Scientific Research Publishing

Advances in Pure Mathematics, 2020, 10, 245-258
https://www.scirp.org/journal/apm
ISSN Online: 2160-0384
ISSN Print: 2160-0368

Some Results on the Space of Bounded Second Variation Functions in the Sense of Shiba

Liliana Pérez, Ebner Pineda, Luz Rodríguez

Escuela Superior Politécnica del Litoral (ESPOL), FCNM, Campus Gustavo Galindo, Guayaquil, Ecuador
Email: lilireperez@gmail.com, lrperez@espol.edu.ec, ebner.pineda@gmail.com, epineda@espol.edu.ec, le.rodriguezq@gmail.com, luzeurod@espol.edu.ec

How to cite this paper: Pérez, L., Pineda, E. and Rodríguez, L. (2020) Some Results on the Space of Bounded Second Variation Functions in the Sense of Shiba. *Advances in Pure Mathematics*, **10**, 245-258.
https://doi.org/10.4236/apm.2020.105015

Received: March 11, 2020
Accepted: May 9, 2020
Published: May 12, 2020

Copyright © 2020 by author(s) and Scientific Research Publishing Inc.
This work is licensed under the Creative Commons Attribution International License (CC BY 4.0).
http://creativecommons.org/licenses/by/4.0/

Open Access

Abstract

In this paper, we study the structure of the space of functions of bounded second variation in the sense of Shiba; an integral representation theorem is also proved and necessary conditions are given for that the space be closed under composition of functions. Another significant result is the proof that this space of bounded second variation in the sense of Shiba is a Banach algebra, which is not immediate as it happens in other spaces of generalized bounded variation.

Keywords

Functional Analysis, Bounded Variation, Second Bounded Variation, Banach Space

1. Introduction

The notion of function of bounded variation, or the functions space of bounded variation on $[a,b]$ ($BV([a,b],\mathbb{R})$) was first introduced by C. Jordan in 1881, [1], when he critically re-examined a faulty proof given by Dirichlet to the famous Fourier's conjecture on trigonometric series expansion of periodic functions, see [2]. Jordan actually extended the Dirichlet's criterium (on convergence of the Fourier series of monotone functions) to the class of BV functions. The interest generated by the classical notion of function of bounded variation has led to some generalizations of the concept, mainly, intended to the search of bigger classes of functions whose elements have pointwise convergent Fourier series or to applications in geometric measure theory, calculus of variations and mathematical physics. As in the classical case, these generalizations have found many applications in the study of certain differential and integral equations (see [3]). Consequently, the study of certain notions of generalized bounded variation takes an important direction in the field of mathematical analysis [4]. Two well-known generalizations are the functions of bounded p-variation and the functions of bounded Φ-variation, due to N. Wiener and L. C. Young respectively. In 1924 Wiener, [5], showed that the Fourier series of function in one variable of finite p-variation converges almost everywhere. In 1937 L. C. Young, [6], developed an integration theory with respect to functions of finite φ-variation and showed that the Fourier series of such functions converges everywhere.

In 1972, Waterman [7] introduced the class of bounded variation functions ΛBV. In 1980, M. Shiba [8] generalizes this class and introduces the class $\Lambda_p BV$ $(1 \le p < \infty)$. This class $\Lambda_p BV$, is the set of all functions $f : [a,b] \to \mathbb{R}$ of Λ_p-bounded variation on $[a,b]$ which definition is as follows:

Definition 1. [9] *Given an interval* $I = [a,b]$ *and a non-decreasing sequence of positive numbers* $\Lambda = \{\lambda_i\}(i = 1,2,\cdots)$ *such that* $\sum(1/\lambda_i)$ *diverges and* $p \ge 1$. *A function* $f : I \to \mathbb{R}$ *is said to be of* Λ_p*-bounded variation on* I ($f \in \Lambda_p BV(I)$) *if*

$$V_{\Lambda_p}(f) = V_\Lambda(f,p,I) = \sup_\xi V_\Lambda(\xi,f,p,I) < \infty,$$

where

$$V_\Lambda(\xi,f,p,I) := \left(\sum_{i=1}^{n} \frac{|f(x_i) - f(x_{i-1})|^p}{\lambda_i}\right)^{1/p}, \tag{1}$$

and the supremum is taking over all partitions $\xi : a = x_0 < x_1 < \cdots < x_n = b$ *of the interval I.*

In [10], the authors introduce the notion of functions of bounded second variation *in the sense of Shiba*, following the line traced by De la Vallée Poussin [11], in 1908, and M. Shiba [8], in 1980. *Inspired by the work done in* [7], *a sequence of positive real numbers,* $\Lambda = \{\lambda_i\}_{i=1}^{\infty}$, *is a* $\mathcal{W}$*-sequence if it is non-decreasing and* $\sum(1/\lambda_i) = +\infty$. Additionally, they use $\Pi_3([a,b])$ to denote the set of the partitions $\pi = \{x_i\}_{i=0}^{n}$ of the interval $[a,b]$, which includes at least three points ($\pi \in \Pi_3[a,b]$), and then define a function of bounded second variation in the sense of Shiba, as follows:

Definition 2. [10] *Let* $1 \le p < \infty$ *and* $\Lambda = \{\lambda_i\}_{i=0}^{\infty}$ *be a* $\mathcal{W}$*-sequence. The* $(\Lambda,2,p)$*-th variation of f on* $[a,b]$ *is defined as*

$$V_{\Lambda,2,p}(f;[a,b]) = V_{\Lambda,2,p}(f) = \sup_\pi \left(\sum_{i=0}^{n-2} \frac{|Q_1(f;x_{i+2},x_{i+1}) - Q_1(f;x_{i+1},x_i)|^p}{\lambda_i}\right)^{1/p},$$

where $Q_1(f;\beta,\alpha) = \dfrac{f(\beta) - f(\alpha)}{\beta - \alpha}$ *and the supremun is taken over all the partitions* $\pi = \{x_i\}_{i=0}^{n} \in \Pi_3([a,b])$.

The sum in the definition 2, is called an approximate sum to $V_{\Lambda,2,p}(f;[a,b])$.

When $V_{\Lambda,2,p}(f;[a,b]) < \infty$, f has $(\Lambda,2,p)$-th bounded variation on $[a,b]$. $\Lambda_p^2 BV([a,b])$ is the space of such functions.

Example 1. *Let* $f : [a,b] \to \mathbb{R}$ *defined by* $f(t) = t^2$. *We will prove that*

$f \in \Lambda_p^2 BV([a,b])$.

