ u(x,0)=u_0(x)=w_0(x)+\delta_0(x), \\ u_t(x,0)=u_1(x)=w_1(x)-\delta_0'(x). \end{cases} \tag{2.3}$$

Notice that the parameter $-q>0$ in system (1.1) becomes $c_0>0$ in system (2.3). Moreover, we have

$$u(0,t)=\frac{1-q}{1+c_0}w(0,t).$$

To recover the state of (2.3), we now design an observer for (2.3) using the known signal $w(0,t)$ and $w_t(0,t-\tau), \tau\in[0,1]$ as

$$
\begin{cases}
\dot{\hat{v}}_1(t) = S_1\hat{v}_1(t) + K_1\left[\hat{u}(0,t) - \dfrac{1-q}{1+c_0}w(0,t)\right], t > 0, \\
\dot{\hat{v}}_2(t) = S_2\hat{v}_2(t) + K_2\left[Q_2\hat{v}_2(t) - r(t)\right], t > 0, \\
\hat{u}_{tt}(x,t) = \hat{u}_{xx}(x,t) + r_1(x)\left[\hat{u}(0,t) - \dfrac{1-q}{1+c_0}w(0,t)\right] \\
\qquad + f(x)Q_1\hat{v}_1(t) + r_2(x)\left[\hat{u}_t(0,t) - \dfrac{1-q}{1+c_0}w_t(0,t)\right], \\
\hat{u}_x(0,t) = c_0\hat{u}_t(0,t) + (c_1 + c_0PK_1)\left[\hat{u}(0,t) - \dfrac{1-q}{1+c_0}w(0,t)\right], \\
\hat{u}_x(1,t) = U(t) + \dfrac{q+c_0}{1+c_0}w_t(0,t-1), \\
\hat{v}_1(0) = \hat{v}_{10} \in C^{n_1}, \hat{v}_2(0) = \hat{v}_{20} \in C^{n_2}, \\
\hat{u}(x,0) = \hat{u}_0(x), \hat{u}_t(x,0) = \hat{u}_1(x),
\end{cases}
\tag{2.4}
$$

where $c_1 > 0$ is a constant, and $r_1(x), r_2(x) \in C[0,1]$, K_1, K_2 are row vectors where K_2 is designed to make the matrix $S_2 + K_2Q_2$ Hurwitz and P, K_1 to be determined later.

Let $\tilde{u} = u - \hat{u}, \tilde{v}_1 = v_1 - \hat{v}_1, \tilde{v}_2 = v_2 - \hat{v}_2$, then we have

$$
\begin{cases}
\dot{\tilde{v}}_1(t) = S_1\tilde{v}_1(t) + K_1\tilde{u}(0,t), t > 0, \\
\dot{\tilde{v}}_2(t) = (S_2 + K_2Q_2)\tilde{v}_2(t), t > 0, \\
\tilde{u}_{tt}(x,t) = \tilde{u}_{xx}(x,t) + f(x)Q_1\tilde{v}_1(t) + r_1(x)\tilde{u}(0,t) + r_2(x)\tilde{u}_t(0,t), \\
\tilde{u}_x(0,t) = c_0\tilde{u}_t(0,t) + (c_1 + c_0PK_1)\tilde{u}(0,t), \\
\tilde{u}_x(1,t) = 0, \\
\tilde{v}_1(0) = \tilde{v}_{10} \in C^{n_1}, \tilde{v}_2(0) = \tilde{v}_{20} \in C^{n_2}, \\
\tilde{u}(x,0) = \tilde{u}_0(x), \tilde{u}_t(x,0) = \tilde{u}_1(x),
\end{cases}
\tag{2.5}
$$

Construct the following transformation

$$
\begin{cases}
\tilde{v}_1(t) = \tilde{v}_1(t), \tilde{v}_2(t) = \tilde{v}_2(t), \\
e(x,t) = \tilde{u}(x,t) + g(x)\tilde{v}_1(t),
\end{cases}
\tag{2.6}
$$

then the observer error system is found to be

$$
\begin{cases}
\dot{\tilde{v}}_1(t) = (S_1 - K_1g(0))\tilde{v}_1(t) + K_1e(0,t), t > 0, \\
\dot{\tilde{v}}_2(t) = (S_2 + K_2Q_2)\tilde{v}_2(t), t > 0, \\
e_{tt}(x,t) = e_{xx}(x,t), \\
e_x(0,t) = c_0e_t(0,t) + c_1e(0,t), \\
e_x(1,t) = 0, \\
\tilde{v}_1(0) = \tilde{v}_{10} \in C^{n_1}, \tilde{v}_2(0) = \tilde{v}_{20} \in C^{n_2}, \\
e(x,0) = e_0(x), e_t(x,0) = e_1(x),
\end{cases}
\tag{2.7}
$$

where $g(x)$ satisfies the boundary value problem (regulator equation) as

$$\begin{cases} \dfrac{\mathrm{d}^2 g(x)}{\mathrm{d}x^2} = g(x)S_1^2 + f(x)Q_1, \\ \left.\dfrac{\mathrm{d}g(x)}{\mathrm{d}x}\right|_{x=0} = c_0 g(0)S_1 + c_1 g(0), \\ \left.\dfrac{\mathrm{d}g(x)}{\mathrm{d}x}\right|_{x=1} = 0. \end{cases} \tag{2.8}$$

We make $P = g(0), r_1(x) = -g(x)S_1K_1, r_2(x) = -g(x)K_1$.

