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Bosonization Approach and Novel Traveling Wave Solutions of the Superfield Gardner Equation

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Abstract

In this paper, the bosonization of the superfield Gardner equation in the case of multifermionic parameters is presented and novel traveling wave solutions are extracted from the coupled bosonic equations by using the mapping and deformation relations. In the case of two-fermionic-parameter bosonization procedure, we provide a special solution in the form of Jacobian elliptic functions. Meanwhile, we discuss and formally derive traveling wave solutions of N fermionic parameters bosonization procedure. This technique can also be applied to treat the $N = 1$ supersymmetry KdV and mKdV systems which are obtained in two limiting cases.

Keywords

Supersymmetry, Superfield Gardner Equation, Bosonizaion, Traveling Wave Solutions

1. Introduction

The supersymmetry (SUSY), applied to treat fermions and bosons in a unified way in elementary particle physics since the concept first arose in 1971 by Ramond, Golfand and Likhtman, has been researched extensively during past four decades [1]-[6]. The starting point of SUSY is the supersymmetric versions of well known KdV equation first by Kupershmidt in 1984 (a simple fermionic but not supersymmetric extension) and later found independently of the work of Manin-Radual on super KP hierarchy [7] [8] [9] [10]. It was pointed out afterwards that the latter is indeed a truly $N = 1$ (N refers to the number of super-symmetries, for $N = 1$ standard, for $N > 1$ extended) sKdV equation which is invariant under supersymmetric transformation [8]. Since then, various properties have been established for its supersymmetric versions, such as Lax representation, bi-Hamiltonian structures, Backlund transformation (BT), Painleve analysis, N-soliton solutions, (non) local conserved quantities, etc. [9]-[15].

The integrability of sKdV equation can be established in the way to supersymmetrize the unique Gardner transformation. For the well known KdV equation

$$u_t - 6uu_x + u_{xxx} = 0 \tag{1}$$

we extend the classical spacetime x,t to a super-spacetime x,t,θ, where θ is a Grassmannian odd number $\theta^2 = 0$. Now we write the $N = 1$ supersymmetric KdV (sKdV) equation accompanied with a fermionic super-variable $\Phi = \Phi(x,t,\theta)$ under the compact form

$$\Phi_t - 3D^2(\Phi D\Phi) + D^6\Phi = 0 \tag{2}$$

Here the covariant super-derivative D is defined by $D = \partial_\theta + \theta\partial_x$. Mathieu found that a unique extension

$$\Phi = \chi + \varepsilon\chi_x + \varepsilon^2\chi(D\chi) \tag{3}$$

of establishing the integrability maps a solution of the superfield Gardner equation

$$\chi_t - 3D^2(\chi D\chi) - 3\varepsilon^2(D\chi)D^2(\chi D\chi) + D^6\chi = 0 \tag{4}$$

into a solution of the sKdV equation [8] [9] [15]. The Gardner equation is also called the extended KdV equation with the variable-sign cubic non-linear term or the combined KdV and mKdV (KdV-mKdV) equation. It is widely used in various branches of physics, such as plasma physics, fluid physics, nonlinear phenomena and quantum field theory, etc., and it also describes a variety of wave phenomena in plasma and solid state [16] [17] [18] [19] [20].

This map was also used to recover an infinite number of conservation laws for the sKdV equation, and construct interesting BT [6]. It is easy to show that such super equation is invariant under the supersymmetry transformation: $x \to x - \eta\theta$, $\theta \to \theta + \eta$ (η is an anticommuting parameter.). The component form of the above equation with the superfield $\chi = \xi + \theta u$ can be rewritten as

$$u_t + u_{xxx} - 6uu_x + 3\xi\xi_{xx} - \varepsilon^2\left[6u^2u_x - 3(\xi\xi_x u)_x\right] = 0 \tag{5a}$$

$$\xi_t + \xi_{xxx} - 3(\xi u)_x - 3\varepsilon^2 u(\xi u)_x = 0 \tag{5b}$$

Note that ξ and u are new setted fermionic and bosonic functions, the usual Gardner equation is recovered by setting the fermionic variable to be absent and the sKdV equation is the limiting case where $\varepsilon \to 0$.

Nonlinear partial differential equations play an important role in nonlinear physics, even nonlinear science. Various effective methods have been proposed to derive explicit or formal solutions. Recently, a simple but powerful bosonization approach, which main idea is to consider fields in a Grassmannian algebra and rewrite a system in a basis of this algebra to arrive at a system of ordinary (commutative) evolution equations, can effectively simplify such systems containing anti-communicating fermionic fields [7] [21] [22]. In [23], B. Ren *et al.* used this approach in the $N = 1$ supersymmetric Burgers (SB) system, and the exact solutions of the usual pure bosonic systems are obtained with the mapping and deformation method and Lie point symmetries theory. In [24], the Lie point symmetries of the supersymmetric KdV-a system are considered and similarity reductions of it are conducted. Several types of similarity reduction solutions of the coupled bosonic equations are also simply obtained. The motivation and purpose of this paper is to show this procedure and outcome of the method by taking the superfield Gardner equation and to acquire novel traveling wave solutions of this equation.

