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Block-Transitive $4-(v,k,4)$ Designs and Ree Groups

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Abstract

This article is a contribution to the study of the automorphism groups of $4-(v,k,\lambda)$ designs. Let $\mathcal{S}=(\mathcal{P},\mathcal{B})$ be a non-trivial $4-(q^3+1,k,4)$ design where $q=3^{2n+1}$ for some positive integer $n\geq 1$, and $G\leq Aut(\mathcal{S})$ is block-transitive. If the socle of G is isomorphic to the simple groups of lie type $^2G_2(q)$, then G is not flag-transitive.

Keywords

Flag-Transitive, Block-Transitive, *t*-Design, Ree Group

1. Introduction

For positive integers $t\leq k\leq v$ and λ, we define a $t-(v,k,\lambda)$ design to be a finite incidence structure $\mathcal{S}=(\mathcal{P},\mathcal{B})$, where $\mathcal{P}$ denotes a set of points, $|\mathcal{P}|=v$, and $\mathcal{B}$ a set of blocks, $|\mathcal{B}|=b$, with the properties that each block is incident with *k* points, and each *t*-subset of $\mathcal{P}$ is incident with λ blocks. A flag of $\mathcal{S}$ is an incident point-block pair (x,B) with *x* is incident with *B*, where $B\in\mathcal{B}$. We consider automorphisms of $\mathcal{S}$ as pairs of permutations on $\mathcal{P}$ and $\mathcal{B}$ which preserve incidence structure. We call a group $G\leq Aut(\mathcal{S})$ of automorphisms of $\mathcal{S}$ flag-transitive (respectively block-transitive, point *t*-transitive, point *t*-homogeneous) if *G* acts transitively on the flags (respectively transitively on the blocks, *t*-transitively on the points, *t*-homogeneously on the points) of $\mathcal{S}$. For short, $\mathcal{S}$ is said to be, e.g., flag-transitive if $\mathcal{S}$ admits a flag-transitive group of automorphisms.

For historical reasons, a $t-(v,k,\lambda)$ design with $\lambda=1$ is called a Steiner *t*-design (sometimes this is also known as a Steiner system). If $t<k<v$ holds, then we speak of a non-trivial Steiner *t*-designs.

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Investigating t-designs for arbitrary λ, but large t, Cameron and Praeger proved the following result:

Theorem 1. ([1]) *Let* $\mathcal{S}=(\mathcal{P},\mathcal{B})$ *be a* $t-(v,k,\lambda)$ *design. If* $G\le Aut(\mathcal{S})$ *acts block-transitively on* $\mathcal{S}$, *then* $t\le 7$, *while if* $G\le Aut(\mathcal{S})$ *acts flag-transitively on* $\mathcal{S}$, *then* $t\le 6$.

Recently, Huber (see [2]) completely classified all flag-transitive Steiner t-designs using the classification of the finite 2-transitive permutation groups. Hence the determination of all flag-transitive and block-transitive t-designs with $\lambda\ge 2$ has remained of particular interest and has been known as a long-standing and still open problem.

The present paper continues the work of classifying block-transitive t-designs. We discuss the block-transitive $4-(v,k,4)$ designs and Ree groups. We get the following result:

Main Theorem. Let $\mathcal{S}=(\mathcal{P},\mathcal{B})$ be a non-trivial $4-(q^3+1,k,4)$ design, where $q=3^{2n+1}$ for some positive integer $n\ge 1$, and $G\le Aut(\mathcal{S})$ is block-transitive. If $Soc(G)$, the socle of G, is ${}^2G_2(q)$, then G is not flag-transitive.

The second section describes the definitions and contains several preliminary results about flag-transitivity and t-designs. In 3 Section, we give the proof of the Main Theorem.

