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Inverse and Saturation Theorems for Some Summation Integral Linear Positive Operators

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Abstract

Inverse and saturation theorems have been studied for linear combination of some summation integral hybrid linear positive operators. Linear approximating technique and Steklov means have been used to obtain better approximation results.

Keywords

Linear Combination, Hybrid Operators, Steklov Means, Order of Approximation

1. Introduction

Development of linear positive operators brings major contribution in the field of approximation theory. Several mathematicians [1] [2] [3] [4] [5] have worked on hybrid linear positive operators. They improved rate of convergence by taking their linear combination. Here in this paper we consider a sequence of hybrid operators, combination of Beta and Baskakov basis functions,

$$\left(B_{(n)} f\right)(x) = \frac{(n-1)}{n} \sum_{i=0}^{\infty} b_{(n,i)}(x) \int_0^{\infty} p_{(n,i)}(t) f(t) \mathrm{d}t, \tag{1}$$

for every, $x \in [0,\infty)$ and $f \in L_p[0,\infty), p \geq 1$

Here,

$$b_{(n,i)}(x) = \frac{1}{B(i+1,n)} \frac{x^i}{(1+x)^{n+i+1}}$$

and

$$p_{(n,i)}(t) = \binom{n+i-1}{i} \frac{t^i}{(1+t)^{n+i}} \tag{2}$$

where, $B(i+1,n)=\dfrac{i!(n-1)!}{(n+1)!}$ is Beta function.

Clearly, the above operators are linear positive operators and reproduce only constant functions.

Also, $B_{(n)}(1,x)=1$.

These operators can be used to approximate Lebesgue integrable functions and can be L_p approximation methods.

The order of approximation for these operators is at its best at $O\left(n^{-1}\right)$. We can improve the order of approximation by taking linear combination of these operators.

Let $d_0,d_1,d_2,\cdots,d_n$ be $k+1$ arbitrary but fixed distinct positive integers. Then linear combination $B_{(n)}(f,k,x)$ of $B_{(d_jn)}(f,x)$, $j=0,1,\cdots,n$ is defined by,

$$B_{(n)}(f,k,x)=\frac{1}{\Delta}\begin{vmatrix} B_{(d_0n)}(f,x) & d_0^{-1} & d_0^{-2} & \cdots & d_0^{-n} \\ B_{(d_1n)}(f,x) & d_1^{-1} & d_1^{-2} & \cdots & d_1^{-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_{(d_nn)}(f,x) & d_n^{-1} & d_n^{-2} & \cdots & d_n^{-n} \end{vmatrix} \tag{3}$$

where Δ is Vandermonde determinant obtained by replacing the operator column of above determinant by entries 1 given by

$$\Delta=\begin{vmatrix} 1 & d_0^{-1} & d_0^{-2} & \cdots & d_0^{-n} \\ 1 & d_1^{-1} & d_1^{-2} & \cdots & d_1^{-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & d_n^{-1} & d_n^{-2} & \cdots & d_n^{-n} \end{vmatrix}$$

Simplification of (3) leads to,

$$B_{(n)}(f,k,x)=\sum\nolimits_{j=0}^{k}C(j,k)B_{(d_jn)}(f,x) \tag{4}$$

where, $C(j,k)=\prod\nolimits_{i=0,i\neq j}^{n}\dfrac{d_j}{d_j-d_i}$ $k\neq 0$ and $C(0,0)=1$

Let $0<a_1<a_3<a_2<b_2<b_3<b_1<\infty$ and $I_i=[a_i,b_i]$, $i=1,2,3$. Also let $[\alpha]$ denote integral part of α.