Let $1 \le p < \infty$, $\Lambda = \{\lambda_i\}_{i=0}^{\infty}$ *a* $\mathcal{W}$*-sequence and* $\pi : a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ *in* $\Pi_3([a,b])$ *a partition.*

Let's consider

$$A = \frac{\left|Q_1(f;t_{i+2},t_{i+1}) - Q_1(f;t_{i+1},t_i)\right|^p}{\lambda_i} = \frac{\left|\frac{t_{i+2}^2 - t_{i+1}^2}{t_{i+2} - t_{i+1}} - \frac{t_{i+1}^2 - t_i^2}{t_{i+1} - t_i}\right|^p}{\lambda_i} = \frac{\left|t_{i+2} - t_i\right|^p}{\lambda_i},$$

thus, using that $(x+y)^{1/p} \le x^{1/p} + y^{1/p}$ *for all* $x, y \ge 0$:

$$\left(\sum_{i=0}^{n-2} A\right)^{1/p} \le \sum_{i=0}^{n-2}\left(\frac{\left|t_{i+2}-t_i\right|^p}{\lambda_i}\right)^{1/p} = \sum_{i=0}^{n-2}\frac{\left|t_{i+2}-t_i\right|}{(\lambda_i)^{1/p}}$$
$$\le \sum_{i=0}^{n-2}\frac{\left|t_{i+2}-t_i\right|}{(\lambda_0)^{1/p}} = \frac{1}{(\lambda_0)^{1/p}}\sum_{i=0}^{n-2}\left|t_{i+2}-t_i\right|$$
$$= \frac{1}{(\lambda_0)^{1/p}}\left[(t_n - t_0) + (t_{n-1} - t_1)\right] = \frac{2(b-a)}{(\lambda_0)^{1/p}}.$$

Taking supremum

$$V_{\Lambda,2,p}(f;[a,b]) = \sup_{\pi}\left(\sum_{i=0}^{n-2}\frac{\left|Q_1(f;t_{i+2},t_{i+1}) - Q_1(f;t_{i+1},t_i)\right|^p}{\lambda_i}\right)^{1/p}$$
$$= \sup_{\pi}\left(\sum_{i=0}^{n-2}\frac{\left|t_{i+2}-t_i\right|^p}{\lambda_i}\right)^{1/p} \le \frac{2(b-a)}{(\lambda_0)^{1/p}} < \infty,$$

therefore $f \in \Lambda_p^2 BV([a,b])$.

In [10], some properties of $\Lambda_p^2 BV([a,b])$ were proved in order to show under what conditions it is guaranteed that an overlay operator acts in that space. However, there are other features that have not been explored and that allow us to improve this space. For this reason we present our following results.

2. Results for Functions of $\Lambda, 2, p$-Bounded Variation

In [10] it was shown that $\Lambda_p^2 BV([a,b])$ is a vector space and that

$$\|f\|_{\Lambda,2,p} = \|f\|_{\infty} + V_{\Lambda,2,p}(f;[a,b])$$

provides a norm for that space.

Next, we will prove the completeness of the space $\Lambda_p^2 BV([a,b])$.

Theorem 1. *Let* $\Lambda = \{\lambda_i\}_{i=0}^{\infty}$ *a* $\mathcal{W}$*-sequence, then* $\left(\Lambda_p^2 BV([a,b]); \|\cdot\|_{\Lambda,2,p}\right)$ *is a Banach space.*

Proof. Let $\Lambda = \{\lambda_i\}_{i\ge 0}$ a $\mathcal{W}$-sequence and $\{f_n\}_{n\ge 1}$ a Cauchy sequence in $\left(\Lambda_p^2 BV([a,b]); \|\cdot\|_{\Lambda,2,p}\right)$. Then, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $m, n \ge N$ we obtain:

$$\| f_n - f_m \|_{\Lambda,2,p} \le \frac{\varepsilon}{2},$$

this is,

$$\| f_n - f_m \|_\infty + V_{\Lambda,2,p}\left(f_n - f_m;[a,b]\right) < \frac{\varepsilon}{2}.$$

Thus, for

$$m,n \ge N, \quad \| f_n - f_m \|_\infty < \frac{\varepsilon}{2} \tag{2}$$

and

$$V_{\Lambda,2,p}\left(f_n - f_m;[a,b]\right) < \frac{\varepsilon}{2}. \tag{3}$$

On the other hand, using (2) we have, for $m,n \ge N$:

$$\left| f_n(t) - f_m(t) \right| \le \| f_n - f_m \|_\infty < \frac{\varepsilon}{2}, \forall t \in [a,b].$$

Therefore, $\{ f_n(t)\}_{n\ge1}$ is a uniform Cauchy sequence on the interval $[a,b]$. Then, there exist a function *f* defined on $[a,b]$ such that $f_n \to f$ uniformly in $[a,b]$, thus

$$n \ge N \Rightarrow \left| f_n(t) - f(t) \right| \le \frac{\varepsilon}{2}, \forall t \in [a,b]. \tag{4}$$

Let's prove that $f \in \Lambda_p^2 BV([a,b])$.

Consider a partition $\pi = \{t_i\}_{i=0}^k \in \Pi_3([a,b])$; that is,

$a = t_0 < t_1 < \cdots < t_{k-1} < t_k = b$, then using (3)

$$\left(\sum_{i=0}^{k-2} \frac{\left| Q_1\left(f_n - f_m; t_{i+2}, t_{i+1}\right) - Q_1\left(f_n - f_m; t_{i+1}, t_i\right)\right|^p}{\lambda_i} \right)^{1/p} \le V_{\Lambda,2,p}\left(f_n - f_m\right) < \frac{\varepsilon}{2}$$

fixed $n \ge N$ and $m \to \infty$ we have:

$$\left(\sum_{i=0}^{k-2} \frac{\left| Q_1\left(f_n - f; t_{i+2}, t_{i+1}\right) - Q_1\left(f_n - f; t_{i+1}, t_i\right)\right|^p}{\lambda_i} \right)^{1/p} \le \frac{\varepsilon}{2}$$

$$\Rightarrow V_{\Lambda,2,p}\left(f_n - f\right) \le \frac{\varepsilon}{2} < \varepsilon. \tag{5}$$

