Advances in Pure Mathematics, 2019, 9, 347-403
http://www.scirp.org/journal/apm
ISSN Online: 2160-0384
ISSN Print: 2160-0368

Tables of Pure Quintic Fields

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How to cite this paper: Mayer, D.C. (2019) Tables of Pure Quintic Fields. *Advances in Pure Mathematics*, **9**, 347-403.
https://doi.org/10.4236/apm.2019.94017

Received: December 11, 2018
Accepted: April 26, 2019
Published: April 29, 2019

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Abstract

By making use of our generalization of Barrucand and Cohn's theory of principal factorizations in pure cubic fields $\mathbb{Q}\left(\sqrt[3]{D}\right)$ and their Galois closures $\mathbb{Q}\left(\zeta_3,\sqrt[3]{D}\right)$ with 3 possible types to pure quintic fields $L=\mathbb{Q}\left(\sqrt[5]{D}\right)$ and their pure metacyclic normal fields $N=\mathbb{Q}\left(\zeta_5,\sqrt[5]{D}\right)$ with 13 possible types, we compile an extensive database with arithmetical invariants of the 900 pairwise non-isomorphic fields N having normalized radicands in the range $2 \le D < 10^3$. Our classification is based on the Galois cohomology of the unit group U_N, viewed as a module over the automorphism group $\mathrm{Gal}(N/K)$ of N over the cyclotomic field $K=\mathbb{Q}(\zeta_5)$, by employing theorems of Hasse and Iwasawa on the Herbrand quotient of the unit norm index $\left(U_K : N_{N/K}(U_N)\right)$ by the number $\#(\mathcal{P}_{N/K}/\mathcal{P}_K)$ of primitive ambiguous principal ideals, which can be interpreted as principal factors of the different $\mathfrak{D}_{N/K}$. The precise structure of the $\mathbb{F}_5$-vector space of differential principal factors is expressed in terms of norm kernels and central orthogonal idempotents. A connection with integral representation theory is established via class number relations by Parry and Walter involving the index of subfield units $(U_N : U_0)$. The statistical distribution of the 13 principal factorization types and their refined splitting into similarity classes with representative prototypes is discussed thoroughly.

Keywords

Pure Quintic Fields, Pure Metacyclic Fields, Units, Galois Cohomology, Differential Principal Factorization Types, Similarity Classes, Prototypes, Class Group Structure

1. Introduction

At the end of his 1975 article on class numbers of pure quintic fields, Parry suggested verbatim: In conclusion, the author would like to say that he believes a numerical study of pure quintic fields would be most interesting ([1] p. 484). Of course, it would have been rather difficult to realize Parry's desire in 1975. But now, 40 years later, we are in the position to use the powerful computer algebra systems PARI/GP [2] and MAGMA [3] [4] [5] for starting an attack against this hard problem. Prepared by [6] [7] [8] [9], this will actually be done in the present paper.

Even in 1991, when we generalized Barrucand and Cohn's theory [10] [11] of principal factorization types from pure cubic fields $\mathbb{Q}\left(\sqrt[3]{D}\right)$ to pure quintic fields $L=\mathbb{Q}\left(\sqrt[5]{D}\right)$ and their pure metacyclic normal closures $N=\mathbb{Q}\left(\zeta_5,\sqrt[5]{D}\right)$ [12], it was still impossible to verify our hypothesis about the distinction between *absolute*, *intermediate* and *relative differential principal factors* (DPF) ([6] (6.3)) and about the values of the *unit norm index* $\left(U_K:N_{N/K}\left(U_N\right)\right)$ ([6] (1.3)) by actual computations.

All these conjectures have been proven by our most recent numerical investigations. Our classification is based on the Hasse-Iwasawa theorem about the Herbrand quotient of the unit group U_N of the Galois closure N of L as a module over the relative group $G=\mathrm{Gal}(N/K)$ with respect to the cyclotomic subfield $K=\mathbb{Q}(\zeta_5)$. It only involves the unit norm index $\left(U_K:N_{N/K}\left(U_N\right)\right)$ and our 13 types of differential principal factorizations ([6] Thm. 1.3), but not the index of subfield units $\left(U_N:U_0\right)$ ([6] § 5) in Parry's class number formula ([6] (5.1)).

We begin with a collection of explicit multiplicity formulas in § 2 which are required for understanding the subsequent extensive presentation of our computational results in twenty tables of crucial invariants in § 3. This information admits the classification of all 900 pure quintic fields $L=\mathbb{Q}\left(\sqrt[5]{D}\right)$ with normalized radicands $2\le D<10^3$ into 13 DPF types and the refined classification into similarity classes with representative prototypes in § 4.

We draw the attention to remaining open questions in § 3.3, and we collect theoretical consequences of our experimental results in § 4.3. The exposition is concluded with a retrospective final § 5.

2. Collection of Multiplicity Formulas

For the convenience of the reader, we provide a summary of formulas for calculating invariants of pure quintic fields $L=\mathbb{Q}\left(\sqrt[5]{D}\right)$ with normalized fifth power free radicands $D>1$ and their associated pure metacyclic normal fields $N=\mathbb{Q}\left(\zeta,\sqrt[5]{D}\right)$ with a primitive fifth root of unity $\zeta=\zeta_5$.

Let f be the class field theoretic conductor of the relatively quintic Kummer extension N/K over the cyclotomic field $K=\mathbb{Q}(\zeta)$. It is also called the conductor of the pure quintic field L. The *multiplicity* $m=m(f)$ of the conductor f indicates the number of non-isomorphic pure metacyclic fields N sharing the common conductor f, or also, according to ([6] Prop. 2.1), the number of normalized fifth power free radicands $D>1$ whose fifth roots generate non-isomorphic pure quintic fields L sharing the common conductor f. The cardinality of a set S is denoted $\#S$.

We adapt the general multiplicity formulas in ([13] Thm. 2, p. 104) to the quintic case $p=5$. If L is a field of species 1a ([6] (2.6) and Exm. 2.2), *i.e.* $f^4=5^6\cdot q_1^4\cdots q_t^4$, then $m=4^t$ where $t:=\#\{q\in\mathbb{P}\mid q\ne 5, q\mid f\}$. The explicit values of m in dependence on t are given in **Table 1**.

If L is a field of species 1b ([6] (2.6) and Exm. 2.2), *i.e.* $f^4=5^2\cdot q_1^4\cdots q_t^4$, then $m=4^u\cdot X_v$ where $u:=\#\{q\in\mathbb{P}\mid q\equiv\pm1,\pm7 \bmod (25), q\mid f\}$, $v:=t-u$ and $X_j:=\frac{1}{5}\left(4^j-(-1)^j\right)$, that is $\left(X_j\right)_{j\ge -1}=\left(\frac{1}{4},0,1,3,13,51,205,\cdots\right)$. The explicit values of m in dependence on u and v are given in **Table 2**.

If L is a field of species 2 ([6] (2.6) and Exm. 2.2), *i.e.* $f^4=5^0\cdot q_1^4\cdots q_t^4$, then $m=4^u\cdot X_{v-1}$ where $u:=\#\{q\in\mathbb{P}\mid q\equiv\pm1,\pm7 \bmod (25), q\mid f\}$, $v:=t-u$ and $X_j:=\frac{1}{5}\left(4^j-(-1)^j\right)$, that is $\left(X_j\right)_{j\ge -1}=\left(\frac{1}{4},0,1,3,13,51,205,\cdots\right)$. The explicit values of m in dependence on u and v are given in **Table 3**.

3. Classification by DPF Types in 20 Numerical Tables

3.1. DPF Types

The following twenty **Tables 6-25** establish a complete classification of all 900 pure metacyclic fields $N=\mathbb{Q}(\zeta,5D)$ with normalized radicands in the range $2\le D<10^3$. With the aid of PARI/GP [2] and MAGMA [5] we have determined the *differential principal factorization type*, T, of each field N by means of other invariants U,A,I,R ([6] Thm. 6.1). After several weeks of CPU time, the date of completion was September 17, 2018.

The possible DPF types are listed in dependence on U,A,I,R in **Table 4**, where the symbol $\times$ in the column η, resp. ζ, indicates the existence of a unit $H\in U_N$, resp. $Z\in U_N$, such that $\eta=N_{N/K}(H)$, resp. $\zeta=N_{N/K}(Z)$. The 5-valuation of the *unit norm index* $\left(U_K:N_{N/K}U_N\right)$ is abbreviated by U ([6] (1.3), (6.3)]. Here, $\eta=\frac{1}{2}\left(1+\sqrt{5}\right)$ denotes the fundamental unit of $K^+=Q\left(\sqrt{5}\right)$.

Table 1. Multiplicity of fields of species 1a.

t	0	1	2	3	4	5
m	1	4	16	64	256	1024

Table 2. Multiplicity of fields of species 1b.

u	v	0	1	2	3	4	5
0	m	0	1	3	13	51	205
1		0	4	12	52	204	820
2		0	16	48	208	816	
3		0	64	192	832		
4		0	256	768			

Table 3. Multiplicity of fields of species 2.

u	v	0	1	2	3	4	5
0	m	0	0	1	3	13	51
1		1	0	4	12	52	204
2		4	0	16	48	208	816
3		16	0	64	192	832	
4		64	0	256	768		

Table 4. Differential principal factorization types, T, of pure metacyclic fields N.

T	U	η	ζ	A	I	R
α_1	2	–	–	1	0	2
α_2	2	–	–	1	1	1
α_3	2	–	–	1	2	0
β_1	2	–	–	2	0	1
β_2	2	–	–	2	1	0
γ	2	–	–	3	0	0
δ_1	1	×	–	1	0	1
δ_2	1	×	–	1	1	0
ε	1	×	–	2	0	0
ζ_1	1	–	×	1	0	1
ζ_2	1	–	×	1	1	0
η	1	–	×	2	0	0
ϑ	0	×	×	1	0	0

3.2. Justification of the Computational Techniques

The steps of the following classification algorithm are ordered by increasing requirements of CPU time. To avoid unnecessary time consumption, the algorithm stops at early stages already, as soon as the DPF type is determined unambiguously. The illustrating subfield lattice of N is drawn in Figure 1.