The rest of this paper is organized as follows. Section 2 and 3 are brief reminders of fairly basic illustrations of the bosonization approach of the superfield Gardner equation with two and three fermionic parameters. In Section 4 we present the N fermionic parameters bosonization case. In Section 5 we give the $N = 1$ supersymmetric KdV and mKdV equations using parallel procedure in two particular cases, and we will also give a short summary.

2. Two-Fermionic-Parameter Bosonization

To get direct comprehension and fixed notations of the bosonizaion approach for superfield Gardner equation with multi-fermionic parameters, we first concentrate on a linear space $G(V)$. Mathematically, such uncomplicated method used for vanishing fermionic fields is based on direct sum of superspace $G(V) = \Lambda_0 \oplus \Lambda_1$, where Λ_0 and Λ_1 represents subspace containing even and odd elements, respectively [25]. Here and in the following we omit the exterior algebra sign $\wedge$ and denote it briefly by ordinary multiplication. Moreover, relevant solutions may involve rich (super) symmetries physically. An exact example is the cases of two and three fermionic parameters bosonization, thereby we can directly derive the usual Gardner equation and coupled equations appear below. For the case of two fermionic parameters θ_1 and θ_2 with $\theta_1^2 = \theta_2^2 = 0$, let the two component fields u and ξ be expanded as

$$u = u_0 + u_1\theta_1\theta_2, \quad \xi = v_1\theta_1 + v_2\theta_2, \tag{6}$$

here $u_i = u_i(x,t)(i = 0,1)$, $v_i = v_i(x,t)(i = 1,2)$ are all usual bosonic functions with respect to spacetime variable x and t, thus we get nonlinear Partial Differential Equations (PDEs) in the component form by using (5)

$$u_{0t} + u_{0xxx} - 6u_0u_{0x} f(u_0) = 0 \tag{7a}$$

$$v_{it} + v_{ixxx} - 3(v_iu_0)_x f(u_0) = 0 \tag{7b}$$

$$u_{1t} + u_{1xxx} - 6(u_0u_1)_x f(u_0) - 6\varepsilon^2 u_0u_{0x}u_1 + F_1 = 0 \tag{7c}$$

where $f(u_0) = 1 + \varepsilon^2 u_0$ and $F_1 = 3\left[f(u_0)(v_1v_{2x} - v_2v_{1x})\right]_x$.

Next we introduce the traveling wave variable $X = kx + \omega t + x_0$ along with the constants of wavenumber k, angular frequency ω and phase x_0, therefore, above equations would be changed to a system consisting of ordinary differential equations(ODEs):

$$\omega u_{0X} + k^3 u_{0XXX} - 6ku_0 u_{0X} f(u_0) = 0 \tag{8a}$$

$$\omega v_{iX} + k^3 v_{iXXX} - 3k(v_i u_0)_X f(u_0) = 0 \, (i = 1,2) \tag{8b}$$

$$\omega u_{1X} + k^3 u_{1XXX} - 6k(u_0 u_1)_X f(u_0) - 6\varepsilon^2 k u_0 u_{0X} u_1 + F_1(X) = 0. \tag{8c}$$

Note that we denote $F_1(X) = 3k^2 \left[f(u_0)(v_1 v_{2X} - v_2 v_{1X}) \right]_X$ in here.

The traveling waves we discuss are only in the usual spacetime x,t but not in the super-spacetime x,t,θ, for example, $\chi(x,t,\theta) = \chi(X + \theta\zeta)$ with Grassmannian constant ζ is different from those in the usual spacetime. In addition to a directly integrable ODE in u_0, the solution of the residual system which are related to third-order linear (non)homogeneous ODEs in u_1, v_1 and v_2, can be obtained through the variable transformation from ordinary coordinates space to phase space on the base of periodic wave solutions of usual Gardner equation. We first solve out u_{0X} from Equation (8a), and the result reads

$$u_{0X} = \frac{a_0}{k^2}\sqrt{k\lambda(z)}, \quad \lambda(z) = k\varepsilon^2 z^4 + 2kz^3 - \omega z^2 - 2C_1 z - C_2 \tag{9}$$

where $a_0^2 = 1$, C_1 and C_2 are two arbitrary integral constants, here and below the new variable z stands for function u_0.

