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Infinite Sets of Solutions and Almost Solutions of the Equation $N \cdot M = reversal(N \cdot M)$ II

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Abstract

Motivated by their intrinsic interest and by applications to the study of numeric palindromes and other sequences of integers, we discover a method for producing infinite sets of solutions and almost solutions of the equation $N \cdot M = reversal(N \cdot M)$, our results are valid in a general numeration base $b > 2$.

Keywords

Palindrome, Numeration Base, Reversal

1. Introduction

In this paper, motivated by their intrinsic interest and by applications to the study of numeric palindromes and other sequences of integers, we discover a method for producing infinite sets of solutions and almost solutions of the equation:

$$N \cdot M = reversal(N \cdot M). \tag{1}$$

where if N is an integer written in base b, *which is understood from the context then reversal(N) is the base b integer obtained from N writing its digits in reverse order.*

An almost solution of (1) is a pair of integers (M, N) for which the equality (1) holds up to a few of digits for which we understand their position. Our results are valid in a general numeration base $b > 2$ and complement the results in [1]. Recently one of us showed in Nitica [2] that, in any numeration base b, for any integer N not divisible by b, the Equation (1) has an infinite set of solutions (N, M). Nevertheless, as one can see from [3], finding explicit values for M can be difficult from a computational point of view, even for small values of N, e.g. $N = 81$. We show in [1] for many numeration bases explicit infinite families of solutions of (1). This families of solutions here complement and are independent of those shown in [1].

Another application of our results may appear in the study of the classes of b-multiplicative and b-additive Ramanujan-Hardy numbers, recently introduced in Nitica [4]. The first class consists of all integers N for which there exists an integer M such that $S_b(N)$, the sum of base b-digits of N, times M, multiplied by the reversal of the product, is equal to N. The second class consists of all integers N for which there exists an integer M such that $S_b(N)$, times M, added to the reversal of the product, is equal to N. As showed in Nitica [2] [4], the solutions of Equation (1) for which we can compute the sum of digits of $S_b(N) \cdot M + reversal(S_b(N) \cdot M)$ or of $S_b(N) \cdot M \cdot reversal(S_b(N) \cdot M)$, can be used to find infinite sets of above numbers.

2. Statements of the Main Results

The heuristics behind our results is that the product of a palindrome by a small integer still preserves some of the symmetric structure of the palindrome; if, in addition, the palindrome has many digits of 9, many times the results observed in base 10 can be carried over to an arbitrary numeration base b replacing 9 by $b-1$.

Let $b \geq 2$ be a numeration base. If x is a string of digits, let $(x)^{\wedge k}$ denote the base b integer obtained by repeating x k-times. Let $[x]_b$ denote the value of the string x in base b.

Next theorem is one of our main results.

Theorem 1. Let $b \geq 2$ be a numeration base. Let $0 < A, B, c, d \leq b$ integers such that $A \cdot B = [cd]_b$ and $c + d = A$. Then,

$$A^{\wedge k} \cdot B = \left[cA^{\wedge k-1} d \right]_b .$$

Proof of Theorem 1 is covered in Section 3. Similar proof to that of Theorem 1 gives also the somewhat stronger statement Theorem 3.

k	$A^{\wedge k}$	$A^{\wedge k} \cdot B$	$[cA^{\wedge k-1}d]_b$
2	99	891	891
3	999	8991	8991
4	9999	89991	89991
5	99999	899991	899991
6	999999	8999991	8999991
7	9999999	89999991	89999991
8	99999999	899999991	899999991

The above table illustrates the result from Theorem 1 if $b = 10$ and $(A, B) = (9, 9)$, $[cd]_b = [81]_{10}$, and $k \in \{2, 3, 4, 5, 6, 7, 8\}$. Note that $9 \times 9 = 81$ and $8 + 1 = 9$.

Theorem 2. Let $b > 2$ numeration base and $k, l > 1$ integers then one has:

$$(b-1)^{\wedge k} \cdot [a_1 a_2 a_3 \cdots a_l]_b = [a_1 a_2 a_3 \cdots a_l]_b [a_1 a_2 a_3 \cdots a_l - 1]_b (b-1)^{\wedge}(k-l) - [b^l - a_1 a_2 a_3 \cdots a_l]_b \quad (2)$$

in particular if *b is odd* and $[a_1 a_2 a_3 \cdots a_l]_b = (b^l - 1)/2$.

Then (2) gives a solution of (1).

The proof of Theorem 2 is done in Section 4.

The following examples illustrate the statement of Theorem 2.

Example:

$$9^{\wedge 130} \cdot [123]_{10} = [122 \quad 9^{\wedge 1327} 83]_{10}$$

$$7^{\wedge 130} \cdot [123]_8 = [1227^{\wedge 127} \quad 489]_8$$

$$9^{\wedge 130} \cdot [123]_{10} = [122 \quad 9^{\wedge 127} 389]_8$$

Theorem 3. let $b > 2$ umeration base. Let $0 < A, B, c, d, \alpha \le b$ integers such that $A \cdot B = [cd]_b$ and $c + d = \alpha$. Then,

$$A^{\wedge k} B = [c \alpha^{\wedge k-1} d]_b = AB^{\wedge k}$$

Next theorem shows for all numeration bases examples of pairs (A, B) that satisfy the hypothesis of Theorem 1.