2. Preliminary Results

The Ree groups ${}^2G_2(q)$ form an infinite family of simple groups of Lie type, and were defined in [3] as subgroups of $GL(7,q)$. Let $GF(q)$ be finite field of q elements, where $q=3^{2n+1}$ for some positive integer $n\ge 1$ (in particular, $q\ge 27$). Let Q is a Sylow 3-subgroup of G, K is a multiplicative group of $GF(q)$ and ${}^2G_2(q)$ is a group of order $q^3(q^3+1)(q-1)$ (see [4]-[6]). Hence ${}^2G_2(q)$ is a group of automorphisms of Steiner $3-(q^3+1,q+1,1)$ design and acts 2-transitive on q^3+1 points (see [7]).

Here we gather notation which are used throughout this paper. For a t-design $\mathcal{S}=(\mathcal{P},\mathcal{B})$ with $G\le Aut(\mathcal{S})$, let r denotes the number of blocks through a given point, G_x denotes the stabilizer of a point $x\in\mathcal{P}$ and G_B the setwise stabilizer of a block $B\in\mathcal{B}$. We define $G_{xB}=G_x\cap G_B$. For integers m and n, let (m,n) denotes the greatest common divisor of m and n, and $m\mid n$ if m divides n.

Lemma 1. ([2]) *Let G act flag-transitively on* $t-(v,k,\lambda)$ *design* $\mathcal{S}=(\mathcal{P},\mathcal{B})$. *Then G is block-transitive and the following cases hold*:

1) $|G|=|G_x||x^G|=|G_x|v$, where $x\in\mathcal{P}$;

2) $|G|=|G_B||B^G|=|G_B|b$, where $B\in\mathcal{B}$;

3) $|G|=|G_{xB}||(x,B)^G|=|G_{xB}|bk$, where $x\in B$.

Lemma 2. ([8]) *Let* $\mathcal{S}=(\mathcal{P},\mathcal{B})$ *is a non-trivial* $t-(v,k,\lambda)$ *design. Then*

$$\lambda(v-t+1)\ge(k-t+2)(k-t+1).$$

Lemma 3. ([8]) *Let* $\mathcal{S}=(\mathcal{P},\mathcal{B})$ *is a non-trivial* $4-(v,k,\lambda)$ *design. Then*

1) $bk=vr$;

2) $b=\dfrac{\lambda v(v-1)(v-2)(v-3)}{k(k-1)(k-2)(k-3)}$.

Corollary 1. *Let* $\mathcal{S}=(\mathcal{P},\mathcal{B})$ *is a non-trivial* $4-(v,k,4)$ *design. If* $v=q^3+1$, *Then* $k<3+2q\sqrt{q}$.

Proof. By Lemma 2, we have $4(v-3)\ge(k-2)(k-3)$. If $v=q^3+1$, then

$$4(q^3-2)\ge(k-2)(k-3).$$

Hence

$$k^2-5k-4q^3+14\le 0.$$

We get

$$k\le\frac{5+\sqrt{16q^3-31}}{2}<3+2q\sqrt{q}.$$

3. Proof of the Main Theorem

Suppose that G acts flag-transitively on $4-(v,k,4)$ design and $v=q^3+1$. Then G is block-transitive and point-transitive. Since $T={}^2G_2(q)\trianglelefteq G\le Aut(T)$, we may assume that $G=T:\langle\alpha\rangle$ and $G=T:(G\cap\langle\alpha\rangle)$ by Dedekind's theorem, where $\alpha: x\to x^3$, $x\in GF(q)$ and α is an automorphism of field $GF(q)$. Let $q=3^f$, $f=2n+1$ is odd, and $|\langle\alpha\rangle|=m$, then $m\mid f$. Obviously, $|G|=q^3(q^3+1)(q-1)m$.

First, we will proof that if $g\in G$ fixes three different points of $\mathcal{P}$, then g must fix at least four points in $\mathcal{P}$.