Lemma 2.1: Assume that S_1 is a diagonalizable matrix. Then the regulator equation (2.8) admits a unique solution.

Considering our previous assumptions that the matrix $S_2 + K_2Q_2$ is Hurwitz and $c_1 > 0$, the PDE-part of system (2.7) is exponentially stable. (2.7) will be exponentially stable if we can show that $S_1 - K_1 g(0)$ is also Hurwitz.

Lemma 2.2: Define a function $N_1^{\mathrm{T}}(s) = -\int_0^1 \cosh(s(1-y)) f(y)\mathrm{d}y$ and ξ_i be the eigenvector of matrix S_1 corresponding to the eigenvalue λ_i of S_1. Then $(g(0), S_1)$ is observable if and only if $N_1^{\mathrm{T}}(\lambda_i)Q_1\xi_i \neq 0, (i = 1, 2, \cdots, n_1)$.

Lemma 2.1 and 2.2 are similar to Lemma 3.1 and 3.2 in [11] respectively, and we omit the details of proof here. Suppose that $\mu_i, (i = 1, 2, \cdots, n_1)$ satisfies the conditions of lemma 2.2, then $(g(0), S_1)$ is observable and consequently K_1 can be identified to make the matrix $S_1 - K_1 g(0)$ Hurwitz.

Theorem 2.1: Define a function $N_1^{\mathrm{T}}(s) = -\int_0^1 \cosh(s(1-y)) f(y)\mathrm{d}y$ satisfies $N_1^{\mathrm{T}}(\lambda_i)Q_1\xi_i \neq 0, (i = 1, 2, \cdots, n_1)$ for all eigen-pairs (λ_i, ξ_i) of S_1. $c_0, c_1 > 0$ are constants, $g(x)$ is the unique solution to (2.8). We make $P = g(0)$ and $r_1(x) = -g(x)S_1K_1, r_2(x) = -g(x)_1 K_1$, matrix $S_1 - K_1 g(0)$ is Hurwitz. Then system (2.7) is well-posed and exponentially stable.

Proof:

We divide system (2.7) into PDE-part and ODE-part, and consider the stability of the solution respectively.

The PDE-part of (2.7) is

$$\begin{cases} e_{tt}(x,t) = e_{xx}(x,t), \\ e_x(0,t) = c_0 e_t(0,t) + c_1 e(0,t), \\ e_x(1,t) = 0, \\ e(x,0) = e_0(x), e_t(x,0) = e_1(x), \end{cases} \tag{2.9}$$

Define $H_0 = H^1(0,1) \times L^2(0,1)$ to be an usual Hilbert space with the following norm induced by the inner product

$$\left\|(f,g)^{\mathrm{T}}\right\|_{H_0}^2 = \int_0^1 \left(\left|f'(x)\right|^2 \mathrm{d}x + \left|g(x)\right|^2\right)\mathrm{d}x + c_0 \left|f(0)\right|^2, \forall (f,g) \in H_0.$$

And $H = C^{n_1} \times C^{n_2} \times H_0$. We can directly come to the conclusion that (2.9) is exponentially stable from [7]. In other words, there exist two constants $M_0, w_0 > 0$ such that

$$\left\|\left(e(\cdot,t), e_t(\cdot,t)\right)^{\mathrm{T}}\right\|_{H_0} \le M_0 \mathrm{e}^{-w_0 t} \left\|\left(e_0(\cdot), e_1(\cdot)\right)^{\mathrm{T}}\right\|_{H_0}, t > 0, \tag{2.10}$$

which implies that

$$\left|e(0,t)\right| \le M_0 \mathrm{e}^{-w_0 t} \left\|\left(e_0(\cdot), e_1(\cdot)\right)^{\mathrm{T}}\right\|_{H_0}, t>0.$$

The ODE-part of (2.7) is governed by

$$\begin{cases} \dot{\tilde{v}}_1(t) = \left(S_1 - K_1 g(0)\right)\tilde{v}_1(t) + K_1 e(0,t), t>0, \\ \dot{\tilde{v}}_2(t) = \left(S_2 + K_2 Q_2\right)\tilde{v}_2(t), t>0, \\ \tilde{v}_1(0) = \tilde{v}_{10} \in C^{n_1}, \tilde{v}_2(0) = \tilde{v}_{20} \in C^{n_2}. \end{cases} \tag{2.11}$$