Algorithm 3.1 (*Classification into* 13 *DPF types.*)

Input: a normalized fifth power free radicand $D \ge 2$.

Step 1: By purely *rational* methods, without any number field constructions, the prime factorization of the radicand D (including the counters $t,u,v;n,s_2,s_4$, § 4.2) is determined. If $D = q \in \mathbb{P}$, $q \equiv \pm 2 \bmod (5)$, $q \not\equiv \pm 7 \bmod (25)$, then N is a Polya field of type ε; stop. If $D = q \in \mathbb{P}$, $q = 5$ or $q \equiv \pm 7 \bmod (25)$, then N is a Polya field of type ϑ; stop.

Step 2: The field L of degree 5 is constructed. The primes $q_1,\cdots,q_T$ dividing the conductor f of N/K are determined, and their overlying prime ideals $\mathfrak{q}_1,\cdots,\mathfrak{q}_T$ in L are computed. By means of at most 5^T principal ideal tests of

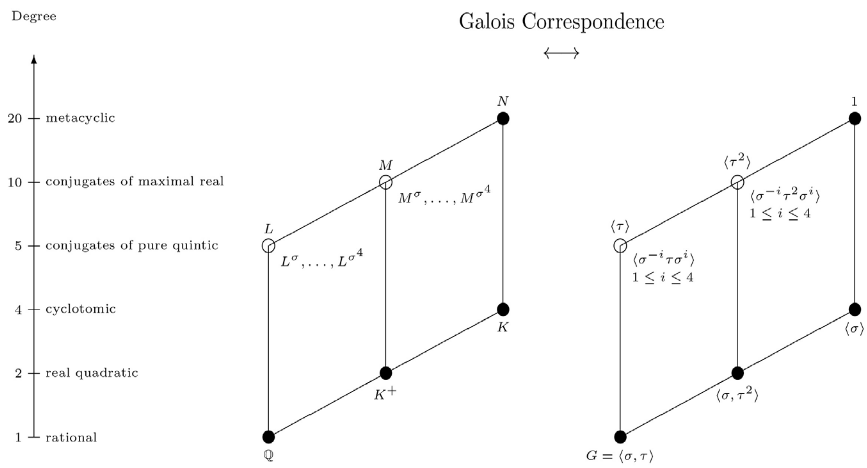

Figure 1. Lattices of subfields of N and of subgroups of $G=\mathrm{Gal}(N/\mathbb{Q})$.

the elements of $\mathcal{I}_{L/\mathbb{Q}}/\mathcal{I}_{\mathbb{Q}}=\bigoplus_{i=1}^{T}\mathbb{F}_5\mathfrak{q}_i$, the number $5^A:=\#\left\{(v_1,\cdots,v_T)\in\mathbb{F}_5^T\mid\prod_{i=1}^{T}\mathfrak{q}_i^{v_i}\in\mathcal{P}_L\right\}$, that is the cardinality of $\mathcal{P}_{L/\mathbb{Q}}/\mathcal{P}_{\mathbb{Q}}$, is determined. If $A=T$, then N is a Polya field. If $A=3$, then N is of type γ; stop. If $A=2$, $s_2=s_4=0$, $v\ge 1$, then N is of type ε; stop. If $A=1$, $s_2=s_4=0$, then N is of type ϑ; stop.

Step 3: If $s_2\ge 1$ or $s_4\ge 1$, then the field M of degree 10 is constructed. For the 2-split primes $\ell_1,\cdots,\ell_{s_2+s_4}\equiv\pm 1\bmod(5)$ among the primes $q_1,\cdots,q_T$ dividing the conductor f of N/K, the overlying prime ideals $\mathcal{L}_1,\mathcal{L}_1^\tau,\cdots,\mathcal{L}_{s_2+s_4},\mathcal{L}_{s_2+s_4}^\tau$ in M are computed. By means of at most $5^{s_2+s_4}$ principal ideal tests of the elements of $\left(\mathcal{I}_{M/K^+}/\mathcal{I}_{K^+}\right)\bigcap\ker\left(N_{M/L}\right)=\bigoplus_{i=1}^{s_2+s_4}\mathbb{F}_5\mathcal{K}_{(\ell_i)}$, where $\mathcal{K}_{(\ell_i)}=\mathcal{L}_i^{1+4\tau}$ for $1\le i\le s_2+s_4$, the number $5^I:=\#\left\{(v_1,\cdots,v_{s_2+s_4})\in\mathbb{F}_5^{s_2+s_4}\mid\prod_{i=1}^{s_2+s_4}\mathcal{K}_{(\ell_i)}^{v_i}\in\mathcal{P}_M\right\}$, that is the cardinality of $\left(\mathcal{P}_{M/K^+}/\mathcal{P}_{K^+}\right)\bigcap\ker\left(N_{M/L}\right)$, is determined. If $I=2$, then N is of type α_3; stop. If $I=1$, $A=2$, then N is of type β_2; stop.

Step 4: If $s_4\ge 1$, then the field N of degree 20 is constructed. For all 4-split primes $\ell_{s_2+1},\cdots,\ell_{s_2+s_4}\equiv+1\bmod(5)$ among the primes $q_1,\cdots,q_T$ dividing the conductor f of N/K, the overlying prime ideals $\mathfrak{L}_{s_2+1},\mathfrak{L}_{s_2+1}^{\tau^2},\mathfrak{L}_{s_2+1}^{\tau},\mathfrak{L}_{s_2+1}^{\tau^3},\cdots,\mathfrak{L}_{s_2+s_4},\mathfrak{L}_{s_2+s_4}^{\tau^2},\mathfrak{L}_{s_2+s_4}^{\tau},\mathfrak{L}_{s_2+s_4}^{\tau^3}$ in N are computed. By means of at most 5^{2s_4} principal ideal tests of the elements of $\left(\mathcal{I}_{N/K}/\mathcal{I}_K\right)\bigcap\ker\left(N_{N/M}\right)=\bigoplus_{i=s_2+1}^{s_2+s_4}\left(\mathbb{F}_5\mathfrak{K}_{1,(\ell_i)}\oplus\mathbb{F}_5\mathfrak{K}_{2,(\ell_i)}\right)$, where $\mathfrak{K}_{1,(\ell_i)}=\mathfrak{L}_i^{1+4\tau^2+2\tau+3\tau^3}$ and $\mathfrak{K}_{2,(\ell_i)}=\mathfrak{L}_i^{1+4\tau^2+3\tau+2\tau^3}$ for $s_2+1\le i\le s_2+s_4$, the number $5^R:=\#\left\{(v_{1,s_2+1},v_{2,s_2+1},\cdots,v_{1,s_2+s_4},v_{2,s_2+s_4})\in\mathbb{F}_5^{2s_4}\mid\prod_{i=s_2+1}^{s_2+s_4}\left(\mathfrak{K}_{1,(\ell_i)}^{v_{1,i}}\mathfrak{K}_{2,(\ell_i)}^{v_{2,i}}\right)\in\mathcal{P}_N\right\}$, that is the cardinality of $\left(\mathcal{P}_{N/K}/\mathcal{P}_K\right)\bigcap\ker\left(N_{N/M}\right)$, is determined. If $R=2$, then N is of type α_1; stop. If $R=1$, $I=1$, then N is of type α_2; stop. If $R=1$, $A=2$, then N is of type β_1; stop.

Step 5: If the type of the field N is not yet determined uniquely, then $U=1$ and there remain the following possibilities. If $v\ge 1$, then N is of type δ_1, if $R=1$, of type δ_2, if $I=1$, and of type ε, if $R=I=0$. If $v=0$, then a fundamental system $(E_j)_{1\le j\le 9}$ of units is constructed for the unit group U_N of the field N of degree 20, and all relative norms of these units with respect to the cyclotomic subfield K are computed. If $N_{N/K}(E_j)=\zeta_5^k$ for some $1\le j\le 9$, $1\le k\le 4$, then N is of type ζ_1, if $R=1$, of type ζ_2, if $I=1$, and of type η, if $R=I=0$. Otherwise the conclusions are the same as for $v\ge 1$.

Output: the DPF type of the field $N=\mathbb{Q}\left(\zeta_5,\sqrt[5]{D}\right)$ and the decision about its Polya property.

Proof. The claims of Step 1 concerning the types ε,ϑ are proved in items (1) and (2) of ([6] Thm. 10.1).

For Step 2, the formulas (4.1) and (4.2) in ([6] Thm. 4.1) give an $\mathbb{F}_5$-basis of the space of absolute differential factors, and the formulas (4.3) and (4.4) in ([6] Cor. 4.1) determine bounds for the $\mathbb{F}_5$-dimension A of the space of *absolute* DPF in the field L of degree 5. The Polya property was characterized in ([6] Thm. 10.5)], the claim concerning type γ follows from ([6] Thm. 6.1), and the claims about the types ε,ϑ from ([6] Thm. 8.1 and Thm. 6.1).

For Step 3, the formulas (4.5) and (4.6) in ([6] Thm. 4.3) give an $\mathbb{F}_5$-basis of the space of intermediate differential factors, and the formulas (4.7) and (4.8) in ([6] Cor. 4.2) determine bounds for the $\mathbb{F}_5$-dimension I of the space of *intermediate* DPF in the field M of degree 10. The claims concerning the types α_3,β_2 are consequences of ([6] Thm. 6.1),

For Step 4, the formulas (4.9) and (4.10) in ([6] Thm. 4.4) give an $\mathbb{F}_5$-basis of the space of relative differential factors, and the formulas (4.11) and (4.12) in ([6] Cor. 4.3) determine bounds for the $\mathbb{F}_5$-dimension R of the space of *relative* DPF in the field N of degree 20. The claims concerning the types $\alpha_1,\alpha_2,\beta_1$ are consequences of ([6] Thm. 6.1).