Let $1\leq p\leq\infty$, $f\in L_p[0,\infty)$. Then for sufficiently small $\eta>0$, the steklov mean $f_{\eta,m}$ of *m*-th order corresponding to *f* is defined as,

$$f_{\eta,m}(t)=\eta^{-m}\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2}\cdots\int_{-\pi/2}^{\pi/2}\left[f(t)+(-1)^{m-1}\Delta^m\sum\nolimits_{i=1}^{mt_i}f(t)\right]\mathrm{d}t_1\mathrm{d}t_2\cdots\mathrm{d}t_m \tag{5}$$

We will use the following results:

a) $f_{\eta,m}$ has derivatives up to order *m*, $f_{\eta,m}^{(m-1)}\in AC[a_1,b_1]$ and $f_{\eta,m}^{(m)}$ exists a.e and belongs to $L_p(a_1,b_1)$.

b) $\left\|f_{\eta,m}^{(r)}\right\|_{L_{p[a_2,b_2]}}\leq G_m\eta^{-r}\omega_r, r=1,2,3,\cdots,m$

c) $\left\|f-f_{\eta,m}\right\|_{L_{p[a_2,b_2]}}\leq G_{m+1}\omega_m\left(f,\eta,p,[a_1,b_1]\right)$

d) $\left\| f_{\eta,m} \right\|_{L_{p[a_2,b_2]}} \le G_{m+2} \left\| f \right\|_{L_p(a_1,b_1)}$

$$\left\| f_{\eta,m}^{(m)} \right\|_{L_{p[a_2,b_2]}} \le G_{m+3} \eta^{-m} \left\| f \right\|_{L_p(a_1,b_1)} \tag{6}$$

where G_i's are certain constants independent of f and η.

The present chapter deals with inverse and saturation results for these operators using linear approximating methods.

Operators (1) can be written as,

$$\left(B_{(n)} f\right)(x) = \int_0^\infty P_{(n)}(x,t) f(t) \mathrm{d}t \tag{7}$$

where the kernel, $P_{(n)}(x,t) = \dfrac{(n-1)}{n} \sum_{i=0}^{\infty} b_{(n,i)}(x) p_{(n,i)}(t)$

2. Some Auxiliary Results

Here, we will present some definitions, results and lemmas which we will be needing in our main theorems.

Definition 2.1.

Jensen's Inequality. It generalizes the statement that the secant line of a convex function lies above the graph of the function. The secant line consists of weighted means of the convex function (for $t \in [0,1]$).

$$f\left(tx_1 + (1-t)x_2\right) \le tf(x_1) + (1-t) f(x_2)$$

Definition 2.2.

Fubini's Theorem. It gives conditions under which it is possible to compute a double integral by usng iterated integral. Order of integration may be switched if the double integral yields a finite answer when the integrand is replaced by its absolute value.

Lemma 2.3. [3] There exists polynomials $s_{(k,j,r)}(x)$ independent of i and n such that,

$$\frac{\mathrm{d}^r}{\mathrm{d}x^r}\left(\frac{x^i}{(1+x)^{n+i}}\right) = \sum_{\substack{2j+k \le r \\ j,k \ge 0}} n^i (i-nx)^j s_{(k,j,r)}(x) \frac{x^{i-r}}{(1+x)^{n+i+r}}$$

Also,

$$\left(x^2+x\right)^r \frac{\mathrm{d}^r}{\mathrm{d}x^r}\left(b_{(n,i)}(x)\right) = \sum_{\substack{2j+k \le r \\ j,k \ge 0}} (n+1)^k \left(i-(n+1)x\right)^j s_{(k,j,r)}(x) b_{(n,i)}(x)$$

Lemma 2.4. For $m \in N_0, n \in N, x \in [0,\infty)$, the mth order central moment is defined by,

$$T_{(n,m)}(x) = B_{(n)}\left((t-x)^m, x\right) = \frac{(n-1)}{n} \sum_{i=0}^{\infty} b_{(n,i)}(x) \int_0^\infty p_{(n,i)}(t)(t-x)^m \mathrm{d}t$$

then,

1) For each $x \in [0,\infty)$, we have, $T_{(n,m)}(x) = O\left(n^{-[(m+1)/2]}\right)$
2) $T_{(n,0)} = 1$

3) $T_{(n,1)} = (3x+1)/(n-2)$

4) $T_{(n,2)} = \dfrac{2\{nx(1+x)+7x^2+5x+1\}}{n^2-5n+6}$

5) For n sufficiently large and $c>2$, $B_{(n)}(|t-x|,x) \le \left[T_{(n,2)}\right]^{1/2} \le \sqrt{\dfrac{Cx^2+Cx}{n}}$