Theorem 4. Let $b \ge 2$ be a numeration base. Then the pairs $(AB) = [(b-1)(b-k)]_b, 1 \le k \le b$ satisfy the hypothesis of Theorem 1.

Proof:

$$[(b-1)(b-k)]_b$$

$$b^2 - bk - b + k = b(b-k-1) + k = [[b-k-1], k]_b$$

$$\Rightarrow b - k - 1 + k = b - 1.$$

Corollary. Let $b \ge 2$ be numeration base. Then $[(b-1)(b-2)]b$.

Consequently, satisfies the hypothesis of Theorem 1, consequently

$$(b-1)^{\wedge k}(b-2) = [(b-3)(b-1)^{\wedge(k-1)} 2]_b.$$

Proof: apply Theorem 4 to the pair $(AB) = (b-1)(b-2)$.

k	$A^{\wedge k}$	$[A^{\wedge k} \cdot B]_b$	$[cA^{\wedge k-1} d]_b$
2	66	$[462]_7$	$[462]_7$
3	666	$[4662]_7$	$[4662]_7$
4	6666	$[46662]_7$	$[46662]_7$
5	66666	$[466662]_7$	$[466662]_7$
6	666666	$[4666662]_7$	$[4666662]_7$
7	6666666	$[46666662]_7$	$[46666662]_7$
8	66666666	$[466666662]_7$	$[466666662]_7$

The above table illustrates the result from Theorem 1 & Theorem 3 if $b = 7$, $b - 1 = 6$, $b - 2 = 5$, $[cd]_b = [42]_7$, thus $A = 6, B = 5$ and $k \in \{2,3,4,5,6,7,8\}$. Note that $[6 \cdot 5]_7 = [42]_7$ and $[4+2]_7 = 6$.

b	(A,B)
2	
3	$(2,2)$
4	$(2,3),(3,2),(3,3)$
5	$(2,3),(2,4),(3,2),(3,4),(4,2),(4,3),(4,4)$
6	$(2,5),(3,5),(4,5),(5,2),(5,3),(5,4),(5,5)$
7	$(2,4),(2,6),(3,3),(3,5),(3,6),(4,2),(4,4),(4,6),$ $(5,3),(5,6),(6,2),(6,3),(6,4),(6,5),(6,6)$
8	$(3,7),(4,7),(5,7),(6,7),(7,2),(7,3),(7,4),(7,5),(7,6),(7,7)$
9	$(2,5),(2,8),(3,4),(3,8),(4,3),(4,5),(4,6),(4,7),(4,8),$ $(5,2),(5,4),(5,6),(5,8),(6,4),(6,5),(6,8),(7,4),(7,8),$ $(8,2),(8,3),(8,4),(8,5),(8,6),(8,7),(8,8)$
10	$(2,9),(3,4),(3,7),(3,9),(4,6),(4,9),(5,9),(6,4),(6,7),$ $(6,9),(7,3),(7,6),(7,9),(8,9),(9,2),(9,3),(9,4),(9,5),$ $(9,6),(9,7),(9,8),(9,9)$

The above table shows all pairs (A,B) that satisfy the hypothesis of Theorem 1 for small numeration bases. We observe that for $b=2$ there are no pairs (A,B) that satisfy the hypothesis of Theorem 1.

3. Proof of Theorem 1

$$\begin{aligned}\sum_{l=1}^{k} Ab^{l}\cdot B &= \sum_{l=1}^{k} A\cdot Bb^{l} = \sum_{l=1}^{k}(cb+d)b^{l} = \sum_{l=1}^{k} c\cdot b^{l+1} + d\cdot\sum_{l=1}^{k} b^{l}\\ &= c\cdot b^{k+1} + \sum_{l=1}^{k-1} c\cdot b + \sum_{l=1}^{k-1} d\cdot b + d\cdot b^{k}\\ &= c\cdot b^{k+1} + \sum_{l=1}^{k-1}(c+d)\cdot b^{l} + d\cdot b^{k}\\ &= c\cdot b^{k+1} + \sum_{l=1}^{k-1} A\cdot b + d\cdot b^{k} = \left[c(A)\wedge^{k-1} d\right]_b\end{aligned}$$

4. Proof of Theorem 2

Using that $(b-1)^{k} = b^{k}-1$ and that $(b-1)^{k-l} = b^{k-l}-1$.

One has that:

$$\begin{aligned}&(b-1)^{k}\cdot[a_1a_2a_3\cdots a_l]_b = \left(b^{k}-1\right)\cdot[a_1a_2a_3\cdots a_l]_b\\ &= \left[+b^{k}a_1a_2a_3\cdots a_l\right]_b - b^{l}\left[a_1a_2a_3\cdots a_l\right]_b\\ &= +\left[+b^{k}a_1a_2a_3\cdots a_l\right]_b - 1 + b^{k} + b^{l} - b^{l}\\ &= +\left[+b^{k}a_1a_2a_3\cdots a_l\right]_b - 1 + b^{l}\left(b^{k-l}-1\right) + \left[b^{l} - a_1a_2a_3\cdots a_l\right]_b\\ &= -1(b-1)\wedge(k-l) - \left[b^{l} - a_1a_2a_3\cdots a_l\right]_b\end{aligned}$$

5. Conclusion

Motivated by possible applications to the study of palindromes and other sequences of integers we discover a method for producing infinite families of integer solutions and almost solutions of the equation $N \cdot M = reversal(N \cdot M)$. Our results complement the results in [1] and are valid in all numeration bases $b > 2$.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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