Concerning Step 5, the signature of N is $(r_1,r_2)=(0,10)$, whence the torsion free Dirichlet unit rank of N is given by $r=r_1+r_2-1=9$. The claims about all types are consequences of ([6] Thm. 6.1), including information on the constitution of the norm group $N_{N/K}(U_N)$.

Remark 3.1 Whereas the execution of Step 1 and 2 in Algorithm 3.1, implemented as a Magma program [5], is a matter of a few seconds on a machine with two Intel XEON 8-core processors and clock frequency 2 GHz, the CPU time for Step 3 lies in the range of several minutes. The time requirement for Step 4 and 5 can reach hours or even days in spite of code optimizations for the calculation of units, in particular the use of the Magma procedures IndependentUnits() and SetOrderUnitsAreFundamental() prior to the call of UnitGroup().

3.3. Open Problems

We conjecture that considerable amounts of CPU time can be saved in our Algorithm 3.1 by computing the logarithmic 5-class numbers $V_F:=v_5(h_F)$ of the fields $F\in\{L,M,N\}$, which admit the determination of the logarithmic indices E, resp. E^+, of subfield units in the Parry [1], resp. Kobayashi [14] [15], class number relation, according to the formulas

$$E = 5 + V_N - 4 \cdot V_L, \quad E^+ = 2 + V_M - 2 \cdot V_L. \tag{3.1}$$

However, first there would be required rigorous proofs of the heuristic connections between E, E^+ and the DPF types in Table 5, where $(E, E^+) = (1,0)$ implies type α_2, $(E, E^+) = (2,0)$ implies type α_3, $(E, E^+) = (4,2)$ implies type δ_1, but $(E, E^+) = (2,1)$ admits types α_1, δ_1, $(E, E^+) = (3,1)$ admits types $\alpha_2, \beta_1, \delta_2$, $(E, E^+) = (4,1)$ admits types β_2, ζ_2, $(E, E^+) = (5,2)$ admits types $\beta_1, \varepsilon, \zeta_1, \vartheta$, and $(E, E^+) = (6,2)$ admits types γ, η. $(E, E^+) = (0,0)$ seems to be impossible.

3.4. Conventions and Notation in the Tables

The *normalized* radicand $D = q_1^{e_1} \cdots q_s^{e_s}$ of a pure metacyclic field *N* of degree 20 is minimal among the powers D^n, $1 \le n \le 4$, with corresponding exponents e_j reduced modulo 5. The normalization of the radicands *D* provides a warranty that all fields are pairwise non-isomorphic ([6] Prop. 2.1).

Prime factors are given for composite radicands *D* only. Dedekind's *species*, S, of radicands is refined by distinguishing $5 \mid D$ (species 1a) and $\gcd(5, D) = 1$ (species 1b) among radicands $D \not\equiv \pm 1, \pm 7 \bmod (25)$ (species 1). By the species and factorization of *D*, the shape of the *conductor f* is determined. We give the fourth power f^4 to avoid fractional exponents. Additionally, the *multiplicity m* indicates the number of non-isomorphic fields sharing a common conductor *f* (§ 2). The symbol V_F briefly denotes the 5-valuation of the order $h_F = \#\mathrm{Cl}(F)$ of the class group $\mathrm{Cl}(F)$ of a number field *F*. By *E* we denote the exponent of the power in the *index of subfield units* $(U_N : U_0) = 5^E$.

Table 5. Logarithmic indices E, E^+ of subfield units for DPF types, T.

T	E	E^+	or	E	E^+
α_1	2	1			
α_2	1	0		3	1
α_3	2	0			
β_1	3	1		5	2
β_2	4	1			
γ	6	2			
δ_1	2	1		4	2
δ_2	3	1			
ε	5	2			
ζ_1	5	2			
ζ_2	4	1			
η	6	2			
ϑ	5	2			

An asterisk denotes the smallest radicand with given Dedekind kind, DPF type and 5-class groups $\mathrm{Cl}_5(F)$, $F \in \{L, M, N\}$. The latter are usually elementary abelian, except for the cases indicated by an additional asterisk (see § 4.4).

Principal factors, P, are listed when their constitution is not a consequence of the other information. According to ([6] Thm. 7.2., item (1)) it suffices to give the rational integer norm of *absolute* principal factors. For *intermediate* principal factors, we use the symbols $\mathcal{K} := \mathcal{L}^{1-\tau} = \alpha\mathcal{O}_M$ with $\alpha \in M$ or $\mathcal{L} = \lambda\mathcal{O}_M$ with a prime element $\lambda \in M$ (which implies $\mathcal{L}^{\tau} = \lambda^{\tau}\mathcal{O}_M$ and thus also $\mathcal{K} = \lambda^{1-\tau}\mathcal{O}_M$). Here, $\left(\mathcal{L}^{1+\tau}\right)^5 = \ell\mathcal{O}_M$ when a prime $\ell \equiv \pm 1 \bmod (5)$ divides the radicand D. For *relative* principal factors, we use the symbols $\mathfrak{K}_1 := \mathfrak{L}^{1+4\tau^2+2\tau+3\tau^3} = A_1\mathcal{O}_N$ and $\mathfrak{K}_2 := \mathfrak{L}^{1+4\tau^2+3\tau+2\tau^3} = A_2\mathcal{O}_N$ with $A_1, A_2 \in N$. Here, $\left(\mathfrak{L}^{1+\tau+\tau^2+\tau^3}\right)^5 = \ell\mathcal{O}_N$ when a prime number $\ell \equiv +1 \bmod (5)$ divides the radicand D. (Kernel ideals in [6] § 7)

The quartet $(1,2,4,5)$ indicates conditions which either enforce a reduction of possible DPF types or enable certain DPF types. The lack of a prime divisor $\ell \equiv \pm 1 \bmod (5)$ together with the existence of a prime divisor $q \not\equiv \pm 7 \bmod (25)$ and $q \neq 5$ of D is indicated by a symbol × for the component 1. In these cases, only the two DPF types γ and ε can occur ([6] Thm. 8.1).

A symbol × for the component 2 emphasizes a prime divisor $\ell \equiv -1 \bmod (5)$ of D and the possibility of intermediate principal factors in M, like $\mathcal{L}$ and $\mathcal{K}$. A symbol × for the component 4 emphasizes a prime divisor $\ell \equiv +1 \bmod (5)$ of D and the possibility of relative principal factors in N, like $\mathfrak{K}_1$ and $\mathfrak{K}_2$. The × symbol is replaced by ⊗ if the facility is used completely, and by (×) if the facility is only used partially.

If D has only prime divisors $q \equiv \pm 1, \pm 7 \bmod (25)$ or $q = 5$, a symbol × is placed in component 5. In these cases, ζ can occur as a norm $N_{N/K}(Z)$ of some unit in $Z \in U_N$. If it actually does, the × is replaced by ⊗ ([6] § 8).

4. Statistical Evaluation and Refinements

4.1. Statistics of DPF Types

The complete statistical evaluation of the following twenty **Tables 6-25** is given in **Table 26**. The first ten columns show the absolute frequencies of pure metacyclic fields $N = \mathbb{Q}\left(\zeta, \sqrt[5]{D}\right)$ with various DPF types for the ranges $2 \le D < n \cdot 100$ with $1 \le n \le 10$. The eleventh column lists the relative percentages of the five most frequent DPF types for the complete range $2 \le D < 10^3$ of normalized radicands.

Among our 13 differential principal factorization types, type γ with 3-dimensional absolute principal factorization, $A = 3$, is clearly dominating with more than one third (36%) of all occurrences in the complete range $2 \le D < 10^3$, followed by type ε with 2-dimensional absolute principal factorization, $A = 2$, which covers nearly one quarter (23%) of all cases. The third place (nearly 18%) is occupied by type β_2 with mixed absolute and intermediate principal factorization, $A = 2$, $I = 1$.

Table 6. 40 pure metacyclic fields with normalized radicands $2 \le D \le 52$.

No.	D	Factors	S	f^4	m	V_L	V_M	V_N	E	$(1,2,4,5)$	T	P
1	*2		1b	$5^2 2^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
2	3		1b	$5^2 3^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
3	*5		1a	5^6	1	0	0	0	5	$(-,-,-,\otimes)$	ϑ	
4	*6	2×3	1b	$5^2 2^4 3^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
5	*7		2	7^4	1	0	0	0	5	$(-,-,-,\otimes)$	ϑ	
6	*10	2×5	1a	$5^6 2^4$	4	0	0	0	5	$(\times,-,-,-)$	ε	
7	*11		1b	$5^2 11^4$	1	1	1	2	3	$(-,-,\otimes,-)$	α_2	$\mathcal{L},\mathfrak{K}_1$
8	12	$2^2\times 3$	1b	$5^2 2^4 3^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
9	13		1b	$5^2 13^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
10	*14	2×7	1b	$5^2 2^4 7^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	
11	15	3×5	1a	$5^6 3^4$	4	0	0	0	5	$(\times,-,-,-)$	ε	
12	17		1b	$5^2 17^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
13	*18	2×3^2	2	$2^4 3^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
14	*19		1b	$5^2 19^4$	1	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{L}$
15	20	$2^2\times 5$	1a	$5^6 2^4$	4	0	0	0	5	$(\times,-,-,-)$	ε	
16	21	3×7	1b	$5^2 3^4 7^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	
17	*22	2×11	1b	$5^2 2^4 11^4$	3	1	1	3	4	$(-,-,(\times),-)$	β_2	$2\times 5,\mathcal{K}$
18	23		1b	$5^2 23^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
19	26	2×13	2	$2^4 13^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
20	28	$2^2\times 7$	1b	$5^2 2^4 7^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	
21	29		1b	$5^2 29^4$	1	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{L}$
22	*30	$2\times 3\times 5$	1a	$5^6 2^4 3^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
23	*31		1b	$5^2 31^4$	1	2	3	5	2	$(-,-,\otimes,-)$	α_1	$\mathfrak{K}_1,\mathfrak{K}_2$
24	*33	3×11	1b	$5^2 3^4 11^4$	3	2	2	4	1	$(-,-,\otimes,-)$	α_2	$\mathcal{K},\mathfrak{K}_2$
25	34	2×17	1b	$5^2 2^4 17^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
26	*35	5×7	1a	$5^6 7^4$	4	0	0	1	6	$(-,-,-,\otimes)$	η	
27	37		1b	$5^2 37^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
28	*38	2×19	1b	$5^2 2^4 19^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$5,\mathcal{K}$
29	39	3×13	1b	$5^2 3^4 13^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
30	40	$2^3\times 5$	1a	$5^6 2^4$	4	0	0	0	5	$(\times,-,-,-)$	ε	
31	41		1b	$5^2 41^4$	1	1	1	2	3	$(-,-,\otimes,-)$	α_2	$\mathcal{L},\mathfrak{K}_2$
32	*42	$2\times 3\times 7$	1b	$5^2 2^4 3^4 7^4$	12	1	2	5	6	$(\times,-,-,-)$	γ	$2\times 5,3\times 5^2$
33	43		2	43^4	1	0	0	0	5	$(-,-,-,\otimes)$	ϑ	
34	44	$2^2\times 11$	1b	$5^2 2^4 11^4$	3	1	1	3	4	$(-,-,(\times),-)$	β_2	$2\times 5,\mathcal{K}$
35	45	$3^2\times 5$	1a	$5^6 3^4$	4	0	0	0	5	$(\times,-,-,-)$	ε	
36	46	2×23	1b	$5^2 2^4 23^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
37	47		1b	$5^2 47^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
38	48	$2^4\times 3$	1b	$5^2 2^4 3^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
39	51	3×17	2	$3^4 17^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
40	52	$2^2\times 13$	1b	$5^2 2^4 13^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	