We have the following recurrence relation,

$$\begin{aligned}(n-m-2)T_{(n,m+1)}(x) &= (x^2+x)T^{(1)}_{(n,m)}(x) + 2m(x^2+x)T_{(n,m-1)}(x) \\ &\quad + \left[2x+m+1+x(2m+1)\right]T_{(n,m)}(x)\end{aligned} \tag{8}$$

Proof. $(x^2+x)b'_{(n,i)}(x) = (i-nx-x)b_{(n,i)}(x)$

$$(t^2+t)\,p'_{(n,i)}(t) = (i-nt)\,p_{(n,i)}(t)$$

Also,

$$\begin{aligned}(x^2+x)T^{(1)}_{(n,m)}(x) &= \frac{(n-1)}{n}\sum_{i=0}^{\infty}(x^2+x)b'_{(n,i)}(x) \\ &\quad \times \int_0^{\infty} p_{(n,i)}(t)(t-x)^m\,\mathrm{d}t - m(x^2+x)T_{(n,m-1)}(x)\end{aligned}$$

This implies,

$$\begin{aligned}&(x^2+x)T^{(1)}_{(n,m)}(x) + m(x^2+x)T_{(n,m-1)}(x) \\ &= \frac{(n-1)}{n}\sum_{i=0}^{\infty}(i-nx-x)b_{(n,i)}(x)\int_0^{\infty} p_{(n,i)}(t)(t-x)^m\,\mathrm{d}t \\ &\quad -\frac{(n-1)}{n}\sum_{i=0}^{\infty}b_{(n,i)}(x)\int_0^{\infty}\{(i-nt)+(nt-nx)-x\}\,p_{(n,i)}(t)(t-x)^m\,\mathrm{d}t \\ &= \frac{(n-1)}{n}\sum_{i=0}^{\infty}b_{(n,i)}(x)\int_0^{\infty}(t^2+t)\,p'_{(n,i)}(t)(t-x)^m\,\mathrm{d}t + nT_{(n,m+1)}(x) - xT_{(n,m)}(x) \\ &= \frac{(n-1)}{n}\sum_{i=0}^{\infty}b_{(n,i)}(x)\int_0^{\infty}\Big[(t-x)+2x(t-x)+(t-x)^2 \\ &\quad +(x^2+x)\Big]p_{(n,i)}(t)(t-x)^m\,\mathrm{d}t + nT_{(n,m+1)}(x) - xT_{(n,m)}(x) \\ &= -(m+1)(2x+1)T_{(n,m)}(x) - (m+2)T_{(n,m+1)}(x) \\ &\quad - m(x^2+x)T_{(n,m-1)}(x) + nT_{(n,m+1)}(x) - xT_{(n,m)}(x)\end{aligned}$$

Hence the result (8).

Lemma 2.5. For sufficiently large n and certain polynomials $Q(p,k,x)$ in x of degree $p/2$ there holds,

$$B_{(n)}\left((t-x)^p,k,x\right) = \frac{1}{n^{(k+1)}}\{Q(p,k,x)+O(1)\}$$

for every $p \in N$.

Proof. From the above lemma 2.4 we have,

$$B_{(n)}\left((t-x)^p,x\right) = \sum_{i=0}^{p/2}P_{(i)}(x)\left\{\frac{1}{n^{\left[\frac{p+1}{2}\right]+i}}\right\}$$

Here, $P_{(i)}$'s are certain polynomials in x of degree at most i.

Now using lemma 2.4 we will have the required result.

Lemma 2.6. [4] [5] Let for $q(t)\in L_1[0,\infty)$ and $n\in N$, we have,

$$\left\|\frac{(n-1)}{n}\int_0^\infty b_{(n,i)}(x)\,p_{(n,i)}(t)\frac{(i-nt)^m}{n^m}q(t)\,\mathrm{d}t\right\|_{L_1[0,\infty)}\le K\frac{1}{n^{m/2}}\left\|q(t)\right\|_{L_1[0,\infty)} \tag{9}$$

where, $q(t)$ has a compact support and K is some constant independent of n and q.