There are some positive constants $M_j, w_j > 0, (j=1,2,3)$ such that the solution $\tilde{v}_1(t), \tilde{v}_2(t)$ of (2.11) has the estimation as

$$\begin{cases} \left\|\tilde{v}_1(t)\right\| = \left\|\mathrm{e}^{[S_1 - K_1 g(0)]t}\tilde{v}_1(0) + \int_0^t \mathrm{e}^{[S_1 - K_1 g(0)](t-s)} K_1 e(0,s)\mathrm{d}s\right\|_{C^{n_1}} \\ \qquad \le \left\|\mathrm{e}^{[S_1 - K_1 g(0)]t}\tilde{v}_1(0)\right\|_{C^{n_1}} + \left\|\int_0^t \mathrm{e}^{[S_1 - K_1 g(0)](t-s)} K_1 e(0,s)\mathrm{d}s\right\|_{C^{n_1}} \\ \qquad \le M_1 \mathrm{e}^{-w_1 t}\left\|\tilde{v}_1(0)\right\|_{C^{n_1}} + M_2 \mathrm{e}^{-w_2 t}\left\|\left(e_0(\cdot), e_1(\cdot)\right)^{\mathrm{T}}\right\|_{C^{n_1}}, \\ \left\|\tilde{v}_2(t)\right\| = \left\|\mathrm{e}^{(S_2 + K_2 Q_2)t}\tilde{v}_2(0)\right\|_{C^{n_2}} \le M_3 \mathrm{e}^{-w_3 t}\left\|\tilde{v}_2(0)\right\|_{C^{n_2}}. \end{cases} \tag{2.12}$$

Thus we have

$$\left\|\left(\tilde{v}_1^{\mathrm{T}}(t), \tilde{v}_2^{\mathrm{T}}(t), e(\cdot,t), e_t(\cdot,t)\right)^{\mathrm{T}}\right\|_H \le M_4 \mathrm{e}^{-w_4 t}\left\|\left(\tilde{v}_1^{\mathrm{T}}(0), \tilde{v}_2^{\mathrm{T}}(0), e_0(\cdot), e_1(\cdot)\right)^{\mathrm{T}}\right\|_H$$

for any constants $M_4, w_4 > 0$.

Define an invertible bounded operator $\mathrm{A}_0 : H \to H$ as

$$\begin{aligned} &\mathrm{A}_0\left(\tilde{v}_1^{\mathrm{T}}(t), \tilde{v}_2^{\mathrm{T}}(t), e(x,t), e_t(x,t)\right)^{\mathrm{T}} \\ &= \left(\tilde{v}_1^{\mathrm{T}}(t), \tilde{v}_2^{\mathrm{T}}(t), e(x,t) - g(x)\tilde{v}_1(t), e_t(x,t) - g(x)\left[\left(S_1 - K_1 g(0)\right)\tilde{v}_1(t) + K_1 e(0,t)\right]\right) \\ &= \left(\tilde{v}_1^{\mathrm{T}}(t), \tilde{v}_2^{\mathrm{T}}(t), \tilde{u}(x,t), \tilde{u}_t(x,t)\right). \end{aligned}$$

Then there exists $M>0$ which is independent of initial value such that

$$\left\|\left(\tilde{v}_1^{\mathrm{T}}(t), \tilde{v}_2^{\mathrm{T}}(t), \tilde{u}(\cdot,t), \tilde{u}_t(\cdot,t)\right)^{\mathrm{T}}\right\|_H \le M\left\|\left(\tilde{v}_1^{\mathrm{T}}(0), \tilde{v}_2^{\mathrm{T}}(0), \tilde{u}_0(x), \tilde{u}_1(x)\right)^{\mathrm{T}}\right\|_H. \quad \square$$

3. Output Regulator Design

We construct a new transformation as

$$\begin{cases} v(t) = v(t), \\ z(x,t) = u(x,t) + h(x)v(t). \end{cases} \tag{3.1}$$

Then $z(x,t)$ is governed by

$$\begin{cases} \dot{v}(t) = Sv(t), t>0, \\ z_{tt}(x,t) = z_{xx}(x,t), \\ z_x(0,t) = c_0 z_t(0,t), \\ z_x(1,t) = U(t) + \delta_x(1,t) + \left.\dfrac{\mathrm{d}h(x)}{\mathrm{d}x}\right|_{x=1} v(t), \\ z(x,0) = z_0(x), \\ z_t(x,0) = z_1(x), \end{cases} \tag{3.2}$$

where $h(x)$ satisfies the BVP as follows

$$\begin{cases} \dfrac{\mathrm{d}^2 h(x)}{\mathrm{d}x^2} = h(x)S^2 + f(x)P_1, \\ \left.\dfrac{\mathrm{d}h(x)}{\mathrm{d}x}\right|_{x=0} = c_0 h(0)S, \\ \dfrac{1+c_0}{q-1} h(0) = P_2. \end{cases} \tag{3.3}$$

Moreover, we have

$$e(t) = \frac{1+c_0}{1-q} z(0,t) - \frac{1+c_0}{1-q} h(0)v(t) - P_2 v(t) = \frac{1+c_0}{1-q} z(0,t) \tag{3.4}$$

according to the second boundary condition in (3.3).

Lemma 3.1: Assume that S is a diagonalizable matrix, then the regulator equation (3.3) admits a unique solution.