Table 7. 45 pure metacyclic fields with normalized radicands $53 \le D \le 104$.

No.	D	Factors	S	f^4	m	V_L	V_M	V_N	E	(1,2,4,5)	T	P
41	53		1b	$5^2 53^4$	1	0	0	0	2	$(\times,-,-,-)$	ε	
42	*55	5×11	1a	$5^6 11^4$	4	1	1	2	3	$(-,-,\otimes,-)$	α_2	$\mathcal{K},\mathfrak{K}_1$
43	56	$2^3\times 7$	1b	$5^2 2^4 7^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	
44	*57	3×19	2	$3^4 19^4$	1	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{K}$
45	58	2×29	1b	$5^2 2^4 29^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$29\times 5^2,\mathcal{K}$
46	59		1b	$5^2 59^4$	1	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{L}$
47	60	$2^2\times 3\times 5$	1a	$5^6 2^4 3^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
48	61		1b	$5^2 61^4$	1	1	1	2	3	$(-,-,\otimes,-)$	α_2	$\mathcal{L},\mathfrak{K}_2$
49	62	2×31	1b	$5^2 2^4 31^4$	3	1	1	3	4	$(-,-,(\times),-)$	β_2	$5,\mathcal{K}$
50	63	$3^2\times 7$	1b	$5^2 3^4 7^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	
51	65	5×13	1a	$5^6 13^4$	4	0	0	0	5	$(\times,-,-,-)$	ε	
52	*66	$2\times 3\times 11$	1b	$5^2 2^4 3^4 11^4$	13	1	2	5	6	$(-,-,\times,-)$	γ	$2\times 5,3\times 5^3$
53	67		1b	$5^2 67^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
54	68	$2^2\times 17$	2	$2^4 17^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
55	69	3×23	1b	$5^2 3^4 23^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
56	*70	$2\times 5\times 7$	1a	$5^6 2^4 7^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
57	71		1b	$5^2 71^4$	1	1	1	2	3	$(-,-,\otimes,-)$	α_2	$\mathcal{L},\mathfrak{K}_1$
58	73		1b	$5^2 73^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
59	74	2×37	2	$2^4 37^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
60	75	3×5^2	1a	$5^6 3^4$	4	0	0	0	5	$(\times,-,-,-)$	ε	
61	76	$2^2\times 19$	2	$2^4 19^4$	1	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{K}$
62	*77	7×11	1b	$5^2 7^4 11^4$	4	1	1	3	4	$(-,-,(\times),-)$	β_2	$11\times 5^3,\mathcal{K}$
63	*78	$2\times 3\times 13$	1b	$5^2 2^4 3^4 13^4$	13	1	2	5	6	$(\times,-,-,-)$	γ	$3,2\times 5^3$
64	79		1b	$5^2 79^4$	1	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{L}$
65	80	$2^4\times 5$	1a	$5^6 2^4$	4	0	0	0	5	$(\times,-,-,-)$	ε	
66	*82	2×41	2	$2^4 41^4$	1	1	1	2	3	$(-,-,\otimes,-)$	α_2	$\mathcal{K},\mathfrak{K}_2$
67	83		1b	$5^2 83^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
68	84	$2^2\times 3\times 7$	1b	$5^2 2^4 3^4 7^4$	12	1	2	5	6	$(\times,-,-,-)$	γ	$2\times 5,3\times 5$
69	85	5×17	1a	$5^6 17^4$	4	0	0	0	5	$(\times,-,-,-)$	ε	
70	86	2×43	1b	$5^2 2^4 43^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	
71	87	3×29	1b	$5^2 3^4 29^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$29,\mathcal{L}$
72	88	$2^3\times 11$	1b	$5^2 2^4 11^4$	3	2	2	4	1	$(-,-,\otimes,-)$	α_2	$\mathcal{K},\mathfrak{K}_2$
73	89		1b	$5^2 89^4$	1	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{L}$
74	90	$2\times 3^2\times 5$	1a	$5^6 2^4 3^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
75	91	7×13	1b	$5^2 7^4 13^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	
76	92	$2^2\times 23$	1b	$5^2 2^4 23^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
77	93	3×31	2	$3^4 31^4$	1	1	1	2	3	$(-,-,\otimes,-)$	α_2	$\mathcal{K},\mathfrak{K}_1$
78	94	2×47	1b	$5^2 2^4 47^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
79	*95	5×19	1a	$5^6 19^4$	4	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{K}$
80	97		1b	$5^2 97^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
81	99	$3^2\times 11$	2	$3^4 11^4$	1	1	1	2	3	$(-,-,\otimes,-)$	α_2	$\mathcal{K},\mathfrak{K}_1$
82	*101		2	101^4	1	1	2	4	5	$(-,-,\otimes,\otimes)$	ζ_1	$\mathfrak{K}_2$
83	102	$2\times 3\times 17$	1b	$5^2 2^4 3^4 17^4$	13	1	2	5	6	$(\times,-,-,-)$	γ	$2,3\times 5$
84	103		1b	$5^2 103^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
85	104	$2^3\times 13$	1b	$5^2 2^4 13^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	

Table 8. 45 pure metacyclic fields with normalized radicands $105 \le D \le 155$.

No.	D	Factors	S	f^4	m	V_L	V_M	V_N	E	$(1,2,4,5)$	T	P
86	105	$3\times5\times7$	1a	$5^6 3^4 7^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
87	106	2×53	1b	$5^2 2^4 53^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
88	107		2	107^4	1	0	0	0	5	$(-,-,-,\otimes)$	ϑ	
89	109		1b	$5^2 109^4$	1	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{L}$
90	*110	$2\times5\times11$	1a	$5^6 2^4 11^4$	16	1	1	3	4	$(-,-,(\times),-)$	β_2	$11,\mathcal{L}$
91	111	3×37	1b	$5^2 3^4 37^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
92	112	$2^4\times7$	1b	$5^2 2^4 7^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	
93	113		1b	$5^2 113^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
94	*114	$2\times3\times19$	1b	$5^2 2^4 3^4 19^4$	13	1	2	5	6	$(-,\times,-,-)$	γ	$2\times5^3,3\times5^3$
95	115	5×23	1a	$5^6 23^4$	4	0	0	0	5	$(\times,-,-,-)$	ε	
96	116	$2^2\times29$	1b	$5^2 2^4 29^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$29\times5,\mathcal{K}$
97	117	$3^2\times13$	1b	$5^2 3^4 13^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
98	118	2×59	2	$2^4 59^4$	1	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{K}$
99	119	7×17	1b	$5^2 7^4 17^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	
100	120	$2^3\times3\times5$	1a	$5^6 2^4 3^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
101	122	2×61	1b	$5^2 2^4 61^4$	3	1	1	3	4	$(-,-,(\times),-)$	β_2	$61\times5^3,\mathcal{K}$
102	*123	3×41	1b	$5^2 3^4 41^4$	3	2	3	6	3	$(-,-,\otimes,-)$	α_2	$\mathcal{K},\mathfrak{K}_2$
103	124	$2^2\times31$	2	$2^4 31^4$	1	1	1	2	3	$(-,-,\otimes,-)$	α_2	$\mathcal{K},\mathfrak{K}_1$
104	*126	$2\times3^2\times7$	2	$2^4 3^4 7^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	
105	127		1b	$5^2 127^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
106	129	3×43	1b	$5^2 3^4 43^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	
107	130	$2\times5\times13$	1a	$5^6 2^4 13^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
108	*131		1b	$5^2 131^4$	1	2	2	4	1	$(-,-,\otimes,-)$	α_2	$\mathcal{L},\mathfrak{K}_2$
109	*132	$2^2\times3\times11$	2	$2^4 3^4 11^4$	3	1	1	3	4	$(-,-,(\times),-)$	β_2	$11,\mathcal{L}$
110	*133	7×19	1b	$5^2 7^4 19^4$	4	1	1	3	4	$(-,\otimes,-,-)$	β_2	$7,\mathcal{L}$
111	134	2×67	1b	$5^2 2^4 67^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
112	136	$2^3\times17$	1b	$5^2 2^4 17^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
113	137		1b	$5^2 137^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
114	138	$2\times3\times23$	1b	$5^2 2^4 3^4 23^4$	13	1	2	5	6	$(\times,-,-,-)$	γ	$3,2\times5^2$
115	*139		1b	$5^2 139^4$	1	1	1	3	4	$(-,\otimes,-,-)$	β_2	$5,\mathcal{L}$
116	*140	$2^2\times5\times7$	1a	$5^6 2^4 7^4$	16	1	2	4	5	$(\times,-,-,-)$	ε	7
117	*141	3×47	1b	$5^2 3^4 47^4$	3	1	2	4	5	$(\times,-,-,-)$	ε	47×5
118	142	2×71	1b	$5^2 2^4 71^4$	3	1	1	3	4	$(-,-,(\times),-)$	β_2	$71\times5^2,\mathcal{K}$
119	143	11×13	2	$11^4 13^4$	1	1	1	2	3	$(-,-,\otimes,-)$	α_2	$\mathcal{K},\mathfrak{K}_1$
120	145	5×29	1a	$5^6 29^4$	4	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{K}$
121	146	2×73	1b	$5^2 2^4 73^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
122	147	3×7^2	1b	$5^2 3^4 7^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	
123	148	$2^2\times37$	1b	$5^2 2^4 37^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
124	*149		2	149^4	1	1	1	2	3	$(-,\otimes,-,\times)$	δ_2	$\mathcal{L}$