On the other hand, since $\{ f_n\}_{n\ge1}$ is a Cauchy sequence in $\Lambda_p^2 BV([a,b])$, there exist $C > 0$ such that $\| f_n \| < C, \forall n \in \mathbb{N}$, this implies that $V_{\Lambda,2,p}\left(f_n\right) \le C, \forall n \in \mathbb{N}$, thus

$$\left(\sum_{i=0}^{k-2} \frac{\left| Q_1\left(f; t_{i+2}, t_{i+1}\right) - Q_1\left(f; t_{i+1}, t_i\right)\right|^p}{\lambda_i} \right)^{1/p}$$

$$= \left(\sum_{i=0}^{k-2} \frac{\left| \lim_{n\to\infty} Q_1\left(f_n; t_{i+2}, t_{i+1}\right) - \lim_{n\to\infty} Q_1\left(f_n; t_{i+1}, t_i\right)\right|^p}{\lambda_i} \right)^{1/p}$$

$$= \lim_{n\to\infty}\left(\sum_{i=0}^{k-2}\frac{\left|Q_1\left(f_n;t_{i+2},t_{i+1}\right)-Q_1\left(f_n;t_{i+1},t_i\right)\right|^p}{\lambda_i}\right)^{1/p}$$

$$\leq \limsup_{n\to\infty} V_{\Lambda,2,p}\left(f_n\right)\leq C<\infty.$$

Therefore,

$$V_{\Lambda,2,p}\left(f\right)=\sup_{\pi}\left(\sum_{i=0}^{k-2}\frac{\left|Q_1\left(f;t_{i+2},t_{i+1}\right)-Q_1\left(f;t_{i+1},t_i\right)\right|^p}{\lambda_i}\right)^{1/p}\leq C<\infty,$$

this is, $f\in\Lambda_p^2BV\left(\left[a,b\right]\right)$.

Now

$$\left\|f_n-f\right\|_{\Lambda,2,p}=\left\|f_n-f\right\|_\infty+V_{\Lambda,2,p}\left(f_n-f\right).$$

From (4) we get:

$$\left\|f_n-f\right\|_\infty\leq\frac{\varepsilon}{2},$$

then, using this result and (5)

$$\left\|f_n-f\right\|_{\Lambda,2,p}\leq\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon, n\geq N.$$

Therefore, the sequence $\left\{f_n\right\}_{n\geq1}$ converges to the function *f* in the norm $\left\|\cdot\right\|_{\Lambda,2,p}$. Thus, $\left(\Lambda_p^2BV\left(\left[a,b\right]\right);\left\|\cdot\right\|_{\Lambda,2,p}\right)$ is a Banach space. □

In the next result we show that $\Lambda_p^2BV\left(\left[a,b\right]\right)$ is a Banach algebra. It should be noted that this result is not inmediate as it happens in other spaces of functions of generalized bounded variation.

Theorem 2. $\left(\Lambda_p^2BV\left(\left[a,b\right]\right);\left\|\cdot\right\|_{\Lambda,2,p}\right)$ *is a Banach algebra.*

Proof. Let $\Lambda=\left\{\lambda_i\right\}_{i=0}^{\infty}$ a $\mathcal{W}$-sequence. Given $g,h\in\Lambda_p^2BV\left(\left[a,b\right]\right)$, let's prove that $gh\in\Lambda_p^2BV\left(\left[a,b\right]\right)$.

Consider a partition $\pi=\left\{t_i\right\}_{i=0}^{n}\in\Pi_3\left(\left[a,b\right]\right)$; that is, $a=t_0<t_1<\cdots<t_{n-1}<t_n=b$. Let's start by studying the expression

$$\begin{aligned}A&=Q_1\left(gh;t_{i+2},t_{i+1}\right)-Q_1\left(gh;t_{i+1},t_i\right)\\&=\frac{(gh)(t_{i+2})-(gh)(t_{i+1})}{t_{i+2}-t_{i+1}}-\frac{(gh)(t_{i+1})-(gh)(t_i)}{t_{i+1}-t_i}\\&=\frac{g(t_{i+2})h(t_{i+2})-g(t_{i+1})h(t_{i+1})}{t_{i+2}-t_{i+1}}-\frac{g(t_{i+1})h(t_{i+1})-g(t_i)h(t_i)}{t_{i+1}-t_i}\end{aligned}$$

by adding the terms $\pm g\left(t_{i+1}\right)h\left(t_{i+2}\right)$, $\pm g\left(t_i\right)h\left(t_{i+1}\right)$, taking $I_{i+1}=\left[t_i,t_{i+1}\right]$ with $i=0,1,\cdots,n-2$ and making $Q_1\left(f;I_{i+1}\right)=Q_1\left(f;t_{i+1},t_i\right)$, we get

$$\begin{aligned}A&=Q_1\left(gh;t_{i+2},t_{i+1}\right)-Q_1\left(gh;t_{i+1},t_i\right)\\&=\frac{\left[g(t_{i+2})-g(t_{i+1})\right]h(t_{i+2})+g(t_{i+1})\left[h(t_{i+2})-h(t_{i+1})\right]}{t_{i+2}-t_{i+1}}\\&\quad-\frac{\left[g(t_{i+1})-g(t_i)\right]h(t_{i+1})+g(t_i)\left[h(t_{i+1})-h(t_i)\right]}{t_{i+1}-t_i}\end{aligned}$$

$$
\begin{aligned}
&= \frac{\left[g(t_{i+2})-g(t_{i+1})\right]}{t_{i+2}-t_{i+1}}h(t_{i+2})-\frac{\left[h(t_{i+1})-h(t_i)\right]}{t_{i+1}-t_i}h(t_{i+1}) \\
&\quad + g(t_{i+1})\frac{\left[g(t_{i+2})-g(t_{i+1})\right]}{t_{i+2}-t_{i+1}}-g(t_i)\frac{\left[h(t_{i+1})-h(t_i)\right]}{t_{i+1}-t_i} \\
&= Q_1(g;I_{i+2})h(t_{i+2})-Q_1(g;I_{i+1})h(t_{i+1}) \\
&\quad + g(t_{i+1})Q_1(h;I_{i+2})-g(t_i)Q_1(h;I_{i+1})
\end{aligned}
$$

now, by adding the following terms $\pm h(t_{i+2})Q_1(g;I_{i+1}),\pm g(t_{i+1})Q_1(h;I_{i+1})$ and grouping these, we get

$$
\begin{aligned}
A &= h(t_{i+2})\left[Q_1(g;I_{i+2})-Q_1(g;I_{i+1})\right]+h(t_{i+2})Q_1(g;I_{i+1})-Q_1(g;I_{i+1})h(t_{i+1}) \\
&\quad + g(t_{i+1})\left[Q_1(h;I_{i+2})-Q_1(h;I_{i+1})\right]+g(t_{i+1})Q_1(h;I_{i+1})-g(t_i)Q_1(h;I_{i+1}) \\
&= h(t_{i+2})\left[Q_1(g;I_{i+2})-Q_1(g;I_{i+1})\right]+g(t_{i+1})\left[Q_1(h;I_{i+2})-Q_1(h;I_{i+1})\right] \\
&\quad + Q_1(g;I_{i+1})\left[h(t_{i+2})-h(t_{i+1})\right]+Q_1(h;I_{i+1})\left[g(t_{i+1})-g(t_i)\right] \\
&= h(t_{i+2})\left[Q_1(g;I_{i+2})-Q_1(g;I_{i+1})\right]+g(t_{i+1})\left[Q_1(h;I_{i+2})-Q_1(h;I_{i+1})\right] \\
&\quad + Q_1(g;I_{i+1})\left[h(t_{i+2})-h(t_{i+1})\right]+\left[h(t_{i+1})-h(t_i)\right]Q_1(g;I_{i+1}) \\
&= h(t_{i+2})\left[Q_1(g;I_{i+2})-Q_1(g;I_{i+1})\right]+g(t_{i+1})\left[Q_1(h;I_{i+2})-Q_1(h;I_{i+1})\right] \\
&\quad + Q_1(g;I_{i+1})\left[h(t_{i+2})-h(t_i)\right].
\end{aligned}
$$