Proof:

Since S is diagonalizable, there exists an invertible matrix $A = (\xi_1, \xi_2, \cdots, \xi_n)$ such that $A^{-1}SA = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$, where ξ_i is the eigenvector of S corresponding to the eigenvalue $\lambda_i, (i = 1, 2, \cdots, n)$ of S. Postmultiplying (3.3) by ξ_i, n ODEs are found to be

$$\begin{cases} \dfrac{\mathrm{d}^2 h_i^*(x)}{\mathrm{d}x^2} = h_i^*(x)\lambda_i^2 + f(x)P_1^*, \\ \left.\dfrac{\mathrm{d}h_i^*(x)}{\mathrm{d}x}\right|_{x=0} = c_0 h_i^*(0)\lambda_i, \\ \dfrac{1+c_0}{q-1} h_i^*(0) = P_2^*. \end{cases} \tag{3.5}$$

Here $h_i^*(x) = h(x)\xi_i, P_1^* = P_1\xi_i, P_2^* = P_2\xi_i$.

Case 1: When $\lambda_i = 0$, the BVP (3.5) becomes

$$\begin{cases} \dfrac{\mathrm{d}^2 h_i^*(x)}{\mathrm{d}x^2} = f(x)P_1^*, \\ \left.\dfrac{\mathrm{d}h_i^*(x)}{\mathrm{d}x}\right|_{x=0} = 0, \\ \dfrac{1+c_0}{q-1} h_i^*(0) = P_2^*. \end{cases} \tag{3.6}$$

A formal simple computation shows that the solution of (3.6) is

$$\begin{aligned} h_i^*(x) &= h_i^*(0) + \left.\frac{\mathrm{d}h_i^*(x)}{\mathrm{d}x}\right|_{x=0} x + \int_0^x (x-y) f(y)\mathrm{d}y P_1^*, \\ &= \frac{q-1}{1+c_0} P_2^* + \int_0^x (x-y) f(y)\mathrm{d}y P_1^*. \end{aligned}$$

Case 2: When $\lambda_i \neq 0$, the BVP (3.5) has a general solution as

$$h_i^*(x) = m_{i1}\sinh(\lambda_i x) + m_{i2}\cosh(\lambda_i x) + \frac{1}{\lambda_i}\int_0^x \sinh(\lambda_i(x-y)) f(y)\mathrm{d}y P_1^*. \tag{3.7}$$

The last two boundary equations in (3.5) conclude that

$$\begin{cases} \lambda_i m_{i1} = c_0 \lambda_i h_i^*(0) = c_0 \lambda_i m_{i2}, \\ \dfrac{1+c_0}{q-1} m_{i2} = P_2^*. \end{cases} \tag{3.8}$$

Then m_{i1}, m_{i2} are determined by solving (3.8) as

$$\begin{cases} m_{i2} = \dfrac{q-1}{1+c_0} P_2^*, \\ m_{i1} = c_0 m_{i2} = \dfrac{c_0(q-1)}{1+c_0} P_2^*. \end{cases}$$

Thus the unique solution to the regulator Equation (3.3) is obtained and

$$h(x) = \left[h_1(x), h_2(x), \cdots, h_n(x)\right]^{\mathrm{T}}. \qquad \square$$

Now we design the output feedback controller for (3.2) as

$$\begin{aligned} U(t) = &-\frac{q+c_0}{1+c_0} w_t(0,t-1) - c_2 \hat{u}(1,t) \\ &-\left[\left.\frac{\mathrm{d}h(x)}{\mathrm{d}x}\right|_{x=1} + c_2 h(1)\right]\left(\hat{v}_1^{\mathrm{T}}(t) \ \ \hat{v}_2^{\mathrm{T}}(t)\right)^{\mathrm{T}}, \end{aligned} \tag{3.9}$$

where $c_2 > 0$ is a constant. Under (3.9), the closed-loop becomes

$$\begin{cases} \dot{\tilde{v}}_1(t) = \left(S_1 - K_1 g(0)\right)\tilde{v}_1(t) + K_1 e(0,t), t > 0, \\ \dot{\tilde{v}}_2(t) = \left(S_2 + K_2 Q_2\right)\tilde{v}_2(t), t > 0, \\ z_{tt}(x,t) = z_{xx}(x,t), \\ z_x(0,t) = c_0 z_t(0,t), \\ z_x(1,t) = -c_2 z(1,t) + c_2 \tilde{u}(1,t) + \left[\left.\dfrac{\mathrm{d}h(x)}{\mathrm{d}x}\right|_{x=1} + c_2 h(1)\right]\left(\tilde{v}_1^{\mathrm{T}}(t) \ \ \tilde{v}_2^{\mathrm{T}}(t)\right)^{\mathrm{T}}, \\ \tilde{v}_1(0) = \tilde{v}_{10} \in C^{n_1}, \tilde{v}_2(0) = \tilde{v}_{20} \in C^{n_2}, \\ z(x,0) = z_0(x), z_t(x,0) = z_1(x). \end{cases} \tag{3.10}$$