Continued

125	150	$2\times 3\times 5^2$	1a	$5^6 2^4 3^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
126	*151		2	151^4	1	1	1	2	3	$(-,-,\otimes,\times)$	α_2	$\mathcal{L},\mathfrak{K}_1$
127	152	$2^3\times 19$	1b	$5^2 2^4 19^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$19\times 5^2,\mathcal{K}$
128	153	$3^2\times 17$	1b	$5^2 3^4 17^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
129	*154	$2\times 7\times 11$	1b	$5^2 2^4 7^4 11^4$	12	2	3	7	4	$(-,-,(\times),-)$	β_2	$2\times 11^2,\mathcal{K}$
130	*155	5×31	1a	$5^6 31^4$	4	2	3	5	2	$(-,-,\otimes,-)$	α_1	$\mathfrak{K}_1,\mathfrak{K}_2$

Table 9. 45 pure metacyclic fields with normalized radicands $156\le D\le 207$.

No.	D	Factors	S	f^4	m	V_L	V_M	V_N	E	$(1,2,4,5)$	T	P
131	156	$2^2\times 3\times 13$	1b	$5^2 2^4 3^4 13^4$	13	1	2	5	6	$(\times,-,-,-)$	γ	$3,13\times 5^2$
132	157		2	157^4	1	0	0	0	5	$(-,-,-,\otimes)$	ϑ	
133	158	2×79	1b	$5^2 2^4 79^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$5,\mathcal{K}$
134	159	3×53	1b	$5^2 3^4 53^4$	3	1	2	4	5	$(\times,-,-,-)$	ε	5
135	161	7×23	1b	$5^2 7^4 23^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	
136	163		1b	$5^2 163^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
137	164	$2^2\times 41$	1b	$5^2 2^4 41^4$	3	1	1	3	4	$(-,-,(\times),-)$	β_2	$41\times 5,\mathcal{K}$
138	165	$3\times 5\times 11$	1a	$5^6 3^4 11^4$	16	1	1	3	4	$(-,-,(\times),-)$	β_2	$5\times 11^2,\mathcal{K}$
139	166	2×83	1b	$5^2 2^4 83^4$	3	1	2	4	5	$(\times,-,-,-)$	ε	5
140	167		1b	$5^2 167^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
141	168	$2^3\times 3\times 7$	2	$2^4 3^4 7^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	
142	170	$2\times 5\times 17$	1a	$5^6 2^4 17^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
143	*171	$3^2\times 19$	1b	$5^2 3^4 19^4$	3	1	2	4	5	$(-,\times,-,-)$	ε	19×5^2
144	172	$2^2\times 43$	1b	$5^2 2^4 43^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	
145	173		1b	$5^2 173^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
146	*174	$2\times 3\times 29$	2	$2^4 3^4 29^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$3^2\times 29,\mathcal{K}$
147	175	$5^2\times 7$	1a	$5^6 7^4$	4	0	0	1	6	$(-,-,-,\otimes)$	η	
148	176	$2^4\times 11$	2	$2^4 11^4$	1	1	1	2	3	$(-,-,\otimes,-)$	α_2	$\mathcal{K},\mathfrak{K}_2$
149	177	3×59	1b	$5^2 3^4 59^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$3,\mathcal{L}$
150	178	2×89	1b	$5^2 2^4 89^4$	3	1	2	4	5	$(-,\times,-,-)$	ε	89×5^2
151	179		1b	$5^2 179^4$	1	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{L}$
152	*180	$2^2\times 3^2\times 5$	1a	$5^6 2^4 3^4$	16	1	2	4	5	$(\times,-,-,-)$	ε	3
153	181		1b	$5^2 181^4$	1	2	2	4	1	$(-,-,\otimes,-)$	α_2	$\mathcal{L},\mathfrak{K}_2$
154	*182	$2\times 7\times 13$	2	$2^4 7^4 13^4$	4	1	2	4	5	$(\times,-,-,-)$	ε	7
155	183	3×61	1b	$5^2 3^4 61^4$	3	1	1	3	4	$(-,-,(\times),-)$	β_2	$61\times 5^3,\mathcal{K}$
156	184	$2^3\times 23$	1b	$5^2 2^4 23^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
157	185	5×37	1a	$5^6 37^4$	4	0	0	0	5	$(\times,-,-,-)$	ε	
158	*186	$2\times 3\times 31$	1b	$5^2 2^4 3^4 31^4$	13	2	3	6	3	$(-,-,\otimes,-)$	β_1	$5,\mathfrak{K}_2$
159	187	11×17	1b	$5^2 11^4 17^4$	3	1	1	3	4	$(-,-,(\times),-)$	β_2	$11\times 5,\mathcal{K}$

Continued

160	188	$2^2\times 47$	1b	$5^2 2^4 47^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
161	*190	$2\times 5\times 19$	1a	$5^6 2^4 19^4$	16	1	1	3	4	$(-,\otimes,-,-)$	β_2	$5,\mathcal{K}$
162	*191		1b	$5^2 191^4$	1	1	2	4	5	$(-,-,\otimes,-)$	β_1	$5,\mathfrak{K}_1$
163	193		2	193^4	1	0	0	0	5	$(-,-,-,\otimes)$	ϑ	
164	194	2×97	1b	$5^2 2^4 97^4$	3	1	2	4	5	$(\times,-,-,-)$	ε	97×5^2
165	195	$3\times 5\times 13$	1a	$5^6 3^4 13^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
166	197		1b	$5^2 197^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
167	198	$2\times 3^2\times 11$	1b	$5^2 2^4 3^4 11^4$	13	1	2	5	6	$(-,-,\times,-)$	γ	$3\times 5^3,11\times 5^3$
168	199		2	199^4	1	1	1	2	3	$(-,\otimes,-,\times)$	δ_2	$\mathcal{L}$
169	201	3×67	2	$3^4 67^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
170	*202	2×101	1b	$5^2 2^4 101^4$	4	1	1	3	4	$(-,-,(\times),-)$	β_2	$2\times 5,\mathcal{K}$
171	*203	7×29	1b	$5^2 7^4 29^4$	4	1	2	4	5	$(-,\times,-,-)$	ε	29×5
172	204	$2^2\times 3\times 17$	1b	$5^2 2^4 3^4 17^4$	13	1	2	5	6	$(\times,-,-,-)$	γ	$3\times 5^4,17\times 5^3$
173	205	5×41	1a	$5^6 41^4$	4	1	1	2	3	$(-,-,\otimes,-)$	α_2	$\mathcal{K},\mathfrak{K}_2$
174	206	2×103	1b	$5^2 2^4 103^4$	3	1	2	4	5	$(\times,-,-,-)$	ε	2×5^2
175	207	$3^2\times 23$	2	$3^4 23^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	

Table 10. 45 pure metacyclic fields with normalized radicands $208\le D\le 259$.

No.	D	Factors	S	f^4	m	V_L	V_M	V_N	E	$(1,2,4,5)$	T	P
176	208	$2^4\times 13$	1b	$5^2 2^4 13^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
177	*209	11×19	1b	$5^2 11^4 19^4$	3	2	3	7	4	$(-,\otimes,\times,-)$	β_2	$11\times 5^2,\mathcal{K}_{(19)}$
178	*210	$2\times 3\times 5\times 7$	1a	$5^6 2^4 3^4 7^4$	64	1	2	5	6	$(\times,-,-,-)$	γ	$7,3^2\times 5$
179	*211		1b	$5^2 211^4$	1	3	5	9	2	$(-,-,\otimes,-)$	δ_1	$\mathfrak{K}_2$
180	212	$2^2\times 53$	1b	$5^2 2^4 53^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
181	213	3×71	1b	$5^2 3^4 71^4$	3	1	1	3	4	$(-,-,(\times),-)$	β_2	$3\times 5^3,\mathcal{K}$
182	214	2×107	1b	$5^2 2^4 107^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	
183	215	5×43	1a	$5^6 43^4$	4	0	0	1	6	$(-,-,-,\otimes)$	η	
184	217	7×31	1b	$5^2 7^4 31^4$	4	1	1	3	4	$(-,-,(\times),-)$	β_2	$5,\mathcal{K}$
185	*218	2×109	2	$2^4 109^4$	1	1	2	4	5	$(-,\times,-,-)$	ε	2
186	219	3×73	1b	$5^2 3^4 73^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
187	220	$2^2\times 5\times 11$	1a	$5^6 2^4 11^4$	16	1	1	3	4	$(-,-,(\times),-)$	β_2	$2\times 5,\mathcal{K}$
188	221	13×17	1b	$5^2 13^4 17^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
189	222	$2\times 3\times 37$	1b	$5^2 2^4 3^4 37^4$	13	1	2	5	6	$(\times,-,-,-)$	γ	$37,2\times 5^2$
190	223		1b	$5^2 223^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
191	226	2×113	2	$2^4 113^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
192	227		1b	$5^2 227^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
193	228	$2^2\times 3\times 19$	1b	$5^2 2^4 3^4 19^4$	13	1	2	5	6	$(-,\times,-,-)$	γ	$3\times 5,19\times 5$
194	229		1b	$5^2 229^4$	1	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{L}$