Lemma 2.7. [4] [5] Let for $q(w)\in L_p[0,\infty)$, $p>1$ and $j\in N\cup\{0\}$, $m>0$, we have

$$\left\|\frac{(n-1)}{n}\int_0^\infty\sum_{i=0}^\infty b_{(n,i)}(x)\,p_{(n,i)}(t)\frac{(i-nx)^k}{n^k}\int_x^t(t-w)^j q(w)\,\mathrm{d}w\mathrm{d}t\right\|_{L_p(a_2,b_2)} \le H\left\{\frac{1}{n^{(k+j+1)/2}}\|h\|_{L_p(a_1,b_1)}+\frac{1}{n^m}\|h\|_{L_p[0,\infty)}\right\} \tag{10}$$

Here, H is some constant independent of n and q.

Lemma 2.8. [2] [4] Let $q\in L_p[0,\infty)$, $p\ge 1$ and $supp h\subset[a_2,b_2]$, then,

$$\left\|B_{(n)}^{(2k+2)}(q,.)\right\|_{L_p(a_2,b_2)}\le In^{k+1}\|q\|_{L_p(a_2,b_2)} \tag{11}$$

Here, I is some constant independent of n and q.

Lemma 2.9. [1] [2] [4] Let $q^{(2k+1)}\in AC(a_2,b_2)$ and $q^{(2k+2)}\in L_p(a_2,b_2)$, then,

$$\left\|B_{(n)}^{(2k+2)}(q,.)\right\|_{L_p(a_2,b_2)}\le Jn^{k+1}\left\|q^{(2k+2)}\right\|_{L_p(a_2,b_2)} \tag{12}$$

Here, J is some constant independent of n and q.

Lemma 2.10. [1] [2] [4] [5] Let $f\in C^{2k+2}(a_1,b_1)$ have a compact support, there holds for $n\to\infty$

$$B_{(n)}(f,k,x)-f(x)=\frac{1}{n^{k+1}}\left\{\sum_{j=0}^{2(k+1)}P_{(j)}(k,x)\,f^{(j)}(x)+O(1)\right\} \tag{13}$$

uniformly in $x\in(a_1,b_1)$, where $P_{(j)}(k,x)$ is a polynomial in x of degree j and does not vanish for all $i=1,2,3,\cdots,2(k+1)$.

3. Inverse Theorem

Theorem 3.1. Let $0<\alpha<2(k+1)$, $f\in L_p[0,\infty)$, $p\ge 1$, and

$$\left\|B_{(n)}(f,k,x)-f(x)\right\|_{L_p(a_1,b_1)}=O\left(\frac{1}{n^{\alpha/2}}\right)$$

then, $\omega_{2(k+1)}(f,\tau,p,[a_2,b_2])=O(\tau^\alpha)$ for $n\to\infty,\tau\to 0$

Proof. Let $fh=\overline{f}$ for all $\gamma\le\tau$ and $a_1<x_1<x_2<x_3<a_2<b_2<y_3<y_2<y_1<b_1$, where, $supp h\subset[x_2,y_2]$ for any function $g\in C_0^{2(k+1)}$ and $h(t)=1$ on $[x_3,y_3]$

$$\Delta_\gamma^{2(k+1)}\overline{f}(x)=\Delta_\gamma^{2(k+1)}\left(\overline{f}(t)-B_{(n)}\left(\overline{f},k,x\right)\right)+\Delta_\gamma^{2(k+1)}\left(B_{(n)}\left(\overline{f},k,x\right)\right)$$

Now solving 2nd term using Jensen's inequality and Fubini's theorem we have

$$\begin{aligned}\left\|\Delta_\gamma^{2(k+1)}\overline{f}(x)_{L_p[x_2,y_2]}\right\| &\le \left\|\Delta_\gamma^{2(k+1)}\left(\overline{f}(t)-B_{(n)}\left(\overline{f},k,x\right)\right)\right\|_{L_p[x_2,y_2]}\\ &\quad+\gamma^{2(k+1)}\left\|B_{(n)}^{2(k+1)}\left(\overline{f},k,x\right)\right\|_{L_p[x_2,y_2+2\gamma(k+1)]}\\ &\le \left\|\Delta_\gamma^{2(k+1)}\left(\overline{f}(t)-B_{(n)}\left(\overline{f},k,x\right)\right)\right\|_{L_p[x_2,y_2]}\\ &\quad+\gamma^{2(k+1)}\left\|B_{(n)}^{2(k+1)}\left(\overline{f}-\overline{f}_{\eta,2(k+1)},k,x\right)\right\|_{L_p[x_2,y_2+2\gamma(k+1)]}\\ &\quad+\left\|B_{(n)}^{2(k+1)}\left(\overline{f}_{\eta,2(k+1)},k,x\right)\right\|_{L_p[x_2,y_2+2\gamma(k+1)]}\end{aligned}$$