To get the mapping relations of u_1, v_1 and v_2, we introduce the variable transformations as follows $u_1(X) = P_1(u_0)$, $v_1(X) = Q_1(u_0)$ and $v_2(X) = Q_2(u_0(X))$. Applying the transformation via Equation (8a), the linear ODEs (8b)-(8c) are reduced to mapping and deformation relations between the traveling wave solutions of the classical Gardner equation and its supersymmetric equation by exploiting the known solutions of classical Gardner equation

$$K_e(P_1) + R_1(z) = 0 \tag{10a}$$

$$K_o(Q_i) = 0, \, i = 1,2 \tag{10b}$$

where linear operators read

$$K_e = \lambda(z)\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \left(2k\varepsilon^2 z^3 + 3kz^2 - \omega z - C_1\right)\frac{\mathrm{d}}{\mathrm{d}z} + \left(\omega - 6k\varepsilon^2 z^2 - 6kz\right) \tag{11a}$$

$$K_o = \lambda(z)\frac{\mathrm{d}^3}{\mathrm{d}z^3} + \left(6k\varepsilon^2 z^3 + 9kz^2 - 3\omega z - 3C_1\right)\frac{\mathrm{d}^2}{\mathrm{d}z^2} + 3kzf(z)\frac{\mathrm{d}}{\mathrm{d}z} - 3kf(z) \tag{11b}$$

and the nonhomogeneous term $R_1(z)$ is

$$R_1(z) = -3a_0 f(z)\left(Q_1\frac{\mathrm{d}Q_2}{\mathrm{d}z} - Q_2\frac{\mathrm{d}Q_1}{\mathrm{d}z}\right)\sqrt{k\lambda(z)} - A_{1,1}. \tag{12}$$

while $A_{1,1}$ is just an arbitrary integral constant. On this basis, the mapping and deformation relations are obtained as

$$P_1 = \sqrt{\lambda(z)}\left[A_{1,3} + \int^z \frac{A_{1,2} + \int^y R_1(y_1)\mathrm{d}y_1}{\lambda(y)^{\frac{3}{2}}}\mathrm{d}y\right] \tag{13a}$$

$$Q_i = B_{i,1}z + B_{i,2}\sqrt{\eta(z)}\sin\left[H(z) + B_{i,3}\right], i = 1,2. \tag{13b}$$

where $A_{1,2}$ and $B_{i,j}(i, j = 1,2)$ are arbitrary constants, and
$H(z) = \int^z \frac{\tau f(y)}{\eta(y)\sqrt{\lambda(y)}}\mathrm{d}y$ with auxiliary function
$\eta(z) = -\tau_1 + \varepsilon^2\omega z^2 + kz^2 - \varepsilon^4 C_1 z^2$. Coefficients for used are defined as
$\tau_1 = C_1 - \varepsilon^2 C_2$, $\tau_2 = -\varepsilon^2 C_1^2 + kC_2 + \omega C_1$ and $\tau = \sqrt{\tau_1\tau_2}$.

Thus, we have constructed the general two-fermionic-parameter traveling wave solutions of the supersymmetric version of Gardner system

$$u = u_0 + \sqrt{\lambda(u_0)}\left[A_{1,3} + \int^{u_0} \frac{A_{1,2} + \int^y R_1(y_1)\mathrm{d}y_1}{\lambda(y)^{\frac{3}{2}}}\mathrm{d}y\right]\theta_1\theta_2 \tag{14a}$$

$$\xi = \sum_{i=1}^{2}\left\{B_{i,1}u_0 + B_{i,2}\sqrt{\eta(u_0)}\sin\left[H(u_0) + B_{i,3}\right]\right\}\theta_i \tag{14b}$$

with the known solution u_0 or Equation (8a) of the usual Gardner equation which have been extensively studied in many literatures [18] [26]. For a special case which takes $B_{i,2} = B_{i,3} = 0\ (i = 1,2)$ to eliminate nonhomogeneous terms, the above traveling wave solution becomes

$$\chi = B_{1,1}u_0\theta_1 + B_{2,1}u_0\theta_2 + \theta\left(u_0 + A_{1,2}u_{0,X}\theta_1\theta_2\right) \tag{15}$$

It is interesting to see that the expression (9) is a trivial type of the symmetries or conservation quantity of standard Gardner equation
$\sigma = A_1 u_{0,\alpha}(\alpha = x,t)$; $T = \sigma$, $\rho = \sigma_{xx} - 6u_0\sigma f(u_0)$. In fact, for any given $u_0(x,t)$ being a solution of the usual Gardner equation, a certain type of solutions of the bosonic-looking equation can be constructed as follows: $v_1 = B_1 u_0$, $v_2 = B_2 u_0$, $u_1 = \sigma(u_0)$, where $\sigma(u_0)$ represents any symmetry of the usual Gardner equation, and we have much freedom to choose u_0 such that it can construct solutions without restricting to the traveling wave type solutions and infinitely many symmetries of the superfield Gardner equations.