Theorem 3.1: For any initial data $(z_0, z_1) \in H_0$, μ, α are positive constants, there exists a unique (weak) solution to the PDE-part of (3.10) such that $\left(z(\cdot,t), z_t(\cdot,t)\right) \in C(0,\infty; H_0)$. Besides, this solution is exponentially stable in the sense that

$$\left\|\left(z(\cdot,t), z_t(\cdot,t)\right)^{\mathrm{T}}\right\|_{H_0} \le \mu \mathrm{e}^{-\alpha t}\left\|\left(z_0(\cdot), z_1(\cdot)\right)^{\mathrm{T}}\right\|_{H_0}, t > 0. \tag{3.11}$$

and

$$\lim_{t\to\infty} z(0,t) = 0, \lim_{t\to\infty}\int_{t-1}^{t} z_t^2(0,s)\mathrm{d}s = 0. \tag{3.12}$$

Proof:

Define an operator $\mathrm{A}_1 : D(\mathrm{A}_1) \to H_0$ for the PDE-part of system (3.10) by

$$\begin{cases} \mathrm{A}_1(f,g)^{\mathrm{T}} = (g, f'')^{\mathrm{T}}, \forall (f,g)^{\mathrm{T}} \in D(\mathrm{A}_1), \\ D(\mathrm{A}_1) = \left\{(f,g)^{\mathrm{T}} \in H_0 \mid \mathrm{A}_1(f,g)^{\mathrm{T}} \in H_0, f'(0) = c_0 g(0), f'(1) = -c_2 f(1)\right\}. \end{cases}$$

Then the PDE-part of (3.10) can be written as an abstract evolutionary equation in H_0 as follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(z(\cdot,t),z_t(\cdot,t)\right)^{\mathrm{T}}=\mathrm{A}_1\left(z(\cdot,t),z_t(\cdot,t)\right)^{\mathrm{T}}+\mathrm{B}\phi(t), \tag{3.13}$$

where

$$\begin{cases}\mathrm{B}=\left(0,\delta(x-1)\right)^{\mathrm{T}}\\ \phi(t)=c_2\tilde{u}(1,t)+\left[\left.\dfrac{\mathrm{d}h(x)}{\mathrm{d}x}\right|_{x=1}+c_2h(1)\right]\left(\tilde{v}_1^{\mathrm{T}}(t)\ \ \tilde{v}_2^{\mathrm{T}}(t)\right)^{\mathrm{T}}\end{cases} \tag{3.14}$$

$\phi(t)$ decays to zero exponentially from Theorem 2.1 and the transformation (2.6). It's well known that A_1 can generate an exponentially stable C_0-semigroup by [14]. In other words, there exist two constants $\mu_1,\alpha_1>0$, such that $\left\|\mathrm{e}^{\mathrm{A}_1t}\right\|\le\mu_1\mathrm{e}^{-\alpha_1t}$. It is a routine exercise that the operator B is admissible for $\mathrm{e}^{\mathrm{A}_1t}$ by [15].

It concludes that for any initial value $(z_0,z_1)\in H_0$, there exists a unique solution $(z,z_t)\in C(0,\infty;H_0)$ to the PDE-part of system (3.10), which has the form of

$$\left(z(\cdot,t),z_t(\cdot,t)\right)^{\mathrm{T}}=\mathrm{e}^{\mathrm{A}_1t}\left(z_0(\cdot),z_1(\cdot)\right)^{\mathrm{T}}+\int_0^t\mathrm{e}^{\mathrm{A}_1(t-s)}\mathrm{B}\phi(s)\mathrm{d}s. \tag{3.15}$$

The first term on the right side of (3.15) can be estimated as

$$\begin{aligned}\left\|\mathrm{e}^{\mathrm{A}_1t}\left(z(\cdot,0),z_t(\cdot,0)\right)^{\mathrm{T}}\right\|_{H_0}&\le\left\|\mathrm{e}^{\mathrm{A}_1t}\right\|\cdot\left\|\left(z(\cdot,0),z_t(\cdot,0)\right)^{\mathrm{T}}\right\|_{H_0}\\&\le\mu_1\mathrm{e}^{-\alpha_1t}\left\|\left(z_0(\cdot),z_1(\cdot)\right)^{\mathrm{T}}\right\|_{H_0}\to0,t\to\infty.\end{aligned} \tag{3.16}$$

The second term on the right side of (3.15) tends to zero exponentially because of the admissibility of B to $\mathrm{e}^{\mathrm{A}_1t}$ and the exponential stability of $\phi(t)$. From Poincare's inequality we have

$$\left|z(0,t)\right|^2\le2\left|z(1,t)\right|^2+2\int_0^1\left|z_x(x,t)\right|^2\mathrm{d}x,$$

Hence $\lim_{t\to\infty}z(0,t)=0$ holds. Next we show that $\lim_{t\to\infty}\int_{t-1}^t z_t^2(0,s)\mathrm{d}s=0$.

First define the Lyapunov function as

$$E_z(t)=\frac{1}{2}\int_0^1\left(z_t^2(x,t)+z_x^2(x,t)\right)\mathrm{d}x+\frac{c_2}{2}z^2(1,t),$$
$$\rho_z(t)=\int_0^1(x-1)z_t(x,t)z_x(x,t)\mathrm{d}x.$$

Notice the fact that $2E_z(t)=\left\|\left(z(\cdot,t),z_t(\cdot,t)\right)\right\|_{H_0}^2$ decays exponentially and $\left|\rho_z(t)\right|\le E_z(t)$.