Applying (6), (11), (12) we have

$$\begin{aligned}&\left\|\Delta_\gamma^{2(k+1)}\overline{f}(x)\right\|_{L_p[x_2,y_2]}\\ &\le\left\|\Delta_\gamma^{2(k+1)}\left(\overline{f}(t)-B_{(n)}\left(\overline{f},k,x\right)\right)\right\|_{L_p[x_2,y_2]}\\ &\quad+L\gamma^{2(k+1)}\left(n^{(k+1)}+\frac{1}{\eta^{2(k+1)}}\right)\omega_{2(k+1)}\left(\overline{f},\eta,p,[x_2,y_2]\right)\end{aligned}$$

For proving our theorem, we will be using mathematical induction:

Step 1: For $\alpha\le 1$

Now we will prove for $n\to\infty$,

$$\left\|\Delta_\gamma^{2(k+1)}\left(\overline{f}(t)-B_{(n)}\left(\overline{f},k,x\right)\right)\right\|_{L_p[x_2,y_2]}=O\left(\frac{1}{n^{\alpha/2}}\right)$$

therefore,

$$\omega_{2(k+1)}\left(\overline{f},\tau,p,[x_2,y_2]\right)=O,\ \tau\to 0$$

$\overline{f}(t)=f(t)$ for $t\in[x_3,y_3]$

$$\begin{aligned}&\left\|B_{(n)}\left(\overline{f},k,x\right)-\overline{f}\right\|_{L_p[x_2,y_2]}\\ &\le\left\|B_{(n)}\left(h(x)\left(f(t)-f(x)\right),k,x\right)\right\|_{L_p[x_2,y_2]}\\ &\quad+\left\|B_{(n)}\left(f(t)\left(h(t)-h(x)\right),k,x\right)\right\|_{L_p[x_2,y_2]}\end{aligned}$$

$$\left\|B_{(n)}\left(\overline{f},k,x\right)-\overline{f}\right\|_{L_p[x_2,y_2]}=O\left(\frac{1}{n^{1/2}}\right)+O\left(\frac{1}{n^{\alpha/2}}\right)$$

Step 2: For $(m-1)<\alpha<m$

Let $[c,d]\subset(a_1,b_1)$

$$\omega_{2(k+1)}\left(f,\tau,p,[c,d]\right)=O\left(\tau^{(m-1)+\beta}\right),\tau\to 0,0<\beta<1$$

Here, we have, $\varphi(t)$ as a characteristic function of $[x_1,y_1]$

Due to smoothness of f we have,

$$\left\|B_{(n)}\left(\bar{f},k,x\right)-\bar{f}\right\|_{L_p[x_2,y_2]}$$
$$\leq \sum_{i=0}^{r-2}\frac{1}{i!}\left\|f^{(i)}(x)B_{(n)}\left((t-x)^i\left(h(t)-h(x)\right),k,x\right)\right\|_{L_p[x_2,y_2]}$$
$$+\frac{1}{(r-2)!}\left\|B_{(n)}\left(\varphi(t)\left(h(t)-h(x)\left(\int_x^t\frac{(t-w)^r}{(t-w)^2}\left(f^{(r-1)}(w)-f^{(r-1)}(x)\right)\mathrm{d}w\right),k,x\right)\right)\right\|_{L_p[x_2,y_2]}$$
$$+\left\|B_{(n)}\left(\left(f(x)-\sum_{i=0}^{r-2}\frac{(t-x)^i}{i!}f^{(i)}(x)\right)\left(1-\varphi(t)\right)\left(h(t)-h(x)\right),k,x\right)\right\|_{L_p[x_2,y_2]}$$
$$=H_1+H_2+H_3$$

where, for $t\in[0,\infty)$ and $x\in[x_2,y_2]$

Using lemma 2.4, we have for $n\to\infty$

$$H_1H_3=O\left(\frac{1}{n^{(k+1)}}\right)$$

Now, using direct theorem in [6]