It is clear that the solution (9) can be expressed by the form of the Jacobian elliptic sine functions, *i.e.*,

$$u_0 = -\frac{1}{2\varepsilon^2} + \frac{a_0 klm}{\varepsilon}sn(X_1, m), \tag{16}$$

where $X_1 = lX$, modulus m and constants $C_i(i = 1,2)$ are related to other known constants through following relations:

$$m = \frac{a_0\sqrt{2k\left(2\varepsilon^2\omega + 3k - 2\varepsilon^2 k^3 l^2\right)}}{2k^2 l\varepsilon},\ C_1 = \frac{\varepsilon^2\omega + k}{2\varepsilon^4},$$
$$C_2 = \frac{\left(4\varepsilon^2 k^3 l^2 - 4\varepsilon^2\omega - 5k\right)\left(4\varepsilon^2 k^2 l^2 - 1\right)}{16\varepsilon^6}. \tag{17}$$

Therefore, we derive special type solutions of the superfield Gardner equation:

$$u = -\frac{1}{2\varepsilon^2} + \frac{a_0 klm}{\varepsilon}S + \theta_1\theta_2 cn(X_1, m)dn(X_1, m)(A_1 + u_1') \tag{18}$$

where

$$u_1' = \frac{1}{C\sqrt{1-m^2S^2}}\Big\{A_2\Big[C\big((E+F)m^2+E-F\big)\sqrt{1-m^2S^2}$$
$$\times\big(m^2S^2-1-m^4C^2\big)S\Big]+A_3\Big[C\big((E+F)m^2+E-F\big)\sqrt{1-m^2S^2} \quad (19)$$
$$+m^4SC^2-4l\varepsilon km^3\left(S^2-\frac{1}{2}\right)+S^3m^2+2lmk\varepsilon-S\Big]\Big\},$$

$A_i\,(i=1,2,3)$ and $B_i\,(i=1,2)$ are all arbitrary integral constants, the Jacobian elliptic functions are denoted as $S=sn(X_1,m)$ and $C=cn(X_1,m)$, the incomplete elliptic integrals are denoted as $E=E(S,m)$ and $F=F(S,m)$. We also note that for the critical modulus $m = 1$, above solutions read:

$$u=-\frac{1}{2\varepsilon^2}+\frac{a_0klm}{\varepsilon}\tanh X_1+\theta_1\theta_2u_1$$
$$\xi=(B_1\theta_1+B_2\theta_2)\left(-\frac{1}{2\varepsilon^2}+\frac{a_0klm}{\varepsilon}\tanh X_1\right) \quad (20)$$

where

$$u_1=\operatorname{sech}^2X_1\left\{A_1+A_2\left[\left(\frac{2}{3}\cosh^3X_1+\cosh X_1\right)\sinh X_1+X_1\right]\right.$$
$$\left.+A_3\left[-\frac{4}{3}\varepsilon kl\cosh^4X_1+\frac{1}{3}\sinh(2X_1)\cosh^2X_1+X_1+\frac{1}{2}\sinh(2X_1)\right]\right\}. \quad (21)$$

3. Three-Fermionic-Parameter Bosonization

For the case of three fermionic parameters $\theta_i\,(i=1,2,3)$ with $\theta_i^2=0\,(i=1,2,3)$, let the two component fields u and ξ be expressed as

$$u=u_0+\sum_{\varepsilon_{ijk}=1}u_i\theta_j\theta_k\,,\quad \xi=\sum_{i=1}^{4}v_i\theta_i\quad(\theta_4=\theta_1\theta_2\theta_3) \quad 22)$$

here $u_i=u_i(x,t)\,(i=0,1,2,3)$ and $v_i=v_i(x,t)\,(i=1,2,3,4)$ are all usual bosonic functions with respect to spacetime variable x and t. The symbol ε_{ijk} is the third-order Levi-Civita tensor. Then from (5), we get PDEs in component form as

$$u_{0t}+u_{0xxx}-6u_0u_{0x}f(u_0)=0 \quad (23a)$$
$$u_{it}+u_{ixxx}-6(u_0u_i)_xf(u_0)-6\varepsilon^2u_0u_{0x}u_i+F_i=0,\ i=1,2,3 \quad (23b)$$
$$v_{it}+v_{ixxx}-3(v_iu_0)_xf(u_0)=0,\ i=1,2,3 \quad (23c)$$
$$v_{4t}+v_{4xxx}-3(v_4u_0)_xf(u_0)+G_4=0 \quad (23d)$$

where the somewhat complex nonhomogeneous terms read

$$F_i=3\left[f(u_0)(v_jv_{kx}-v_kv_{jx})\right]_x,\ \varepsilon_{ijk}=1 \quad (24a)$$
$$G_4=-3\left[f(u_0)\sum_{i=1}^{3}u_iv_i\right]_x-3\varepsilon^2u_0\sum_{i=1}^{3}u_iv_{ix} \quad (24b)$$