Then differentiate $\rho_z(t)$ along the solution to system (3.2) to give

$$\begin{aligned}\dot{\rho}_z(t)&=\frac{x-1}{2}\left.\left(z_t^2(x,t)+z_x^2(x,t)\right)\right|_0^1-\frac{1}{2}\int_0^1\left(z_t^2(x,t)+z_x^2(x,t)\right)\mathrm{d}x\\&=\frac{1}{2}\left(z_t^2(0,t)+z_x^2(0,t)\right)-\frac{1}{2}\int_0^1\left(z_t^2(x,t)+z_x^2(x,t)\right)\mathrm{d}x.\end{aligned} \tag{3.17}$$

Finally integrating (3.17) from $t-1$ to t with respect to t and obtain

$$
\begin{aligned}
&\frac{1}{2}\int_{t-1}^{t}\left(z_t^2(0,s)+z_x^2(0,s)\right)\mathrm{d}s \\
&=\rho_z(t)-\rho_z(t-1)+\frac{1}{2}\int_{t-1}^{t}\int_0^1\left(z_t^2(x,s)+z_x^2(x,s)\right)\mathrm{d}x\mathrm{d}s \qquad (3.18)\\
&\le E_z(t)+E_z(t-1)+\int_{t-1}^{t}E_z(s)\mathrm{d}s,
\end{aligned}
$$

which decays exponentially from the exponential stability of $E_z(t)$.

Hence $\lim_{t\to\infty}\int_{t-1}^{t}z_t^2(0,s)\mathrm{d}s=0$ holds. □

The closed-loop of system (1.1) corresponding to (1.3), (2.1), (2.4) and (3.9) in the state space $\mathrm{X}=H\times H^1(0,1)\times H$ yields to

$$
\begin{cases}
\dot{v}_1(t)=S_1v_1(t),t>0,\\
\dot{v}_2(t)=S_2v_2(t),t>0,\\
w_{tt}(x,t)=w_{xx}(x,t)+f(x)Q_1v_1(t),x\in(0,1),t>0,\\
w_x(0,t)=-qw_t(0,t),\\
w_x(1,t)=\hat{u}_x(1,t)-\dfrac{q+c_0}{1+c_0}w_t(0,t-1),\\
\delta_t(x,t)=-\delta_x(x,t),\\
\delta(0,t)=-\dfrac{q+c_0}{1+c_0}w(0,t),\\
\dot{\hat{v}}_1(t)=S_1\hat{v}_1(t)+K_1\left[\hat{u}(0,t)-\dfrac{1-q}{1+c_0}w(0,t)\right],t>0,\\
\dot{\hat{v}}_2(t)=S_2\hat{v}_2(t)+K_2\left[Q_2\hat{v}_2(t)-r(t)\right],t>0,\\
\hat{u}_{tt}(x,t)=\hat{u}_{xx}(x,t)-g(x)S_1K_1\left[\hat{u}(0,t)-\dfrac{1-q}{1+c_0}w(0,t)\right]\\
\qquad\qquad +f(x)Q_1\hat{v}_1(t)-g(x)K_1\left[\hat{u}_t(0,t)-\dfrac{1-q}{1+c_0}w_t(0,t)\right],\\
\hat{u}_x(0,t)=c_0\hat{u}_t(0,t)+\left(c_1+c_0g(0)K_1\right)\left[\hat{u}(0,t)-\dfrac{1-q}{1+c_0}w(0,t)\right],\\
\hat{u}_x(1,t)=-c_2\hat{u}(1,t)-\left[\left.\dfrac{\mathrm{d}h(x)}{\mathrm{d}x}\right|_{x=1}+c_2h(1)\right]\left(\hat{v}_1^{\mathrm{T}}(t)\ \ \hat{v}_2^{\mathrm{T}}(t)\right)^{\mathrm{T}},\\
v_{10},\hat{v}_{10}\in C^{n_1},v_{20},\hat{v}_{20}\in C^{n_2},\delta(x,0)=\delta_0(x),\\
w(x,0)=w_0(x),w_t(x,0)=w_1(x),\\
\hat{u}(x,0)=\hat{u}_0(x),\hat{u}_t(x,0)=\hat{u}_1(x).
\end{cases}
\qquad (3.19)
$$