Suppose that $g\in G$, $|Fix_{\mathcal{P}}(g)|\ge 3$, $x\in Fix_{\mathcal{P}}(g)$. Let P is a normal Sylow 3-subgroup of G_x. Then $\mathcal{P}$ is transitive on $\mathcal{P}-\{x\}$. By $v=q^3+1$, we have $|P|=|\mathcal{P}-\{x\}|=q^3$. Hence P acts regularly on $\mathcal{P}-\{x\}$. There exist $h\in P$ such that $z=y^h$, where for all $y,z\in\mathcal{P}-\{x\}$. Since $g\in G_x$, $h\in P$ and P is a normal Sylow 3-subgroup of G_x, we have $h^{-1}ghg^{-1}\in P$. On the other hand,

$$z^{h^{-1}ghg^{-1}}=y^{ghg^{-1}}=y^{hg^{-1}}=z^{g^{-1}}=z.$$

So $h^{-1}ghg^{-1}=1$, that is $gh=hg$. Hence $h\in C=C_P(g)$. We get that C is transitive on $Fix_{\mathcal{P}}(g)-\{x\}$. Hence $|Fix_{\mathcal{P}}(g)-\{x\}|\,\big|\,|C|$. By $C\le P$, we have $|Fix_{\mathcal{P}}(g)-\{x\}|\,\big|\,|P|$. Note that $|P|=q^3=3^{3f}$, so $|Fix_{\mathcal{P}}(g)-\{x\}|\,\big|\,3^{3f}$. Hence $|Fix_{\mathcal{P}}(g)-\{x\}|\equiv 1(\mathrm{mod}\,2)$. It follows that $|Fix_{\mathcal{P}}(g)|\equiv 0(\mathrm{mod}\,2)$. This means that g must fix at least four points in $\mathcal{P}$.

Now, we can continue to prove our main theorem. Obviously, α fixes three points of $\mathcal{P}$ which are $0,1,\infty$. Then $\langle\alpha\rangle\le G_{0,1,\infty}$. Hence α must fix at least five points in $\mathcal{P}$. Since G acts block-transitively on $4-(v,k,4)$ design, we can find four blocks, let B_1, B_2, B_3 and B_4, containing four points which is fixed by α. If α exchange B_1, B_2, B_3 and B_4, then $2\,\big|\,|\langle\alpha\rangle|$ which is impossible. Thus α must fix B_1, B_2, B_3 and B_4. We have $G\cap\langle\alpha\rangle\le G_{0B_1}=G_{0B_2}=G_{0B_3}=G_{0B_4}$. Therefore T acts also flag-transitively on $4-(q^3+1,k,4)$ design. We may assume $G=T$ and $|G|=q^3(q^3+1)(q-1)$.

Since G acts flag-transitively on $4-(q^3+1,k,4)$ design, then G is point-transitive. By Lemma 1(1), we get

$$|G_x|=\frac{|G|}{v}=\frac{q^3(q^3+1)(q-1)}{q^3+1}=q^3(q-1).$$

Again by Lemma 3(2) and Lemma 1(3),

$$b=\frac{4v(v-1)(v-2)(v-3)}{k(k-1)(k-2)(k-3)}=\frac{v|G_x|}{k|G_{xB}|}.$$

Thus

$$|G_{xB}|=\frac{(k-1)(k-2)(k-3)|G_x|}{4(v-1)(v-2)(v-3)}=\frac{(k-1)(k-2)(k-3)q^3(q-1)}{4q^3(q^3-1)(q^3-2)}=\frac{(k-1)(k-2)(k-3)}{4(q^2+q+1)(q^3-2)}.$$

By Lemma 2,

$$4|G_{xB}|(q^2+q+1)(q^3-2)=(k-1)(k-2)(k-3)\le(k-1)\cdot 4(v-3)=4(k-1)(q^3-2),$$

Again by Corollary 1,

$$1\le|G_{xB}|\le\frac{k-1}{q^2+q+1}\le\frac{2+2q\sqrt{q}}{q^2+q+1}<1\,(q\ge 27),$$

This is impossible.

This completes the proof the Main Theorem.

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