Thus, using that g,h and $Q_1(f;\cdot,\cdot)$ are bounded (see [10]),

$$
\begin{aligned}
|A|^p &\le 2^p\Big|h(t_{i+2})\left[Q_1(g;I_{i+2})-Q_1(g;I_{i+1})\right] \\
&\qquad + g(t_{i+1})\left[Q_1(h;I_{i+2})-Q_1(h;I_{i+1})\right]\Big|^p+2^p\left|Q_1(g;I_{i+1})\left[h(t_{i+2})-h(t_i)\right]\right|^p \\
&\le 4^p\left|h(t_{i+2})\right|^p\left|Q_1(g;I_{i+2})-Q_1(g;I_{i+1})\right|^p \\
&\qquad + 4^p\left|g(t_{i+1})\right|^p\left|Q_1(h;I_{i+2})-Q_1(h;I_{i+1})\right|^p+2^p\left|Q_1(g;I_{i+1})\right|^p\left|h(t_{i+2})-h(t_i)\right|^p \\
&\le 4^p\|h\|_\infty^p\left|Q_1(g;I_{i+2})-Q_1(g;I_{i+1})\right|^p+4^p\|g\|_\infty^p\left|Q_1(h;I_{i+2})-Q_1(h;I_{i+1})\right|^p \\
&\qquad + 2^p\left|Q_1(g;I_{i+1})\right|^p\left|Q_1(h;t_{i+2},t_i)\right|^p\left|t_{i+2}-t_i\right|^p \\
&\le 4^p\|h\|_\infty^p\left|Q_1(g;I_{i+2})-Q_1(g;I_{i+1})\right|^p+4^p\|g\|_\infty^p\left|Q_1(h;I_{i+2})-Q_1(h;I_{i+1})\right|^p \\
&\qquad + 2^pK_g^pK_h^p\,p(b-a)^{p-1}\left|t_{i+2}-t_i\right|.
\end{aligned}
$$

Therefore, taking $K=K_gK_h$

$$
\begin{aligned}
&\sum_{i=0}^{n-2}\frac{\left|Q_1(gh;t_{i+2},t_{i+1})-Q_1(gh;t_{i+1},t_i)\right|^p}{\lambda_i} \\
&\le 4^p\|h\|_\infty^p\sum_{i=0}^{n-2}\frac{\left|Q_1(g;I_{i+2})-Q_1(h;I_{i+1})\right|^p}{\lambda_i} \\
&\quad + 4^p\|g\|_\infty^p\sum_{i=0}^{n-2}\frac{\left|Q_1(h;I_{i+2})-Q_1(h;I_{i+1})\right|^p}{\lambda_i}+2^pK^p\,p(b-a)^{p-1}\sum_{i=0}^{n-2}\frac{\left|t_{i+2}-t_i\right|}{\lambda_i} \\
&\le 4^p\|h\|_\infty^p\sum_{i=0}^{n-2}\frac{\left|Q_1(g;I_{i+2})-Q_1(h;I_{i+1})\right|^p}{\lambda_i}
\end{aligned}
$$

$$+4^p\|g\|_\infty^p\sum_{i=0}^{n-2}\frac{\left|Q_1\left(h;I_{i+2}\right)-Q_1\left(h;I_{i+1}\right)\right|^p}{\lambda_i}+\frac{p}{\lambda_0}2^{p+1}K^p\left(b-a\right)^p,$$

now, using that $\left(a+b\right)^{1/p}\le a^{1/p}+b^{1/p}$ with $a,b\ge 0, p\ge 1$ and property of supremum

$$\sup_\pi\left(\sum_{i=0}^{n-2}\frac{\left|Q_1\left(gh;t_{i+2},t_{i+1}\right)-Q_1\left(gh;t_{i+1},t_i\right)\right|^p}{\lambda_i}\right)^{1/p}$$

$$\le 4\|h\|_\infty\sup_\pi\left(\sum_{i=0}^{n-2}\frac{\left|Q_1\left(g;I_{i+2}\right)-Q_1\left(h;I_{i+1}\right)\right|^p}{\lambda_i}\right)^{1/p}$$

$$+4\|g\|_\infty\sup_\pi\left(\sum_{i=0}^{n-2}\frac{\left|Q_1\left(g;I_{i+2}\right)-Q_1\left(h;I_{i+1}\right)\right|^p}{\lambda_i}\right)^{1/p}+2K\left(b-a\right)\left(\frac{2p}{\lambda_0}\right)^{1/p}$$

$$=4\|h\|_\infty V_{\Lambda,2,p}\left(g;\left[a,b\right]\right)+4\|g\|_\infty V_{\Lambda,2,p}\left(h;\left[a,b\right]\right)+2K\left(b-a\right)\left(\frac{2p}{\lambda_0}\right)^{1/p}.$$

So,

$$V_{\Lambda,2,p}\left(gh;\left[a,b\right]\right)=\sup_\pi\left(\sum_{i=0}^{n-2}\frac{\left|Q_1\left(gh;t_{i+2},t_{i+1}\right)-Q_1\left(gh;t_{i+1},t_i\right)\right|^p}{\lambda_i}\right)^{1/p}<\infty$$

which implies that $gh\in\Lambda_p^2BV\left(\left[a,b\right]\right)$ and therefore the space $\Lambda_p^2BV\left(\left[a,b\right]\right)$ is an algebra.

In the next *results* we obtain, an integral representation of the space $\Lambda_p^2BV\left(\left[a,b\right]\right)$. □

Theorem 3. *If* $F\in\Lambda_pBV\left(\left[a,b\right]\right)$ *and we define* $f\left(x\right)=\int_a^x F\left(t\right)\mathrm{d}t, \forall x\in\left[a,b\right]$, *then* $f\in\Lambda_p^2BV\left(\left[a,b\right]\right)$ *and*

$$V_{\Lambda,2,p}\left(f;\left[a,b\right]\right)\le V_{\Lambda,1,p}\left(F;\left[a,b\right]\right).$$

Proof. Let $\Lambda=\left\{\lambda_i\right\}_{i\ge 0}$ a $\mathcal{W}$-sequence and define $f\left(x\right)=\int_a^x F\left(t\right)\mathrm{d}t, \forall x\in\left[a,b\right]$, where $F\in\Lambda_pBV\left(\left[a,b\right]\right)$.