4. Main Results

Considering the closed-loop (3.19) in the state space X and define an operator A_2 for (3.19) which satisfies

$$
\begin{cases}
\mathrm{A}_2\left(\varphi_1,\varphi_2,f_1,g_1,h_1,\psi_1,\psi_2,f_2,g_2\right)^{\mathrm{T}}=\Bigg(S_1\varphi_1,S_2\varphi_2,g_1,f_1''+fQ_1\varphi_1,\\
\quad -h_1',S_1\psi_1+K_1\left[f_2(0)-\dfrac{1-q}{1+c_0}f_1(0)\right],S_2\psi_2+K_2Q_2\left(\psi_2-\varphi_2\right),g_2,\\
\quad f_2''+r_1\left[f_2(0)-\dfrac{1-q}{1+c_0}f_1(0)\right]+r_2\left[g_2(0)-\dfrac{1-q}{q+c_0}h_1'(0)\right]^{\mathrm{T}}+fQ_1\psi_1\Bigg)^{\mathrm{T}},\\
\quad \forall\left(\varphi_1,\varphi_2,f_1,g_1,h_1,\psi_1,\psi_2,f_2,g_2\right)^{\mathrm{T}}\in D\left(\mathrm{A}_2\right),\\
D\left(\mathrm{A}_2\right)=\Big\{\left(\varphi_1,\varphi_2,f_1,g_1,h_1,\psi_1,\psi_2,f_2,g_2\right)^{\mathrm{T}}\in \mathrm{X}\mid \mathrm{A}_2\left(\varphi_1,\varphi_2,f_1,g_1,\right.\\
\quad \left.h_1,\psi_1,\psi_2,f_2,g_2\right)^{\mathrm{T}}\in \mathrm{X},\ f_1'(0)=-qg_1(0),\\
\quad f_1'(1)=f_2'(1)-h_1'(1),\ h_1(0)=-\dfrac{q+c_0}{1+c_0}f_1(0),\\
\quad f_2'(0)=c_0g_2(0)+\left(c_1+c_0g(0)K_1\right)\left[f_2(0)-\dfrac{1-q}{1+c_0}f_1(0)\right],\\
\quad f_2'(1)=-c_2f_2(1)-\left[\left.\dfrac{\mathrm{d}h(x)}{\mathrm{d}x}\right|_{x=1}+c_2h(1)\right]\left(\psi_1^{\mathrm{T}}\ \ \psi_2^{\mathrm{T}}\right)^{\mathrm{T}}\Big\}.
\end{cases}
$$

Then (3.19) can be written as an abstract evolutionary equation in X as

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(v_1(t),v_2(t),w(\cdot,t),w_t(\cdot,t),\delta(\cdot,t),\hat{v}_1(t),\hat{v}_2(t),\hat{u}(\cdot,t),\hat{u}_t(\cdot,t)\right)^{\mathrm{T}}\\
&=\mathrm{A}_2\left(v_1(t),v_2(t),w(\cdot,t),w_t(\cdot,t),\delta(\cdot,t),\hat{v}_1(t),\hat{v}_2(t),\hat{u}(\cdot,t),\hat{u}_t(\cdot,t)\right)^{\mathrm{T}},
\end{aligned}
\tag{4.1}
$$

Now we discuss the closed-loop system (4.1) in X.

Theorem 4.1: Define a function $N_1^{\mathrm{T}}(s)=-\int_0^1\cosh\left(s(1-y)\right)f(y)\mathrm{d}y$ satisfies $N_1^{\mathrm{T}}(\lambda_i)Q_1\xi_i\neq 0,\left(i=1,2,\cdots,n_1\right)$ for all eigen-pairs (λ_i,ξ_i) of S_1. $c_0,c_1>0$ are constants, $g(x)$ is the unique solution to BVP (2.8). We make $P=g(0)$ and $r_1(x)=-g(x)S_1K_1,r_2(x)=-g(x)_1K_1$, matrix $S_1-K_1g(0)$ is Hurwitz. Then for any $\left(v_1(0),v_2(0),w_0(\cdot),w_1(\cdot),\delta_0(\cdot),\hat{v}_1(0),\hat{v}_2(0),\hat{u}_0(\cdot),\hat{u}_1(\cdot)\right)\in \mathrm{X}$, system (4.1) admits a unique bounded solution $\left(v_1,v_2,w,w_t,\delta,\hat{v}_1,\hat{v}_2,\hat{u},\hat{u}_t\right)\in C(0,\infty;\mathrm{X})$. Moreover, the tracking error $e(t)\to 0$ exponentially when t tends to infinity.

Proof:

Define an invertible bounded operator $\mathrm{A}_3:\mathrm{X}\to\mathrm{X}$ by

$$
\begin{aligned}
&\mathrm{A}_3\left(v_1(t),v_2(t),w(\cdot,t),w_t(\cdot,t),\delta(\cdot,t),\hat{v}_1(t),\hat{v}_2(t),\hat{u}(\cdot,t),\hat{u}_t(\cdot,t)\right)^{\mathrm{T}}\\
&=\left(v_1(t),v_2(t),w(\cdot,t)+\delta(\cdot,t)+h(x)v(t),w_t(\cdot,t)+\delta_t(\cdot,t)\right.\\
&\quad +h(x)\dot{v}(t),\delta(\cdot,t),v_1(t)-\hat{v}_1(t),v_2(t)-\hat{v}_2(t),\\
&\quad \left.u(\cdot,t)-\hat{u}(\cdot,t),u_t(\cdot,t)-\hat{u}_t(\cdot,t)\right)^{\mathrm{T}}\\
&=\left(v_1(t),v_2(t),z(\cdot,t),z_t(\cdot,t),\delta(\cdot,t),\tilde{v}_1(t),\tilde{v}_2(t),\tilde{u}(\cdot,t),\tilde{u}_t(\cdot,t)\right)^{\mathrm{T}}
\end{aligned}
\tag{4.2}
$$