$$\int_{x_2}^{y_2}\left|B_{(n)}\left(\varphi(t)\left(h(t)-h(x)\left(\int_x^t\frac{(t-w)^r}{(t-w)^2}\left(f^{(r-1)}(w)-f^{(r-1)}(x)\right)\mathrm{d}w\right),x\right)\right)\right|^p$$
$$\leq R\int_{x_2}^{y_2}\int_{x_1}^{y_1}V_{(n)}(x,t)\frac{|t-x|^{rp}}{|t-x|}\int_x^t\varphi(w)\left|f^{(r-1)}(w)-f^{(r-1)}(x)\right|^p\mathrm{d}w\mathrm{d}t\mathrm{d}x$$
$$\leq R\sum_{l=1}^{r}\int_{x_2}^{y_2}\left\{\int_{x+\frac{l}{\sqrt{n}}}^{x+\frac{(l+1)}{\sqrt{n}}}V_{(n)}(x,t)\frac{n^{2p}}{l^{4p}}\frac{|t-x|^{(r+4)p}}{|t-x|}\right.$$
$$\times\int_x^{x+\frac{(l+1)}{\sqrt{n}}}\varphi(w)\left|f^{(r-1)}(w)-f^{(r-1)}(x)\right|^p\mathrm{d}w\mathrm{d}t$$
$$+\int_{x-\frac{(l+1)}{\sqrt{n}}}^{x-\frac{l}{\sqrt{n}}}V_{(n)}(x,t)\frac{n^{2p}}{l^{4p}}\frac{|t-x|^{(r+4)p}}{|t-x|}$$
$$\left.\times\int_{x-\frac{(l+1)}{\sqrt{n}}}^{x}\varphi(w)\left|f^{(r-1)}(w)-f^{(r-1)}(x)\right|^p\mathrm{d}w\mathrm{d}t\right\}\mathrm{d}x$$
$$+\int_{x_2}^{y_2}\int_{x_2-\left(\frac{1}{\sqrt{n}}\right)}^{y_2+\left(\frac{1}{\sqrt{n}}\right)}V_{(n)}(x,t)\frac{|t-x|^{rp}}{|t-x|}\int_{x-\left(\frac{1}{\sqrt{n}}\right)}^{x+\left(\frac{1}{\sqrt{n}}\right)}\varphi(w)\left|f^{(r-1)}(w)-f^{(r-1)}(t)\right|^p\mathrm{d}w\mathrm{d}t\mathrm{d}x$$
$$\leq R\left\{\sum_{l=1}^{r}\frac{n^{(-rp+1)/2}}{l^{4p}}\int_0^{\frac{(l+1)}{\sqrt{n}}}\left(\omega\left(f^{(r-1)},w,p,[x_1,y_1]\right)\right)^p\mathrm{d}w\right.$$
$$\left.+n^{(-rp+1)/2}\int_0^{\frac{1}{\sqrt{n}}}\left(\omega\left(f^{(r-1)},w,p,[x_1,y_1]\right)\right)^p\mathrm{d}w\right\}$$

Using lemma 2.4 and interchanging integration we have

$$\leq R\left\{\sum_{l=1}^{r}\frac{n^{(-rp+1)/2}}{l^{4p}}\int_0^{\frac{(l+1)}{\sqrt{n}}}\left(O\left(\omega^{\beta}\right)\right)^p\mathrm{d}w+n^{(-rp+1)/2}\int_0^{\frac{1}{\sqrt{n}}}\left(O\left(\omega^{\beta}\right)\right)^p\mathrm{d}w\right\}$$

We have for $n \to \infty$

$$H_2 = O\left(\frac{1}{n^{(r+\beta)/2}}\right)$$

Combining all the results

$$H_1 H_2 H_3 = O\left(\frac{1}{n^{(k+1)}}\right) O\left(\frac{1}{n^{(r+\beta)/2}}\right)$$

Thus we have the theorem.