Introducing the traveling wave variable $X = kx + \omega t + x_0$ with constants k, ω and x_0 and variable transformations $u_i = P_i\left(u_0\left(X\right)\right)\left(i = 0,1,2,3\right)$ and $v_i = Q_i\left(u_0\left(X\right)\right)\left(i = 1,2,3,4\right)$, similar to the previous case, the above bosonization system becomes following ODEs

$$K_e\left(P_i\right) + R_i = 0, i = 1,2,3 \tag{25a}$$

$$K_o\left(Q_i\right) = 0, i = 1,2,3 \tag{25b}$$

$$K_o\left(Q_4\right) + G_4' = 0 \tag{25c}$$

where nonhomogeneous terms are

$$R_i = -3a_0 f\left(z\right)\sqrt{k\lambda\left(z\right)}\left(Q_j\frac{\mathrm{d}Q_k}{\mathrm{d}z} - Q_k\frac{\mathrm{d}Q_j}{\mathrm{d}z}\right) - A_{i,1}, \varepsilon_{ijk} = 1 \tag{26a}$$

$$G_4' = -3k\frac{\mathrm{d}}{\mathrm{d}z}\left[f\left(z\right)\sum_{i=1}^{3}u_i v_i\right] - 3\varepsilon^2 kz\sum_{i=1}^{3}u_i\frac{\mathrm{d}v_i}{\mathrm{d}z} \tag{26b}$$

with arbitrary integral constants $A_{i,1}\left(i = 1,2,3\right)$. Mapping and deformation relations with respect to u_0 via traveling wave variable are constructed as

$$P_i = \sqrt{\lambda\left(z\right)}\left[A_{i,3} + \int^z \frac{A_{i,2} + \int^y R_i\left(y_1\right)\mathrm{d}y_1}{\lambda\left(y\right)^{\frac{3}{2}}}\mathrm{d}y\right]\left(i = 1,2,3\right) \tag{27a}$$

$$Q_i = B_{i,1}z + B_{i,2}\sqrt{\eta\left(z\right)}\sin\left[H\left(z\right) + B_{i,3}\right] \tag{27b}$$

$$\begin{aligned} Q_4 = {} & \frac{\varepsilon^2 z}{\tau_1}\int E_4\left(y\right)\mathrm{d}y + \sqrt{\eta\left(z\right)}\int^z \frac{E_4\left(y\right)}{\sqrt{\eta\left(y\right)}}\left\{\frac{\left(ky^2 - C_1\right)f\left(y\right)}{\tau\sqrt{\lambda\left(y\right)}}\sin\left[H\left(z\right) - H\left(y\right)\right]\right. \\ & \left. -\frac{\varepsilon^2 y}{\tau_1}\cos\left[H\left(z\right) - H\left(y\right)\right]\right\}\mathrm{d}y + B_{4,1}z + B_{4,2}\sqrt{\eta\left(z\right)}\sin\left[H\left(z\right) + B_{4,3}\right], \end{aligned} \tag{27c}$$

where $A_{i,j}\left(i = 1,2,3; j = 2,3\right)$ and $B_{i,j}\left(i, j = 1,2,3\right)$ are some arbitrary integral constants, and functioin $E_4\left(z\right) = -\int^z G_4'\left(y\right)\mathrm{d}y + r_4$ with integral constant r_4. Therefore, we have obtained the three-fermionic-parameter traveling wave solutions of the superfield Gardner system. While one of the Grassmann numbers $\theta_i\left(i = 1,2,3\right)$ tends to zero, the solution turns back to above section. Similar to the two-fermionic-parameter case, we write a special type solution

$$u = u_0 + \sigma_1\left(u_0\right)\theta_2\theta_3 + \sigma_2\left(u_0\right)\theta_3\theta_1 - d_3^{-1}\left[d_1\sigma_1\left(u_0\right) + d_2\sigma_2\left(u_0\right)\right]\theta_1\theta_2 \tag{28a}$$

$$\xi = \left(d_1\theta_1 + d_2\theta_2 + d_3\theta_3 + d_4\theta_1\theta_2\theta_3\right)u_0 \tag{28b}$$

where $d_i\left(i = 1,2,3,4\right)$ are some constants, $\sigma_i\left(u_0\right)\left(i = 1,2\right)$ are arbitrary symmetries of the usual Gardner equation, and u_0 is an arbitrary solution of the usual Gardner equation.

4. N-Fermionic-Parameter Bosonization

Motivated by above sections, we repeat same procedure to get traveling wave solutions of the superfield Gardner equation via bosonizaion approach with N fermionic parameters here. The component fields u and ξ can be expressed as

$$u = u_0 + \sum_{n=1}^{N_1} \sum_{M_{2n}} u_{i_1 \cdots i_{2n}} \theta_{i_1} \cdots \theta_{i_{2n}}, \quad \xi = \sum_{n=1}^{N_2} \sum_{M_{2n-1}} v_{i_1 \cdots i_{2n}} \theta_{i_1} \cdots \theta_{i_{2n-1}} \tag{29}$$