An equivalent system of (3.19) is found to be

$$
\begin{cases}
\dot{v}_1(t)=S_1v_1(t),t>0,\\
\dot{v}_2(t)=S_2v_2(t),t>0,\\
d(t)=Q_1v_1(t),\\
r(t)=Q_2v_2(t),\\
z_{tt}(x,t)=z_{xx}(x,t),\\
z_x(0,t)=c_0z_t(0,t),\\
z_x(1,t)=-c_2z(1,t)+c_2\tilde{w}(1,t)+\left[\left.\dfrac{\mathrm{d}h(x)}{\mathrm{d}x}\right|_{x=1}+c_2h(1)\right]\left(\tilde{v}_1^{\mathrm{T}}(t)\ \ \tilde{v}_2^{\mathrm{T}}(t)\right)^{\mathrm{T}},\\
\delta_t(x,t)=-\delta_x(x,t),\\
\delta(0,t)=-\dfrac{q+c_0}{1+c_0}w(0,t),\\
\dot{\tilde{v}}_1(t)=S_1\tilde{v}_1(t)+K_1\tilde{u}(0,t),t>0,\\
\dot{\tilde{v}}_2(t)=(S_2+K_2Q_2)\tilde{v}_2(t),t>0,\\
\tilde{u}_{tt}(x,t)=\tilde{u}_{xx}(x,t)+f(x)Q_1\tilde{v}_1(t)-g(x)S_1K_1\tilde{u}(0,t)-g(x)K_1\tilde{u}_t(0,t),\\
\tilde{u}_x(0,t)=c_0\tilde{u}_t(0,t)+(c_1+c_0g(0)K_1)\tilde{u}(0,t),\\
\tilde{u}_x(1,t)=0,\\
v_{10},\hat{v}_{10}\in C^{n_1},v_{20},\tilde{v}_{20}\in C^{n_2},\delta(x,0)=\delta_0(x),\\
z(x,0)=z_0(x),z_t(x,0)=z_1(x),\\
\tilde{u}(x,0)=\tilde{u}_0(x),\tilde{u}_t(x,0)=\tilde{u}_1(x).
\end{cases}
\tag{4.3}
$$

We can obtain the solution to the transport system (2.1) explicitly as

$$
\delta(x,t)=\begin{cases}\delta_0(x-t) & x\ge t,\\ \dfrac{q+c_0}{q-1}z(0,t-x)-\dfrac{q+c_0}{q-1}h(0)v(t) & x<t.\end{cases}
\tag{4.4}
$$

Let $\varepsilon_0=\sup_{t>0}\left|\dfrac{q+c_0}{q-1}h(0)v(t)\right|\in R$ being a designed parameter, then we have the estimation as

$$
|\delta(0,t)|\le\left|\frac{q+c_0}{q-1}\right||z(0,t)|+\varepsilon_0\le\left|\frac{q+c_0}{q-1}\right|\left\|(z(\cdot,t),z_t(\cdot,t))\right\|_{H_0}+\varepsilon_0,
$$

and

$$
\begin{aligned}
&\int_0^1\left(\delta_t^2(x,t)+\delta_x^2(x,t)\right)\mathrm{d}x\\
&\le 2\left(\frac{q+c_0}{q-1}\right)^2\int_0^1|z_t(0,t-x)|^2\,\mathrm{d}x+4\varepsilon_0\frac{q+c_0}{q-1}\int_0^1|z_t(0,t-x)|\,\mathrm{d}x+2\varepsilon_0^2\\
&=2\left(\frac{q+c_0}{q-1}\right)^2\int_{t-1}^t|z_t(0,s)|^2\,\mathrm{d}s+4\varepsilon_0\frac{q+c_0}{q-1}\int_{t-1}^t|z_t(0,s)|\,\mathrm{d}s+2\varepsilon_0^2.
\end{aligned}
$$

Conclusion (3.12) together with $\left\|(z(\cdot,t),z_t(\cdot,t))\right\|_{H_0}$ exponentially, imply that the solution to (2.1) is bounded as $t\to\infty$.

According to Theorem 2.1 and the well-posedness and exponential stability of system (3.10), (4.3) admits a unique bounded solution in X and so does the closed-loop (3.19) by the invertible transformation (4.2). As a result, the tracking error

$$e(t) = w(0,t) - r(t) = \frac{1+c_0}{1-q} z(0,t) \to 0$$

exponentially in the light of (3.12). □

5. Concluding Remarks

In this paper, the output regulation problem for 1-D anti-stable wave equation is solved. The original system has the anti-damping at the position $x = 0$ which is anti-collocated with the control, and also subjects to the distributed disturbance with unknown intensity generated by an external system. By proposing an observer-based feedback controller for (1.1), the following objectives are achieved: 1) keep all the states of internal-loop bounded; 2) recover the system state from input and output; 3) regulate the output tracks the given reference signal exponentially.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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