Let's consider $\pi=\left\{t_i\right\}_{i=0}^n\in\Pi_3\left(\left[a,b\right]\right)$ a partition of $\left[a,b\right]$. Making

$$\begin{aligned}A&=Q_1\left(f;t_{i+2},t_{i+1}\right)-Q_1\left(f;t_{i+1},t_i\right)\\&=\frac{f\left(t_{i+2}\right)-f\left(t_{i+1}\right)}{t_{i+2}-t_{i+1}}-\frac{f\left(t_{i+1}\right)-f\left(t_i\right)}{t_{i+1}-t_i}\\&=\frac{\int_a^{t_{i+2}}F\left(t\right)\mathrm{d}t-\int_a^{t_{i+1}}F\left(t\right)\mathrm{d}t}{t_{i+2}-t_{i+1}}-\frac{\int_a^{t_{i+1}}F\left(t\right)\mathrm{d}t-\int_a^{t_i}F\left(t\right)\mathrm{d}t}{t_{i+1}-t_i}\\&=\frac{\int_{t_{i+1}}^{t_{i+2}}F\left(t\right)\mathrm{d}t}{t_{i+2}-t_{i+1}}-\frac{\int_{t_i}^{t_{i+1}}F\left(t\right)\mathrm{d}t}{t_{i+1}-t_i},\end{aligned}$$

by change of variable in each integral, for example, $s=\dfrac{t-t_{i+1}}{t_{i+2}-t_{i+1}}$ we have:

$$\begin{aligned}A&=\int_0^1F\left(t_{i+1}+s\left(t_{i+2}-t_{i+1}\right)\right)\mathrm{d}s-\int_0^1F\left(t_i+s\left(t_{i+1}-t_i\right)\right)\mathrm{d}s\\&=\int_0^1\left[F\left(t_{i+1}+s\left(t_{i+2}-t_{i+1}\right)\right)-F\left(t_i+s\left(t_{i+1}-t_i\right)\right)\right]\mathrm{d}s\end{aligned}$$

now, using the Jensen inequality to the convex function $h(t)=t^p$, $p\ge 1$: $h\left(\int_0^1 f(t)\,\mathrm{d}t\right)\le\int_0^1 h(f(t))\,\mathrm{d}t$, with $f\ge 0$, and lineality of the integral

$$\left(\sum_{i=0}^{n-2}\frac{|A|^p}{\lambda_i}\right)^{1/p}=\left(\sum_{i=0}^{n-2}\frac{\left|\int_0^1\left[F\left(t_{i+1}+s\left(t_{i+2}-t_{i+1}\right)\right)-F\left(t_i+s\left(t_{i+1}-t_i\right)\right)\right]\mathrm{d}s\right|^p}{\lambda_i}\right)^{1/p}$$

$$\le\left(\int_0^1\sum_{i=0}^{n-2}\frac{\left|F\left(t_{i+1}+s\left(t_{i+2}-t_{i+1}\right)\right)-F\left(t_i+s\left(t_{i+1}-t_i\right)\right)\right|^p}{\lambda_i}\mathrm{d}s\right)^{1/p}.$$

Fixed *s*, with $0\le s\le 1$, let's notice that

$$t_i\le t_i+s\left(t_{i+1}-t_i\right)\le t_{i+1}\le t_{i+1}+s\left(t_{i+2}-t_{i+1}\right)$$

are elements of a partition of $[a,b]$, that is,

$$\pi_s: a=t_0\le t_0+s\left(t_1-t_0\right)<\cdots<t_i+s\left(t_{i+1}-t_i\right)<\cdots<t_{n-1}+s\left(t_n-t_{n-1}\right)\le t_n=b.$$

Now,

$$\sum_{i=0}^{n-2}\frac{\left|F\left(t_{i+1}+s\left(t_{i+2}-t_{i+1}\right)\right)-F\left(t_i+s\left(t_{i+1}-t_i\right)\right)\right|^p}{\lambda_i}$$

$$\le\sup_{\pi'}\sum_{i=1}^{k}\frac{\left|F\left(x_i\right)-F\left(x_{i-1}\right)\right|^p}{\lambda_i}=V_{\Lambda,1,p}^p\left(F;[a,b]\right),$$

where $\pi': a=x_0\le x_1\le\cdots\le x_k=b$ is any partition of $[a,b]$.

Thus

$$\left(\int_0^1\sum_{i=0}^{n-2}\frac{\left|F\left(t_{i+1}+s\left(t_{i+2}-t_{i+1}\right)\right)-F\left(t_i+s\left(t_{i+1}-t_i\right)\right)\right|^p}{\lambda_i}\mathrm{d}s\right)^{1/p}$$

$$\le\left(\int_0^1 V_{\Lambda,1,p}^p\left(F;[a,b]\right)\mathrm{d}s\right)^{1/p}=V_{\Lambda,1,p}\left(F;[a,b]\right).$$

So, we obtain

$$\left(\sum_{i=0}^{n-2}\frac{\left|Q_1\left(f;t_{i+2},t_{i+1}\right)-Q_1\left(f;t_{i+1},t_i\right)\right|^p}{\lambda_i}\right)^{1/p}\le V_{\Lambda,1,p}\left(F;[a,b]\right),$$

taking supremum over all partitions of $[a,b]$ we have:

$$V_{\Lambda,2,p}\left(f;[a,b]\right)\le V_{\Lambda,1,p}\left(F;[a,b]\right).$$ □

Following the ideas of Giménez, Merentes and Rivas [12], we get the next results.

Lemma 1. *Let* $\Lambda=\{\lambda_i\}_{i=0}^{\infty}$ *a* $\mathcal{W}$*-sequence, D a dense subset of* $[a,b]$ *and let* $F:D\to\mathbb{R}$ *be a function such that there is a constant* $K>0$ *with*

$$\sum_{i=0}^{n-1}\frac{\left|F\left(t_{i+1}\right)-F\left(t_i\right)\right|^p}{\lambda_i}\le K,$$

for any finite collection $\xi: a\le t_0<t_1<\cdots<t_n\le b$ *in D. then* $F_D\left(x^+\right)$ *exists for all* $x\in[a,b)\setminus D$, *donde*

$$F_D\left(x^+\right)=\lim_{\substack{h\to 0^+\\ x+h\in D}} F\left(x+h\right).$$

An analogous result holds for $F_D\left(x^+\right)$ ($x\in[a,b)\setminus D$), *which is similarly defined.*