Here and below we denote by M_k the set of multi-indices which satisfies $1 \le i_1 < i_2 < \cdots < i_k \le N$, $N_1 = \left[\frac{N}{2}\right]$ and $N_2 = \left[\frac{N+1}{2}\right]$ the upper bound of summations. The elements $u_{i_1 \cdots i_{2n}}$ and $v_{i_1 \cdots i_{2n-1}}$ are bosonic smooth real or complex valued functions defined in the commutative algebra. Thus the super Gardner model (5) is transformed to a new pure bosonic-looking system with 2^N coupled nonlinear PDEs:

$$u_{0t} + u_{0xxx} - 6u_0 u_{0x} - 6\varepsilon^2 {u_0}^2 u_{0x} = 0 \tag{30a}$$

$$L_e\left(u_{i_1 \cdots i_{2n}}\right) + F_{i_1 \cdots i_{2n}} = 0 \tag{30b}$$

$$L_o\left(v_{i_1 \cdots i_{2n-1}}\right) + G_{i_1 \cdots i_{2n-1}} = 0 \tag{30c}$$

Operators related to Gâteaux derivative of Equation (4) or Equations (5) with an operator decomposition read

$$U'[\chi]\big|_{\chi \to \chi_0} = L_e + \theta L_o, \quad \chi_0 : u \to u_0, \xi \to 0 \tag{31a}$$

$$L_e = U'[u]\big|_{\chi \to \chi_0} = \partial_t + \partial_x^3 - 6\partial_x u_0 - 6\varepsilon^2 \partial_x u_0^2 \tag{31b}$$

$$L_o = U'[\xi]\big|_{\chi \to \chi_0} = \partial_t + \partial_x^3 - 3\partial_x u_0 - 3\varepsilon^2 u_0 \partial_x u_0 \tag{31c}$$

and nonhomogeneous terms are

$$F_{i_1 \cdots i_{2n}} = \begin{cases} 3T\left[\xi, \xi_x\right]_x + 3\varepsilon^2\left(u_0 T\left[\xi, \xi_x\right]\right)_x, \ \text{for } n = 1, \\ -3T\left[u, u\right]_x + 3T\left[\xi, \xi_x\right]_x - 6\varepsilon^2\left\{\left(u_0 T\left[u, u\right]\right)_x + T\left[u, u, u_x\right]\right\} \\ \quad + 3\varepsilon^2\left\{u_0 T\left[\xi, \xi_x\right] + T\left[\xi, \xi_x, u\right]\right\}_x, \ \text{for } n = 2, \cdots, N_1; \end{cases}$$

$$G_{i_1 \cdots i_{2n-1}} = \begin{cases} 0, \ \text{for } n = 1, \\ -3T\left[\xi, u\right]_x - 3\varepsilon^2\left\{T\left[\left(u_0 u\right)_x, \xi\right] + 2u_0 T\left[u, \xi_x\right]\right\}, \ \text{for } n = 2, \\ -3T\left[\xi, u\right]_x - 3\varepsilon^2\left\{T\left[\left(u_0 u\right)_x, \xi\right] + T\left[u, u_x, \xi\right]\right. \\ \quad \left. + 2u_0 T\left[u, \xi_x\right] + T\left[u, u, \xi_x\right]\right\}, \ \text{for } n = 3, \cdots, N_2. \end{cases} \tag{32}$$

We have also used the shorthand notations

$$T\left[x_1, \cdots, x_s\right] = \sum_J \varepsilon_{j_1 \cdots j_m} \prod_{l=1}^{s} \left(x_l\right)_{\mu_{k_l}}$$

and

$$J = \left\{\left(j_1, \cdots, j_m\right) | 1 \le j_{k_l+1} < \cdots < j_{k_{l+1}} \le m, n_l \le k_l \le n_l', \sum_{l=1}^{s} k_l = m, j_{h_1} \ne j_{h_2}\left(h_1 \ne h_2\right)\right\},$$

with

$$\mu_{k_l} = i_{j_{k_{l-1}+1}} \cdots i_{j_{k_l - k_{l-1}}}, \quad n_l = 2 - \left|x_l\right|, \quad n_l' = m - \sum_{h=1, h \ne l}^{s} n_h$$

and $m = 2n$ for even case; $m = 2n-1$ for odd case. The parity function $|f|$ is defined in [25]. The symbol $\varepsilon_{j_1 \cdots j_m}$ is the generalized Levi-Civita anti-symmetric tensor in m dimensions. Equation (30a) apparently suggests that it is the standard Gardner equation. Each solution in the case of $n-1$ fermionic parameters is a portion of solutions of the system consisting of n homologous Grassmannian variables. Introducing the traveling wave variable and variable transformations, mapping transformations are arranged as follows:

1) Equation of u_0: $u_{0X}=\frac{a_0}{k^2}\sqrt{k\lambda(z)}$ (33a)

2) Equations of $u_{i_1\cdots i_{2n}}$: $K_e(P)-R_{i_1\cdots i_{2n}}=0$ (33b)

3) Equations of $v_{i_1\cdots i_{2n-1}}$: $K_o(Q)+G'_{i_1\cdots i_{2n-1}}=0$ (33c)

Here the nonhonogenerous terms are $G'_{i_1\cdots i_{2n-1}}(z)=u_{0X}^{-1}G_{i_1\cdots i_{2n-1}}(z)$, each usual bosonic functions $u_{i_1\cdots i_{2n}}$ and $v_{i_1\cdots i_{2n-1}}$ are represented by $P=P(u_0(X))$ and $Q=Q(u_0(X))$, respectively.