Continued

195	230	$2\times5\times23$	1a	$5^6 2^4 23^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
196	*231	$3\times7\times11$	1b	$5^2 3^4 7^4 11^4$	12	1	2	2	6	$(-,-,\times,-)$	γ	$11, 7\times5^2$
197	232	$2^3\times29$	2	$2^4 29^4$	1	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{K}$
198	233		1b	$5^2 233^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
199	234	$2\times3^2\times13$	1b	$5^2 2^4 3^4 13^4$	13	1	2	5	6	$(\times,-,-,-)$	γ	$3, 2\times5$
200	235	5×47	1a	$5^6 47^4$	4	0	0	0	5	$(\times,-,-,-)$	ε	
201	236	$2^2\times59$	1b	$5^2 2^4 59^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$2\times5^2, \mathcal{K}$
202	237	3×79	1b	$5^2 3^4 79^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$3\times5, \mathcal{K}$
203	238	$2\times7\times17$	1b	$5^2 2^4 7^4 17^4$	12	1	2	5	6	$(\times,-,-,-)$	γ	$5, 2^4\times7$
204	239		1b	$5^2 239^4$	1	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{L}$
205	240	$2^4\times3\times5$	1a	$5^6 2^4 3^4$	16	1	2	4	5	$(\times,-,-,-)$	ε	3
206	241		1b	$5^2 241^4$	1	1	1	2	3	$(-,-,\otimes,-)$	α_2	$\mathcal{L}, \mathfrak{K}_2$
207	244	$2^2\times61$	1b	$5^2 2^4 61^4$	3	2	2	4	1	$(-,-,\otimes,-)$	α_2	$\mathcal{K}, \mathfrak{K}_1$
208	245	5×7^2	1a	$5^6 7^4$	4	0	0	1	6	$(-,-,-,\otimes)$	η	
209	246	$2\times3\times41$	1b	$5^2 2^4 3^4 41^4$	13	1	2	5	6	$(-,-,\times,-)$	γ	$3, 2\cdot5^2$
210	*247	13×19	1b	$5^2 13^4 19^4$	3	1	2	5	6	$(-,\times,-,-)$	γ	
211	248	$2^3\times31$	1b	$5^2 2^4 31^4$	3	1	1	3	4	$(-,-,(\times),-)$	β_2	$5, \mathcal{K}$
212	249	3×83	2	$3^4 83^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
213	251		2	251^4	1	1	1	2	3	$(-,-,\otimes,\times)$	α_2	$\mathcal{L}, \mathfrak{K}_2$
214	252	$2^2\times3^2\times7$	1b	$5^2 2^4 3^4 7^4$	12	1	2	5	6	$(\times,-,-,-)$	γ	$3, 2\times5$
215	*253	11×23	1b	$5^2 11^4 23^4$	3	1	2	4	5	$(-,-,\otimes,-)$	β_1	$11, \mathfrak{K}_1$
216	254	2×127	1b	$5^2 2^4 127^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
217	255	$3\times5\times17$	1a	$5^6 3^4 17^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
218	257		2	257^4	1	0	0	0	5	$(-,-,-,\otimes)$	ϑ	
219	258	$2\times3\times43$	1b	$5^2 2^4 3^4 43^4$	12	1	2	5	6	$(\times,-,-,-)$	γ	$2, 3\times5$
220	*259	7×37	1b	$5^2 7^4 37^4$	4	1	2	5*	6	$(\times,-,-,-)$	γ	

Table 11. 45 pure metacyclic fields with normalized radicands $260 \le D \le 307$.

No.	D	Factors	S	$\mathfrak{f}^4$	m	V_L	V_M	V_N	E	$(1,2,4,5)$	T	P
221	260	$2^2\times5\times13$	1a	$5^6 2^4 13^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
222	261	$3^2\times29$	1b	$5^2 3^4 29^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$29, \mathcal{L}$
223	262	2×131	1b	$5^2 2^4 131^4$	3	1	1	3	4	$(-,-,(\times),-)$	β_2	$131\times5, \mathcal{K}$
224	263		1b	$5^2 263^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
225	264	$2^3\times3\times11$	1b	$5^2 2^4 3^4 11^4$	13	1	2	5	6	$(-,-,\times,-)$	γ	$2\times5, 2\times11$
226	265	5×53	1a	$5^6 53^4$	4	0	0	0	5	$(\times,-,-,-)$	ε	
227	*266	$2\times7\times19$	1b	$5^2 2^4 7^4 19^4$	12	1	2	5	6	$(-,\times,-,-)$	γ	$5, 2^3\times7$
228	267	3×89	1b	$5^2 3^4 89^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$89\times5^2, \mathcal{K}$
229	268	$2^2\times67$	2	$2^4 67^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	

Continued

230	269		1b	$5^2 269^4$	1	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{L}$
231	270	$2\times3^3\times5$	1a	$5^6 2^4 3^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
232	271		1b	$5^2 271^4$	1	1	2	4	5	$(-,-,\otimes,-)$	β_1	$5,\mathfrak{K}_1$
233	272	$2^4\times17$	1b	$5^2 2^4 17^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
234	*273	$3\times7\times13$	1b	$5^2 3^4 7^4 13^4$	12	2	4	8	5	$(\times,-,-,-)$	ε	$7\times13^2\times5^4$
235	274	2×137	2	$2^4 137^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
236	*275	$5^2\times11$	1a	$5^6 11^4$	4	2	2	4	1	$(-,-,\otimes,-)$	α_2	$\mathcal{K},\mathfrak{K}_2$
237	*276	$2^2\times3\times23$	2	$2^4 3^4 23^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
238	277		1b	$5^2 277^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
239	278	2×139	1b	$5^2 2^4 139^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$139\times5,\mathcal{K}$
240	279	$3^2\times31$	1b	$5^2 3^4 31^4$	3	1	1	3	4	$(-,-,(\times),-)$	β_2	$5,\mathcal{K}$
241	280	$2^3\times5\times7$	1a	$5^6 2^4 7^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
242	*281		1b	$5^2 281^4$	1	3	5*	9*	2	$(-,-,\otimes,-)$	α_1	$\mathfrak{K}_1,\mathfrak{K}_2$
243	282	$2\times3\times47$	2	$2^4 3^4 47^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
244	283		1b	$5^2 283^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
245	284	$2^2\times71$	1b	$5^2 2^4 71^4$	3	1	1	3	4	$(-,-,(\times),-)$	β_2	$2\times5^3,\mathcal{K}$
246	*285	$3\times5\times19$	1a	$5^6 3^4 19^4$	16	1	2	5	6	$(-,\times,-,-)$	γ	
247	*286	$2\times11\times13$	1b	$5^2 2^4 11^4 13^4$	13	2	3	7	4	$(-,-,(\times),-)$	β_2	$11^2\times5,\mathcal{K}$
248	*287	7×41	1b	$5^2 7^4 41^4$	4	2	2	4	1	$(-,-,\otimes,-)$	α_2	$\mathcal{K},\mathfrak{K}_1$
249	*290	$2\times5\times29$	1a	$5^6 2^4 29^4$	16	1	2	4	5	$(-,\times,-,-)$	ε	29
250	291	3×97	1b	$5^2 3^4 97^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
251	292	$2^2\times73$	1b	$5^2 2^4 73^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
252	293		2	293^4	1	0	0	0	5	$(-,-,-,\otimes)$	ϑ	
253	294	$2\times3\times7^2$	1b	$5^2 2^4 3^4 7^4$	12	1	2	5	6	$(\times,-,-,-)$	γ	$3,7^2\times5$
254	295	5×59	1a	$5^6 59^4$	4	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{K}$
255	296	$2^3\times37$	1b	$5^2 2^4 37^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
256	297	$3^3\times11$	1b	$5^2 3^4 11^4$	3	1	1	3	4	$(-,-,(\times),-)$	β_2	$11\times5,\mathcal{K}$
257	*298	2×149	1b	$5^2 2^4 149^4$	4	1	2	4	5	$(-,\times,-,-)$	ε	5
258	299	13×23	2	$13^4 23^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
259	*301	7×43	2	$7^4 43^4$	4	0	0	1	6	$(-,-,-,\otimes)$	η	
260	*302	2×151	1b	$5^2 2^4 151^4$	4	1	2	4	5	$(-,-,\otimes,-)$	β_1	$2,\mathfrak{K}_1$
261	303	3×101	1b	$5^2 3^4 101^4$	4	1	1	3	4	$(-,-,(\times),-)$	β_2	$101\times5,\mathcal{K}$
262	304	$2^4\times19$	1b	$5^2 2^4 19^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$19\times5^4,\mathcal{K}$
263	305	5×61	1a	$5^6 61^4$	4	2	2	4	1	$(-,-,\otimes,-)$	α_2	$\mathcal{K},\mathfrak{K}_1$
264	306	$2\times3^2\times17$	1b	$5^2 2^4 3^4 17^4$	13	1	2	5	6	$(\times,-,-,-)$	γ	$3\times5,3\times17$
265	307		2	307^4	1	0	0	0	5	$(-,-,-,\otimes)$	ϑ	

Table 12. 45 pure metacyclic fields with normalized radicands $308 \le D \le 357$.