Proof. We will prove that $F_D\left(x^+\right)$ exists for all $x\in[a,b)\setminus D$, the case $F_D\left(x^-\right)$ is treated analogously. We will proceed by contradiction. Suppose that this is not true, that is, suppose that exists $x_0\in[a,b)\setminus D$ such that

$$\lim_{\substack{h\to 0^+\\ x_0+h\in D}} F\left(x_0+h\right)=\lim_{\substack{s\to x_0^+\\ s\in D}} F\left(s\right) \text{ does not exist.}$$

Let

$$\alpha=\limsup_{\substack{x\to x_0^+\\ x\in D}} F\left(x\right) \quad\text{and}\quad \beta=\liminf_{\substack{x\to x_0^+\\ x\in D}} F\left(x\right),$$

thus $\alpha>\beta$. Then, there exists two sequences $\left(x_n\right)_{n\in\mathbb{N}}$, $\left(y_n\right)_{n\in\mathbb{N}}$ in *D*, such that

$$x_n,y_n>x_0,\forall n\in\mathbb{N},\quad x_n\to x_0,y_n\to y_0,$$

and

$$\lim_{n\to\infty}F\left(x_n\right)=\alpha \quad\text{and}\quad \lim_{n\to\infty}F\left(y_n\right)=\beta.$$

If α and β are finite, consider $\varepsilon=\frac{\alpha-\beta}{2}>0$; otherwise we take any $\varepsilon>0$. Thus, there exist $N\in\mathbb{N}$ such that

$$n\ge N\Rightarrow\left|F\left(x_n\right)-F\left(y_n\right)\right|>\varepsilon$$

this implies that

$$\sum_{n=N+1}^{N+l}\frac{\left|F\left(x_n\right)-F\left(y_n\right)\right|^p}{\lambda_n}>\sum_{n=N+1}^{N+l}\frac{\varepsilon^p}{\lambda_n},\forall l\in\mathbb{N}$$

$$\Rightarrow K>\varepsilon^p\sum_{n=N+1}^{N+l}\frac{1}{\lambda_n},\forall l\in\mathbb{N}$$

which contradicts $\sum_{i=0}^{\infty}\frac{1}{\lambda_i}=+\infty$. □

Theorem 4. *If* $f\in\Lambda_p^2BV\left(\left[a,b\right]\right)$ *then there exists a function* $F\in\Lambda_pBV\left(\left[a,b\right]\right)$ *such that* $f\left(x\right)=\int_a^x F\left(t\right)\mathrm{d}t,\forall x\in\left[a,b\right]$, *and*

$$V_{\Lambda,2,p}\left(f;\left[a,b\right]\right)\le V_{\Lambda,1,p}\left(F;\left[a,b\right]\right)\le 2V_{\Lambda,2,p}\left(f;\left[a,b\right]\right).$$

Proof. Let $\Lambda=\left\{\lambda_i\right\}_{i\ge 0}$ a $\mathcal{W}$-sequence and $p\ge 1$. As $f\in\Lambda_p^2BV\left(\left[a,b\right]\right)$, *f* is absolutely continuos in $\left[a,b\right]$ (see Lemma 4 in [10]), which implies that *f* is strongly diferentiable a.e., with derivative strongly measurable.

Let E be a set of zero Lebesgue measure such that f' exists at every point of the set $D=\left[a,b\right]\setminus E$, then D is dense in $\left[a,b\right]$. Given $m\in\mathbb{N}$, choose $m+1$ ordered points $a\le x_0<x_1<\cdots<x_m\le b$ in *D*. Now consider $m+2$ positive numbers: $h_0,h_1,\cdots,h_m$ and ξ such that $x_m-h_m,x_{m-1}+\xi,\cdots,x_k+h_k,k=0,1,\cdots,m-1$ are in D with

$$x_0 < x_0 + h_0 < x_1 < x_1 + h_1 < \cdots < x_{m-1} + h_{m-1} < x_{m-1} + \xi < x_m - h_m < x_m.$$

Making

$$A = \sum_{i=0}^{m-2} \frac{\left|Q_1\left(f; x_{i+1} + h_{i+1}, x_{i+1}\right) - Q_1\left(f; x_i + h_i, x_i\right)\right|^p}{\lambda_i} + \frac{\left|Q_1\left(f; x_m, x_m - h_m\right) - Q_1\left(f; x_{m-1} + \xi, x_{m-1} + h_{m-1}\right)\right|^p}{\lambda_{m-1}}, \tag{6}$$

adding and subtracting appropriate terms within the absolute value in each term and using discrete Minkowski inequality:

$$\left(\sum_{k=0}^{n} \left|a_k + b_k\right|^p\right)^{1/p} \le \left(\sum_{k=0}^{n} \left|a_k\right|^p\right)^{1/p} + \left(\sum_{k=0}^{n} \left|b_k\right|^p\right)^{1/p}, \forall n \in \mathbb{N},$$

we have

$$\begin{aligned} A^{1/p} \le & \left(\sum_{i=0}^{m-2} \frac{\left|Q_1\left(f; x_{i+1} + h_{i+1}, x_{i+1}\right) - Q_1\left(f; x_{i+1}, x_i + h_i\right)\right|^p}{\lambda_i} \right. \\ & \left. + \frac{\left|Q_1\left(f; x_m, x_m - h_m\right) - Q_1\left(f; x_m - h_m, x_{m-1} + \xi\right)\right|^p}{\lambda_{m-1}}\right)^{1/p} \\ & + \left(\sum_{i=0}^{m-2} \frac{\left|Q_1\left(f; x_{i+1}, x_i + h_i\right) - Q_1\left(f; x_i + h_i, x_i\right)\right|^p}{\lambda_i} \right. \\ & \left. + \frac{\left|Q_1\left(f; x_m - h_m, x_{m-1} + \xi\right) - Q_1\left(f; x_{m-1} + \xi, x_{m-1} + h_{m-1}\right)\right|^p}{\lambda_{m-1}}\right)^{1/p} \\ \le & V_{\Lambda,2,p}(f) + V_{\Lambda,2,p}(f) = 2V_{\Lambda,2,p}(f), \end{aligned}$$

this implies that $A \le 2^p V_{\Lambda,2,p}^p(f)$.

Taking the limits in (6), as $\xi \to 0$ and $h_k \to 0, k = 0,1,\cdots,m$, and using the above inequality, we get

$$\left(\sum_{i=0}^{m-1} \frac{\left|f'(x_{i+1}) - f'(x_i)\right|^p}{\lambda_i}\right)^{1/p} \le 2V_{\Lambda,2,p}(f). \tag{7}$$

If $a = x_0$ then we obtain $f_D'(a^+)$ instead of $f'(a)$ in (7). Thus, the derivate f' satisfies the conditions of Lemma 1. Now, let us define $F:[a,b] \to \mathbb{R}$, as

$$F(x) = \begin{cases} f'(x) & \text{if } x \in D \\ f_D'(x^-) & \text{if } x \in E \setminus \{a\} \\ f_D'(a^+) & \text{if } x = a \in E. \end{cases}$$

By construction, $f' = F$ a.e., let's prove that $F \in \Lambda_p BV([a,b])$.