Finally, the mapping and deformation relations of $u_{i_1\cdots i_{2n}}$ and $v_{i_1\cdots i_{2n-1}}$ with N fermionic parameters can be formally rewritten as

$$P=\sqrt{\lambda(z)}\left[A_{i_1\cdots i_{2n},3}+\int^z\frac{A_{i_1\cdots i_{2n},2}+\int^y R_{i_1\cdots i_{2n}}(y_1)\mathrm{d}y_1}{\lambda(y)^{\frac{3}{2}}}\mathrm{d}y\right] \tag{34a}$$

$$\begin{aligned}Q={}&B_{i_1\cdots i_{2n-1},1}z+B_{i_1\cdots i_{2n-1},2}\sqrt{\eta(z)}\sin\left[H(z)+B_{i_1\cdots i_{2n-1},3}\right]\\&+\frac{\varepsilon^2 z}{\tau_1}\int E_{i_1\cdots i_{2n-1}}(y)\mathrm{d}y+\sqrt{\eta(z)}\times\int^z\frac{E_{i_1\cdots i_{2n-1}}(y)}{\sqrt{\eta(y)}}\left\{\frac{(ky^2-C_1)f(y)}{\tau\sqrt{\lambda(y)}}\right.\\&\left.\times\sin\left[H(z)-H(y)\right]-\frac{\varepsilon^2 y}{\tau_1}\cos\left[H(z)-H(y)\right]\right\}\mathrm{d}y\end{aligned} \tag{34b}$$

in which auxiliary functions are

$$R_{i_1\cdots i_{2n}}(z)=-\int^z F_{i_1\cdots i_{2n}}(y)\mathrm{d}y-A_{i_1\cdots i_{2n},1} \tag{35a}$$

$$E_{i_1\cdots i_{2n-1}}(z)=-\int^z G'_{i_1\cdots i_{2n-1}}(y)\mathrm{d}y+r_{i_1\cdots i_{2n-1}} \tag{35b}$$

$A_{i_1\cdots i_{2n},j}$, $B_{i_1\cdots i_{2n-1},j}\,(j=1,2,3)$ and $r_{i_1\cdots i_{2n-1}}$ are some arbitrary integral constants.

5. Discussion and Summary

With relationships at hand, in case of preserving the square and cubic non-linear terms in Equations (5) or taking limitation $\varepsilon\to 0$, one can conveniently acquire the corresponding result of the integrable $N=1$ sKdV regardless of some integral constants. Similarly to Section 4, the general traveling wave solutions of the $N=1$ sKdV Equation (2) with N fermionic parameters are

$$u_{i_1\cdots i_{2n}}=\sqrt{2ku_0^3-\omega u_0^2-2C_1u_0-C_2}\left[A_{i_1\cdots i_{2n},3}+\int^z\frac{A_{i_1\cdots i_{2n},2}+\int^y R_{i_1\cdots i_{2n}}(y_1)\mathrm{d}y_1}{\left(2ky^3-\omega y^2-2C_1y-C_2\right)^{\frac{3}{2}}}\mathrm{d}y\right] \tag{36a}$$

$$
\begin{aligned}
\xi_{i_1\cdots i_{2n-1}} &= B_{i_1\cdots i_{2n-1},1}u_0 + B_{i_1\cdots i_{2n-1},2}\sqrt{ku_0^2 - C_1}\sin\left[H(u_0) + B_{i_1\cdots i_{2n-1},3}\right] \\
&\quad + \sqrt{ku_0^2 - C_1}\int^{u_0} \frac{\sin\left[H(u_0) - H(y)\right]E_{i_1\cdots i_{2n-1}}\sqrt{ky^2 - C_1}}{\sqrt{C_1(C_2k + C_1\omega)\left(2ky^3 - \omega y^2 - 2C_1y - C_2\right)}}\mathrm{d}y,
\end{aligned} \tag{36b}
$$

where

$$
H(z) = \int^z \frac{\sqrt{C_1(kC_2 + \omega C_1)}}{\left(ky^2 - C_1\right)\sqrt{2ky^3 - \omega y^2 - 2C_1y - C_2}}\mathrm{d}y .
$$