4. Saturation Result

Theorem 4.1. For $f \in L_p[0,\infty)$, $1 \le p < \infty$ and $\left\| B_{(n)}(f,k,x) - f \right\|_{L_p(a_1,b_1)}$, then, f coincides almost everywhere with a function F on $[a_2,b_2]$ having $2(k+1)$ th derivative such that,

a) $F^{(2k+1)} \in A.C(a_2,b_2), \forall p > 1$

b) $F^{2(k+1)} \in L_p(a_2,b_2), \forall p > 1$

c) $F^{(2k+1)} \in L_p(a_2,b_2), \forall p = 1$

d) $F^{(2k)} \in A.C(a_2,b_2), \forall p = 1$

for $a_1 < x_1 < x_2 < a_2 < b_2 < y_2 < y_1 < b_1$

Proof. Let $q \in C_0^{2(k+1)}, supp\, q \subset (a_2,b_2)$ such that $q = 1$ on $[x_1,y_1]$ and $fq = \overline{f}$, then for $0 < \alpha < 2(k+1)$, we have,

$$\left\| B_{(n)}(f,k,x) - f(x) \right\|_{L_p(x_1,y_1)} = O\left(\frac{1}{n^{\alpha/2}}\right)$$

$$\omega_{2(k+1)}\left(f,\tau,p,[x_2,y_2]\right) = O\left(\tau^{\alpha}\right)$$

Now,

$$\left\| B_{(n)}(\overline{f},k,x) - \overline{f}(x) \right\|_{L_{p[x_2,y_2]}}$$
$$\le \left\| B_{(n)}(f,k,x) - f(x) \right\|_{L_{p[x_2,y_2]}} + \left\| B_{(n)}(\overline{f} - f,k,x) \right\|_{L_{p[x_2,y_2]}}$$

Here, $\left\| B_{(n)}(\overline{f} - f,k,x) \right\|_{L_{p[x_2,y_2]}}$ is arbitrarily small, so we have,

$$\left\| B_{(n)}(\overline{f},k,x) - \overline{f}(x) \right\|_{L_{p[x_2,y_2]}} = O\left(\frac{1}{n^{(k+1)}}\right), \quad n \to \infty$$

Case 1: for $p > 1$

Consider a sequence $\{n_j\}$ and a function $h(x) \in L_{p[x_2,y_2]}$ such that for every $g \in C_0^{2(k+1)}$ and $supp\, g \subset (a_1,b_1)$, we have from Alaoglu's theorem,

$$\lim_{n_j \to \infty} n_j^{k+1} \left\langle B_{(n)}(\overline{f},k,x) - \overline{f}, g \right\rangle = \langle h, g \rangle \tag{14}$$

By lemma 2.10, we have

$$\begin{aligned}
&\lim_{n_j \to \infty} n_j^{k+1} \left\langle B_{(n)}\left(\bar{f}, k, x\right) - \bar{f}, g \right\rangle \\
&= \lim_{n_j \to \infty} n_j^{k+1} \left(\lim_{n_j \to \infty} n_j^{k+1} \left\langle B_{(n)}\left(g, k, x\right) - g, \bar{f} \right\rangle \right) \\
&= \left\langle \sum_{i=1}^{2(k+1)} P_{(i)}\left(k, x\right) \frac{\partial^i}{\partial x^i} \bar{f}, g \right\rangle \\
&= \left\langle g, \sum_{i=1}^{2(k+1)} P_{(i)}^{*}\left(k_i, x\right) \frac{\partial^i}{\partial x^i} \bar{f} \right\rangle
\end{aligned} \tag{15}$$

Comparing (14) and (15) we have

$$h = \sum_{i=1}^{2(k+1)} P_{(j)}\left(k, x\right) \frac{\partial^i}{\partial x^i} \bar{f}$$

Here, $\bar{f}^{(2k+1)} \in A.C\left(x_2, y_2\right)$

So, $F^{(2k+1)} \in A.C\left(a_2, b_2\right)$

Similarly, $\bar{f}^{(2k+1)} \in A.C\left(x_2, y_2\right)$

So, $F^{(2k+1)} \in A.C\left(a_2, b_2\right)$

We have (i) and (ii)

Case 2: for $p = 1$

Proceeding in the similar manner as above, we get (iii) and (iv).

Hence the theorem.

5. Conclusion

We have improved order of approximation by taking suitable linear combinations. Inverse and saturation results have been developed for our hybrid operators.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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