Let $\pi: a = t_0 < t_1 < \cdots < t_n = b$ be a partition of $[a,b]$. We need to consider several cases.

Case 1. If $t_i \in D, \forall i = 0,1,\cdots,n$ then by (7) we get

$$\left(\sum_{i=0}^{n-1}\frac{\left|F\left(t_{i+1}\right)-F\left(t_i\right)\right|^p}{\lambda_i}\right)^{1/p}\le 2V_{\Lambda,2,p}\left(f\right).$$

Case 2. Suppose that there is just one $i\in\{1,2,\cdots,n\}$ such that $t_i\notin D$. Let's choose $s_i\in D$ with $t_{i-1}<s_i<t_i$. Let's consider now, $\pi_1: a=t_0<t_1<\cdots<t_{i-1}<s_i<t_{i+1}<\cdots<t_n=b$ a new partition where all points are in D. So, by the case 1,

$$\sum_{j=0}^{i-2}\frac{\left|F\left(t_{j+1}\right)-F\left(t_j\right)\right|^p}{\lambda_j}+\frac{\left|F\left(s_i\right)-F\left(t_{i-1}\right)\right|^p}{\lambda_{i-1}}$$

$$+\frac{\left|F\left(t_{i+1}\right)-F\left(s_i\right)\right|^p}{\lambda_i}+\sum_{j=i+1}^{n-1}\frac{\left|F\left(t_{j+1}\right)-F\left(t_j\right)\right|^p}{\lambda_j}\le 2V_{\Lambda,2,p}\left(f\right),$$

taking limit as $s_i\to t_i^-, s_i\in D$, we have $F\left(s_i\right)=f'\left(s_i\right)\to f'\left(t_i^-\right)=F\left(t_i\right)$. Thus,

$$\frac{\left|F\left(s_i\right)-F\left(t_{i-1}\right)\right|^p}{\lambda_{i-1}}\to\frac{\left|F\left(t_i\right)-F\left(t_{i-1}\right)\right|^p}{\lambda_{i-1}}\quad\text{and}$$

$$\frac{\left|F\left(t_{i+1}\right)-F\left(s_i\right)\right|^p}{\lambda_i}\to\frac{\left|F\left(t_{i+1}\right)-F\left(t_i\right)\right|^p}{\lambda_i}$$

therefore

$$\left(\sum_{j=0}^{n-1}\frac{\left|F\left(t_{i+1}\right)-F\left(t_i\right)\right|^p}{\lambda_i}\right)^{1/p}\le 2V_{\Lambda,2,p}\left(f\right).$$

Case 3. If exists *j* elements in $\{t_1,t_2,\cdots,t_n\}$ that are not in *D*, we proceed as in case 2, *j* times to get

$$\left(\sum_{j=0}^{n-1}\frac{\left|F\left(t_{i+1}\right)-F\left(t_i\right)\right|^p}{\lambda_i}\right)^{1/p}\le 2V_{\Lambda,2,p}\left(f\right).$$

Case 4. If exists *j* elements in $\{t_0,t_1,\cdots,t_{n-1}\}$ that are not in *D*, we proceed completely analogously as in cases 2 and 3, to we get

$$\left(\sum_{j=0}^{n-1}\frac{\left|F\left(t_{i+1}\right)-F\left(t_i\right)\right|^p}{\lambda_i}\right)^{1/p}\le 2V_{\Lambda,2,p}\left(f\right).$$

In any case, we conclude that

$$F\in\Lambda_pBV\left(\left[a,b\right]\right)\quad\text{and}\quad V_{\Lambda,1,p}\left(F;\left[a,b\right]\right)\le 2V_{\Lambda,2,p}\left(f;\left[a,b\right]\right).$$

By the result of the Theorem 3,

$$V_{\Lambda,2,p}\left(f;\left[a,b\right]\right)\le V_{\Lambda,1,p}\left(F;\left[a,b\right]\right)\le 2V_{\Lambda,2,p}\left(f;\left[a,b\right]\right).\qquad\square$$

In the following result we give necessary conditions for that the composition of functions belongs to the space $\Lambda_p^2BV\left(\left[a,b\right]\right)$.

Recall that, under certain conditions, the composition of two functions *f* and *g*, denoted by $f\circ g$, is defined as $\left(f\circ g\right)\left(x\right)=f\left(g\left(x\right)\right)$.

Theorem 5. *If* $g \in \Lambda_p^2 BV([a,b])$ *is a strictly increasing function and* $f \in \Lambda_p^2 BV([g(a), g(b)])$ *then*

$$f \circ g \in \Lambda_p^2 BV([a,b]).$$

Proof. Let $\Lambda = \{\lambda_i\}_{i \ge 0}$ be a $\mathcal{W}$-sequence, $\pi = \{t_i\}_{i=0}^n \in \Pi_3([a,b])$ a partition of $[a,b]$. Then

$$\frac{|Q_1(f \circ g; t_{i+2}, t_{i+1}) - Q_1(f \circ g; t_{i+1}, t_i)|^p}{\lambda_i}$$

$$= \frac{\left| \frac{f(g(t_{i+2})) - f(g(t_{i+1}))}{t_{i+2} - t_{i+1}} - \frac{f(g(t_{i+1})) - f(g(t_i))}{t_{i+1} - t_i} \right|^p}{\lambda_i}$$

$$= \frac{\left| \frac{f(g(t_{i+2})) - f(g(t_{i+1}))}{g(t_{i+2}) - g(t_{i+1})} \times \frac{g(t_{i+2}) - g(t_{i+1})}{t_{i+2} - t_{i+1}} - \frac{f(g(t_{i+1})) - f(g(t_i))}{g(t_{i+1}) - g(t_i)} \times \frac{g(t_{i+1}) - g(t_i)}{t_{i+1} - t_i} \right|^p}{\lambda_i},$$

by adding and subtracting the term:

$$\frac{f(g(t_{i+2})) - f(g(t_{i+1}))}{g(t_{i+2}) - g(t_{i+1})} \times \frac{g(t_{i+1}) - g(t_i)}{t_{i+1} - t_i},$$

we can group and get the following:

$$\frac{f(g(t_{i+2})) - f(g(t_{i+1}))}{g(t_{i+2}) - g(t_{i+1})} \times \left[\frac{g(t_{i+2}) - g(t_{i+1})}{t_{i+2} - t_{i+1}} - \frac{g(t_{i+1}) - g(t_i)}{t_{i+1} - t_i} \right]$$
$$+ \frac{g(t_{i+1}) - g(t_i)}{t_{i+1} - t_i} \times \left[\frac{f(g(t_{i+2})) - f(g(t_{i+1}))}{g(t_{i+2}) - g(t_{i+1})} - \frac{f(g(t_{i+1})) - f(g(t_i))}{g(t_{i+1}) - g(t_i)} \right],$$

so, using the inequality $|x+y|^p \le 2^p \left(|x|^p + |y|^p \right)$ we have that:

$$\frac{|Q_1(f \circ g; t_{i+2}, t_{i+1}) - Q_1(f \circ g; t_{i+1}, t_i)|^p}{\lambda_i}$$

$$\le 2^p \left| Q_1(f; g(t_{i+2}), g(t_{i+1})) \right|^p \times \frac{\left| \frac{g(t_{i+2}) - g(t_{i+1})}{t_{i+2} - t_{i+1}} - \frac{g(t_{i+1}) - g(t_i)}{t_{i+1} - t_i} \right|^p}{\lambda_i}$$
$$+ 2^p \left| Q_1(g; t_{i+1}, t_i) \right|^p \times \frac{\left| \frac{f(g(t_{i+2})) - f(g(t_{i+1}))}{g(t_{i+2}) - g(t_{i+1})} - \frac{f(g(t_{i+1})) - f(g(t_i))}{g(t_{i+1}) - g(t_i)} \right|^p}{\lambda_i}$$

$$\le 2^p (M_1)^p \times \frac{\left| \frac{g(t_{i+2}) - g(t_{i+1})}{t_{i+2} - t_{i+1}} - \frac{g(t_{i+1}) - g(t_i)}{t_{i+1} - t_i} \right|^p}{\lambda_i}$$
$$+ 2^p (M_2)^p \times \frac{\left| \frac{f(g(t_{i+2})) - f(g(t_{i+1}))}{g(t_{i+2}) - g(t_{i+1})} - \frac{f(g(t_{i+1})) - f(g(t_i))}{g(t_{i+1}) - g(t_i)} \right|^p}{\lambda_i},$$

where, $M_1 = \sup\{Q_1(f;\beta,\alpha);\alpha,\beta\in[a,b]\}$ y $M_2 = \sup\{Q_1(g;\beta,\alpha);\alpha,\beta\in[a,b]\}$, are finite, see Lemma 3 in [10].

This implies that

$$\left(\sum_{i=0}^{n-2}\frac{\left|Q_1(f\circ g;t_{i+2},t_{i+1})-Q_1(f\circ g;t_{i+1},t_i)\right|^p}{\lambda_i}\right)^{1/p}$$

$$\le 2M_1\times\left(\sum_{i=0}^{n-2}\frac{\left|\frac{g(t_{i+2})-g(t_{i+1})}{t_{i+2}-t_{i+1}}-\frac{g(t_{i+1})-g(t_i)}{t_{i+1}-t_i}\right|^p}{\lambda_i}\right)^{1/p}$$

$$+2M_2\times\left(\sum_{i=0}^{n-2}\frac{\left|\frac{f(g(t_{i+2}))-f(g(t_{i+1}))}{g(t_{i+2})-g(t_{i+1})}-\frac{f(g(t_{i+1}))-f(g(t_i))}{g(t_{i+1})-g(t_i)}\right|^p}{\lambda_i}\right)^{1/p},$$

thus,

$$V_{\Lambda,2,p}(f\circ g;[a,b])\le 2M_1V_{\Lambda,2,p}(g;[a,b])+2M_2V_{\Lambda,2,p}(f;[g(a),g(b)))<\infty$$

hence

$$f\circ g\in\Lambda_p^2BV([a,b]).$$

□

3. Conclusions

In this article, we proved important results in the space of functions of bounded second variation in the sense of Shiba, we study the structure of the space and we obtained that this is a Banach algebra, we also proved an integral representation theorem and we obtained necessary conditions for that the space be closed under composition of functions.

Finally, these results characterize the space of the second bounded variation and are indispensable to initiate other investigations, such as the study of the nonlinear integral equations used for example to describe physical phenomena and dynamic models of chemical reactors, as well as to establish compactness arguments in the study of equations system solutions applied to polymer models.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Jordan, C. (1881) Sur la serie de Fourier. *Comptes rendus de l'Académie des Sciences Paris*, **2**, 228-230.

[2] Lakatos, I. (1976) Proofs and Refutations, The Logic of Mathematical Discovery.

Cambridge University Press. https://doi.org/10.1017/CBO9781139171472

[3] Appell, J., Banas, J. and Merentes, N. (2014) Bounded Variation and Around, De Gruyter, Boston, Mass, USA. https://doi.org/10.1515/9783110265118

[4] Adispavic, M. (1986) Concept of Generalized Bounded Variation and the Theory of Fourier Series. *International Journal of Mathematics and Mathematical Sciences*, **9**, 223-244. https://doi.org/10.1155/S0161171286000285

[5] Wiener, N. (1924) The Quadratic Variation of a Function and Its Fourier Coefficients. *Journal of Mathematics and Physics Massachusetts*, **3**, 72-94. https://doi.org/10.1002/sapm19243272

[6] Young, L.C. (1937) Sur une généralisation de la notion de variation de pussance piéme bornée au sens de M. Wiener, et sur la convergence des séries de Fourier. *Comptes rendus de l'Académie des Sciences Paris*, **204**, 470-472.

[7] Waterman, D. (1972) On the Convergence of Fourier Series of Functions of Bounded Variation. *Studia Mathematica*, **44**, 107-117. https://doi.org/10.4064/sm-44-2-107-117

[8] Shiba, M. (1980) On the Absolute Convergence of Fourier Series of Functions Class $\Lambda_p BV$. *Sci. Rep. Fukushima Univ No.* 30, 7-10.

[9] Vyas, R.G. (2006) Properties of Functions of Generalized Bounded Variation. *Matematički Vesnik*, **58**, 91-96.

[10] Giménez, J., Merentes, N., Pineda, E. and Rodríguez, L. (2016) Uniformly Bounded Superposition Operators in the Space of Second Bounded Variation Functions in the Sense of Shiba. *Publicaciones en Ciencias y Tecnología*, **10**, 49-58.

[11] De La Vallée Poussin, Ch.J. (1908) Sur l'intégrale de Lebesgue. *Transactions of the American Mathematical Society*, **16**, 435-501. https://doi.org/10.1090/S0002-9947-1915-1501024-5

[12] Giménez, J., Merentes, N. and Rivas, S. (2012) Integral Representation of Functions of Bounded Second Φ-Variation in the Sense of Schramm. *Opuscula Mathematica*, **32**, 137-151. https://doi.org/10.7494/OpMath.2012.32.1.137