No.	D	Factors	S	f^4	m	V_L	V_M	V_N	E	$(1,2,4,5)$	T	P
266	308	$2^2 \times 7 \times 11$	1b	$5^2 2^4 7^4 11^4$	12	1	2	5	6	$(-,-,\times,-)$	γ	$11 \times 5, 7 \times 5^2$
267	309	3×103	1b	$5^2 3^4 103^4$	3	1	2	4	5	$(\times,-,-,-)$	ε	103×5^2
268	310	$2 \times 5 \times 31$	1a	$5^6 2^4 31^4$	16	1	1	3	4	$(-,-,(\times),-)$	β_2	$5, \mathcal{K}$
269	311		1b	$5^2 311^4$	1	1	1	2	3	$(-,-,\otimes,-)$	α_2	$\mathcal{L}, \mathfrak{K}_2$
270	312	$2^3 \times 3 \times 13$	1b	$5^2 2^4 3^4 13^4$	13	1	2	5	6	$(\times,-,-,-)$	γ	$5, 2^4 \times 13$
271	313		1b	$5^2 313^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
272	314	2×157	1b	$5^2 2^4 157^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	
273	315	$3^2 \times 5 \times 7$	1a	$5^6 3^4 7^4$	16	1	2	4	5	$(\times,-,-,-)$	ε	7
274	316	$2^2 \times 79$	1b	$5^2 2^4 79^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$2 \times 5, \mathcal{K}$
275	317		1b	$5^2 317^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
276	318	$2 \times 3 \times 53$	2	$2^4 3^4 53^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
277	*319	11×29	1b	$5^2 11^4 29^4$	3	2	2	5	2	$(-,\otimes,(\times),-)$	α_3	$\mathcal{K}_{(11)}, \mathcal{K}_{(29)}$
278	321	3×107	1b	$5^2 3^4 107^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	
279	322	$2 \times 7 \times 23$	1b	$5^2 2^4 7^4 23^4$	12	2	4	8	5	$(\times,-,-,-)$	ε	$7 \times 23^2 \times 5^3$
280	323	17×19	1b	$5^2 17^4 19^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$19 \times 5^2, \mathcal{K}$
281	325	$5^2 \times 13$	1a	$5^6 13^4$	4	0	0	0	5	$(\times,-,-,-)$	ε	
282	326	2×163	2	$2^4 163^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
283	327	3×109	1b	$5^2 3^4 109^4$	3	1	2	5	6	$(-,\times,-,-)$	γ	
284	328	$2^3 \times 41$	1b	$5^2 2^4 41^4$	3	1	1	3	4	$(-,-,(\times),-)$	β_2	$2 \times 41 \times 5, \mathcal{K}$
285	*329	7×47	1b	$5^2 7^4 47^4$	4	1	2	4	5	$(\times,-,-,-)$	ε	47
286	*330	$2 \times 3 \times 5 \times 11$	1a	$5^6 2^4 3^4 11^4$	64	2	4	9	6	$(-,-,\times,-)$	γ	$3 \times 11, 5 \times 11^3$
287	331		1b	$5^2 331^4$	1	1	1	2	3	$(-,-,\otimes,-)$	α_2	$\mathcal{L}, \mathfrak{K}_1$
288	332	$2^2 \times 83$	2	$2^4 83^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
289	333	$3^2 \times 37$	1b	$5^2 3^4 37^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
290	334	2×167	1b	$5^2 2^4 167^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
291	335	5×67	1a	$5^6 67^4$	4	0	0	0	5	$(\times,-,-,-)$	ε	
292	336	$2^4 \times 3 \times 7$	1b	$5^2 2^4 3^4 7^4$	12	2	4	8	5	$(\times,-,-,-)$	ε	2×5^4
293	337		1b	$5^2 337^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
294	339	3×113	1b	$5^2 3^4 113^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
295	340	$2^2 \times 5 \times 17$	1a	$5^6 2^4 17^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
296	*341	11×31	1b	$5^2 11^4 31^4$	3	3	5	9	2	$(-,-,\otimes,-)$	α_1	$\mathfrak{K}_{(11),1}\mathfrak{K}^3_{(31),2}$, $\mathfrak{K}_{(11),2}\mathfrak{K}_{(31),1}$
297	342	$2 \times 3^2 \times 19$	1b	$5^2 2^4 3^4 19^4$	13	1	2	5	6	$(-,\times,-,-)$	γ	$19, 2 \times 5^2$
298	344	$2^3 \times 43$	1b	$5^2 2^4 43^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	
299	345	$3 \times 5 \times 23$	1a	$5^6 3^4 23^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
300	346	2×173	1b	$5^2 2^4 173^4$	3	1	2	4	5	$(\times,-,-,-)$	ε	173×5
301	347		1b	$5^2 347^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
302	*348	$2^2 \times 3 \times 29$	1b	$5^2 2^4 3^4 29^4$	13	2	4	8	5	$(-,\times,-,-)$	ε	$2 \times 3 \times 5$
303	349		2	349^4	1	1	1	2	3	$(-,\otimes,-,\times)$	δ_2	$\mathcal{L}$

Continued

304	350	$2\times5^2\times7$	1a	$5^6 2^4 7^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
305	351	$3^3\times13$	2	$3^4 13^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
306	353		1b	$5^2 353^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
307	354	$2\times3\times59$	1b	$5^2 2^4 3^4 59^4$	13	1	2	5	6	$(-,\times,-,-)$	γ	$5, 2\times3^2$
308	355	5×71	1a	$5^6 71^4$	4	1	1	2	3	$(-,-,\otimes,-)$	α_2	$\mathcal{K}, \mathfrak{K}_1$
309	356	$2^2\times89$	1b	$5^2 2^4 89^4$	3	1	2	4	5	$(-,\times,-,-)$	ε	89×5^4
310	357	$3\times7\times17$	2	$3^4 7^4 17^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	

Table 13. 45 pure metacyclic fields with normalized radicands $358 \le D \le 408$.

No.	D	Factors	S	f^4	m	V_L	V_M	V_N	E	$(1,2,4,5)$	T	P
311	358	2×179	1b	$5^2 2^4 179^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$2, \mathcal{L}$
312	359		1b	$5^2 359^4$	1	1	1	3	4	$(-,\otimes,-,-)$	β_2	$5, \mathcal{L}$
313	360	$2^3\times3^2\times5$	1a	$5^6 2^4 3^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
314	362	2×181	1b	$5^2 2^4 181^4$	3	1	1	3	4	$(-,-,(\times),-)$	β_2	$181\times5, \mathcal{K}$
315	364	$2^2\times7\times13$	1b	$5^2 2^4 7^4 13^4$	12	1	2	5	6	$(\times,-,-,-)$	γ	$2, 7\times13$
316	365	5×73	1a	$5^6 73^4$	4	0	0	0	5	$(\times,-,-,-)$	ε	
317	366	$2\times3\times61$	1b	$5^2 2^4 3^4 61^4$	13	2	3	6	3	$(-,-,\otimes,-)$	β_1	$3\times61^2\times5^4, \mathfrak{K}_1$
318	367		1b	$5^2 367^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
319	368	$2^4\times23$	2	$2^4 23^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
320	369	$3^2\times41$	1b	$5^2 3^4 41^4$	3	2	3	6	3	$(-,-,\otimes,-)$	α_2	$\mathcal{K}, \mathfrak{K}_2$
321	370	$2\times5\times37$	1a	$5^6 2^4 37^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
322	371	7×53	1b	$5^2 7^4 53^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	
323	372	$2^2\times3\times31$	1b	$5^2 2^4 3^4 31^4$	13	1	2	5	6	$(-,-,(\times),-)$	γ	$5, 2\times31$
324	373		1b	$5^2 373^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
325	374	$2\times11\times17$	2	$2^4 11^4 17^4$	3	1	1	3	4	$(-,-,(\times),-)$	β_2	$11, \mathcal{L}$
326	376	$2^3\times47$	2	$2^4 47^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
327	*377	13×29	1b	$5^2 13^4 29^4$	3	2	3	6	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{K}$
328	378	$2\times3^3\times7$	1b	$5^2 2^4 3^4 7^4$	12	1	2	5	6	$(\times,-,-,-)$	γ	$2\times5^2, 3^3\times7^2$
329	*379		1b	$5^2 379^4$	1	1	2	4	5	$(-,\times,-,-)$	ε	5
330	380	$2^2\times5\times19$	1a	$5^6 2^4 19^4$	16	1	1	3	4	$(-,\otimes,-,-)$	β_2	$19, \mathcal{L}$
331	381	3×127	1b	$5^2 3^4 127^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
332	382	2×191	2	$2^4 191^4$	1	1	1	2	3	$(-,-,\otimes,-)$	α_2	$\mathcal{K}, \mathfrak{K}_1$
333	383		1b	$5^2 383^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
334	*385	$5\times7\times11$	1a	$5^6 7^4 11^4$	16	1	1	3	4	$(-,-,(\times),-)$	β_2	$7\times11^2, \mathcal{K}$
335	386	2×193	1b	$5^2 2^4 193^4$	4	1	2	4	5	$(\times,-,-,-)$	ε	2