Due to the case of taking limitation $\varepsilon \to \pm 1$ and the absence of non-linear quadratic term $D^2(\chi D\chi)$ in Equation (4), traveling wave solutions of the usual integrable $N = 1$ smKdV Equation [10] [27]

$$
\Phi_t - 3(D\Phi)D^2(\Phi D\Phi) + D^6\Phi = 0 \tag{37}
$$

with N-fermionic-parametric Bosonization procedure can be derived as

$$
\Phi = \xi + \theta u = \sum_{n=1}^{N_2}\sum_{M_{2n-1}} v_{i_1\cdots i_{2n-1}}\theta_{i_1}\cdots\theta_{i_{2n-1}} + \theta\left(u_0 + \sum_{n=1}^{N_1}\sum_{M_{2n}} u_{i_1\cdots i_{2n}}\theta_{i_1}\cdots\theta_{i_{2n}}\right) \tag{38}
$$

where

$$
u_{i_1\cdots i_{2n}} = \sqrt{kz^4 - \omega z^2 - 2C_1z - C_2}\left[A_{i_1\cdots i_{2n},2} + \int^z \frac{A_{i_1\cdots i_{2n},3} + \int^y R_{i_1\cdots i_{2n}}(y_1)\mathrm{d}y_1}{\left(ky^4 - \omega y^2 - 2C_1y - C_2\right)^{\frac{3}{2}}}\mathrm{d}y\right] \tag{39a}
$$

$$
\begin{aligned}
v_{i_1\cdots i_{2n-1}} &= B_{i_1\cdots i_{2n-1},1}u_0 + B_{i_1\cdots i_{2n-1},2}u_0\sinh\left[H(u_0) + B_{i_1\cdots i_{2n-1},3}\right] \\
&\quad + \frac{u_0}{C_2}\int^{u_0} E_{i_1\cdots i_{2n-1}}\left\{\cosh\left[H(u_0) - H(y)\right]\right. \\
&\quad \left. - \frac{\sqrt{C_2}}{\sqrt{\omega y^2 + 2C_1y + C_2 - ky^4}}\sinh\left[H(u_0) - H(y)\right] - 1\right\}\mathrm{d}y,
\end{aligned} \tag{39b}
$$

with

$$
F_{i_1\cdots i_{2n}} = \begin{cases} 3\left(u_0T[\xi,\xi_x]\right)_x, \text{ for } n = 1, \\ -6\left(u_0T[u,u]\right)_{xx} - 6T[u,u,u_x]_x + 3u_0T[\xi,\xi_x] \\ \quad + 3T[\xi,\xi_x,u], \text{ for } n = 2,\cdots,N_1; \end{cases} \tag{40a}
$$

$$
G_{i_1\cdots i_{2n-1}} = \begin{cases} 0, \text{ for } n = 1, \\ -3T\left[(u_0u)_x,\xi\right] - 6u_0T[u,\xi_x], \text{ for } n = 2, \\ -3T\left[(u_0u)_x,\xi\right] - 3T[u,u_x,\xi] - 6u_0T[u,\xi_x] \\ \quad - 3T[u,u,\xi_x], \text{ for } n = 3,\cdots,N_2, \end{cases} \tag{40b}
$$

and

$$
H(z) = \int^z \frac{1}{y}\sqrt{\frac{C_2}{\omega y^2 + 2C_1y + C_2 - ky^4}}\mathrm{d}y .
$$

In summary, the bosonization approach with multi-fermionic parameters to deal with supersymmetric systems is developed in the super Gardner equation with the role of traveling wave solution. The procedure and technique are also available for $N = 1$ sKdV and smKdV equations derived from two particular cases. We expect this procedure exhibited in our paper could be successfully applied or formulated in the $N = 1$ supersymmetric sine-Gordon equation, especially in the $N = 2$ version of KdV (SKdVa) equations [8] [25] [28] [29] [30]. For example, in the case of two fermionic parameters, letting $u = u_0 + u_1 s_1 s_2$, $\phi = \phi_1 s_1 + \phi_2 s_2$, $\psi = \psi_1 s_1 + \psi_2 s_2$ in bosonic field $\Phi = \frac{u}{2} + \xi\phi + \theta\psi - \xi\theta \sin\frac{u}{2}$, the simple traveling wave solutions of the $N = 1$ supersymmetric Sine-Gordon equation $D_x D_t \Phi = \sin\Phi$ are

$$u = u_0 + \sqrt{E + \cos u_0}\left(\int \frac{4D\cos\frac{u_0}{2} + B_1}{2\left(E + \cos u_0\right)^{\frac{3}{2}}} \mathrm{d}u_0 + B_2\right) s_1 s_2 \tag{41a}$$

$$S = \sum_{i=1}^{2}\left\{A_{i1}\sin\frac{u_0}{2} + A_{i2}\cos\left[\arctan\frac{2\sin\frac{u_0}{2}}{\sqrt{2\left(E + \cos u_0\right)}}\right]\right\} s_i \tag{41b}$$

where $D_x = \partial_\xi + \xi\partial_x$ and $D_t = \partial_\theta + \theta\partial_t$ are the usual super derivatives, S represents ϕ and ψ, E, B_1, B_2, A_{i1} and A_{i2} are integral constants, u_0 is a solution of the standard sine-Gordon equation.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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