Continued

336	387	$3^2\times 43$	1b	$5^2 3^4 43^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	
337	388	$2^2\times 97$	1b	$5^2 2^4 97^4$	3	1	2	4	5	$(\times,-,-,-)$	ε	97×5^4
338	389		1b	$5^2 389^4$	1	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{L}$
339	*390	$2\times 3\times 5\times 13$	1a	$5^6 2^4 3^4 13^4$	64	1	2	5	6	$(\times,-,-,-)$	γ	2, 5
340	391	17×23	1b	$5^2 17^4 23^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
341	393	3×131	2	$3^4 131^4$	1	1	1	2	3	$(-,-,\otimes,-)$	α_2	$\mathcal{K},\mathfrak{K}_1$
342	394	2×197	1b	$5^2 2^4 197^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
343	395	5×79	1a	$5^6 79^4$	4	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{K}$
344	396	$2^2\times 3^2\times 11$	1b	$5^2 2^4 3^4 11^4$	13	2	3	6	3	$(-,-,\otimes,-)$	β_1	$11\times 5^2,\mathfrak{K}_2$
345	397		1b	$5^2 397^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
346	*398	2×199	1b	$5^2 2^4 199^4$	4	1	1	3	4	$(-,\otimes,-,-)$	β_2	$5,\mathcal{K}$
347	*399	$3\times 7\times 19$	2	$3^4 7^4 19^4$	4	1	1	3	4	$(-,\otimes,-,-)$	β_2	$3\times 19^2,\mathcal{K}$
348	*401		2	401^4	1	2	3	5	2	$(-,-,\otimes,\times)$	α_1	$\mathfrak{K}_1,\mathfrak{K}_2$
349	402	$2\times 3\times 67$	1b	$5^2 2^4 3^4 67^4$	13	1	2	5	6	$(\times,-,-,-)$	γ	$3\times 5^3, 2\times 5^3$
350	403	13×31	1b	$5^2 13^4 31^4$	3	1	1	3	4	$(-,-,(\times),-)$	β_2	$5,\mathcal{K}$
351	404	$2^2\times 101$	1b	$5^2 2^4 101^4$	4	1	1	3	4	$(-,-,(\times),-)$	β_2	$2\times 5,\mathcal{K}$
352	405	$3^4\times 5$	1a	$5^6 3^4$	4	0	0	0	5	$(\times,-,-,-)$	ε	
353	406	$2\times 7\times 29$	1b	$5^2 2^4 7^4 29^4$	12	1	2	5	6	$(-,\times,-,-)$	γ	$7, 2\times 5^3$
354	407	11×37	2	$11^4 37^4$	1	1	1	2	3	$(-,-,\otimes,-)$	α_2	$\mathcal{K},\mathfrak{K}_2$
355	408	$2^3\times 3\times 17$	1b	$5^2 2^4 3^4 17^4$	13	1	2	5	6	$(\times,-,-,-)$	γ	$17, 3\times 5^3$

Table 14. 45 pure metacyclic fields with normalized radicands $409\le D\le 458$.

No.	D	Factors	S	f^4	m	V_L	V_M	V_N	E	$(1,2,4,5)$	T	P
356	409		1b	$5^2 409^4$	1	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{L}$
357	410	$2\times 5\times 41$	1a	$5^6 2^4 41^4$	16	1	1	3	4	$(-,-,(\times),-)$	β_2	$2\times 41^2,\mathcal{K}$
358	411	3×137	1b	$5^2 3^4 137^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
359	412	$2^2\times 103$	1b	$5^2 2^4 103^4$	3	1	2	4	5	$(\times,-,-,-)$	ε	2×5^2
360	413	7×59	1b	$5^2 7^4 59^4$	4	1	1	3	4	$(-,\otimes,-,-)$	β_2	$59\times 5,\mathcal{K}$
361	414	$2\times 3^2\times 23$	1b	$5^2 2^4 3^4 23^4$	13	1	2	5	6	$(\times,-,-,-)$	γ	2, 3
362	415	5×83	1a	$5^6 83^4$	4	0	0	0	5	$(\times,-,-,-)$	ε	
363	417	3×139	1b	$5^2 3^4 139^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$139\times 5,\mathcal{K}$
364	*418	$2\times 11\times 19$	2	$2^4 11^4 19^4$	3	2	3	6	3	$(-,\otimes,\otimes,-)$	α_2	$\mathcal{K}_{(19)},\mathfrak{K}_{(11),1}$
365	419		1b	$5^2 419^4$	1	1	1	3	4	$(-,\otimes,-,-)$	β_2	$5,\mathcal{L}$
366	420	$2^2\times 3\times 5\times 7$	1a	$5^6 2^4 3^4 7^4$	64	1	2	5	6	$(\times,-,-,-)$	γ	$2, 3^2\times 7$

Continued

367	*421		1b	$5^2 421^4$	1	1	2	3	4	$(-,-,\otimes,-)$	δ_1	$\mathfrak{K}_2$
368	*422	2×211	1b	$5^2 2^4 211^4$	3	1	2	4	5	$(-,-,\times,-)$	ε	2×5^2
369	423	$3^2\times 47$	1b	$5^2 3^4 47^4$	3	1	2	4	5	$(\times,-,-,-)$	ε	3×5^3
370	424	$2^3\times 53$	2	$2^4 53^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
371	425	$5^2\times 17$	1a	$5^6 17^4$	4	0	0	0	5	$(\times,-,-,-)$	ε	
372	426	$2\times 3\times 71$	2	$2^4 3^4 71^4$	3	1	1	3	4	$(-,-,(\times),-)$	β_2	$2^2\times 71,\mathcal{K}$
373	427	7×61	1b	$5^2 7^4 61^4$	4	1	1	3	4	$(-,-,(\times),-)$	β_2	$61\times 5^2,\mathcal{K}$
374	428	$2^2\times 107$	1b	$5^2 2^4 107^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	
375	429	$3\times 11\times 13$	1b	$5^2 3^4 11^4 13^4$	13	2	3	7	4	$(-,-,(\times),-)$	β_2	$13\times 5,\mathcal{K}$
376	430	$2\times 5\times 43$	1a	$5^6 2^4 43^4$	16	1	2	4	5	$(\times,-,-,-)$	ε	43
377	431		1b	$5^2 431^4$	1	1	1	2	3	$(-,-,\otimes,-)$	α_2	$\mathcal{L},\mathfrak{K}_1$
378	433		1b	$5^2 433^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
379	434	$2\times 7\times 31$	1b	$5^2 2^4 7^4 31^4$	12	1	2	5	6	$(-,-,\times,-)$	γ	$5,2\times 31^2$
380	435	$3\times 5\times 29$	1a	$5^6 3^4 29^4$	16	1	1	3	4	$(-,\otimes,-,-)$	β_2	$3\times 5^3,\mathcal{K}$
381	436	$2^2\times 109$	1b	$5^2 2^4 109^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$109,\mathcal{L}$
382	437	19×23	1b	$5^2 19^4 23^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$19\times 5^2,\mathcal{K}$
383	438	$2\times 3\times 73$	1b	$5^2 2^4 3^4 73^4$	13	1	2	5	6	$(\times,-,-,-)$	γ	3, 5
384	439		1b	$5^2 439^4$	1	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{L}$
385	440	$2^3\times 5\times 11$	1a	$5^6 2^4 11^4$	16	1	1	3	4	$(-,-,(\times),-)$	β_2	$2\times 5,\mathcal{K}$
386	442	$2\times 13\times 17$	1b	$5^2 2^4 13^4 17^4$	13	1	2	5	6	$(\times,-,-,-)$	γ	$2\times 5^4,13\times 5^2$
387	443		2	443^4	1	0	0	0	5	$(-,-,-,\otimes)$	ϑ	
388	444	$2^2\times 3\times 37$	1b	$5^2 2^4 3^4 37^4$	13	1	2	5	6	$(\times,-,-,-)$	γ	$5,2^3\times 3$
389	445	5×89	1a	$5^6 89^4$	4	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{K}$
390	446	2×223	1b	$5^2 2^4 223^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
391	447	3×149	1b	$5^2 3^4 149^4$	4	1	1	3	4	$(-,\otimes,-,-)$	β_2	$149\times 5,\mathcal{K}$
392	449		2	449^4	1	1	1	2	3	$(-,\otimes,-,\times)$	δ_2	$\mathcal{L}$
393	*451	11×41	2	$11^4 41^4$	1	2	3	6	3	$(-,-,\otimes,-)$	α_2	$\mathcal{K}_{(11)}\mathcal{K}_{(41)}^4$, $\mathfrak{K}_{(11),1}\mathfrak{K}_{(41),2}^3$
394	452	$2^2\times 113$	1b	$5^2 2^4 113^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
395	453	3×151	1b	$5^2 3^4 151^4$	4	1	1	3	4	$(-,-,(\times),-)$	β_2	$3\times 5^2,\mathcal{K}$
396	454	2×227	1b	$5^2 2^4 227^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
397	455	$5\times 7\times 13$	1a	$5^6 7^4 13^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
398	456	$2^3\times 3\times 19$	1b	$5^2 2^4 3^4 19^4$	13	1	2	5	6	$(-,\times,-,-)$	γ	$2\times 5,19$
399	457		2	457^4	1	0	0	0	5	$(-,-,-,\otimes)$	ϑ	
400	458	2×229	1b	$5^2 2^4 229^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$229\times 5,\mathcal{K}$

Table 15. 45 pure metacyclic fields with normalized radicands $459 \le D \le 508$.

No.	D	Factors	S	f^4	m	V_L	V_M	V_N	E	$(1,2,4,5)$	T	P
401	459	$3^3 \times 17$	1b	$5^2 3^4 17^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
402	460	$2^2 \times 5 \times 23$	1a	$5^6 2^4 23^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
403	461		1b	$5^2 461^4$	1	1	2	3	4	$(-,-,\otimes,-)$	δ_1	$\mathfrak{K}_1$
404	*462	$2 \times 3 \times 7 \times 11$	1b	$5^2 2^4 3^4 7^4 11^4$	52	2	4	9	6	$(-,-,\times,-)$	γ	$5^2 \times 7$, $3 \times 5 \times 11^2$
405	463		1b	$5^2 463^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
406	464	$2^4 \times 29$	1b	$5^2 2^4 29^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$5, \mathcal{K}$
407	*465	$3 \times 5 \times 31$	1a	$5^6 3^4 31^4$	16	2	3	7*	4	$(-,-,(\times),-)$	β_2	$5, \mathcal{K}$
408	466	2×233	1b	$5^2 2^4 233^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
409	467		1b	$5^2 467^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
410	468	$2^2 \times 3^2 \times 13$	2	$2^4 3^4 13^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
411	469	7×67	1b	$5^2 7^4 67^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	
412	470	$2 \times 5 \times 47$	1a	$5^6 2^4 47^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
413	471	3×157	1b	$5^2 3^4 157^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	
414	472	$2^3 \times 59$	1b	$5^2 2^4 59^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$2 \times 5, \mathcal{K}$
415	*473	11×43	1b	$5^2 11^4 43^4$	4	2	3	7*	4	$(-,-,(\times),-)$	β_2	$11, \mathcal{L}$
416	474	$2 \times 3 \times 79$	2	$2^4 3^4 79^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$3, \mathcal{K}$
417	475	$5^2 \times 19$	1a	$5^6 19^4$	4	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{K}$
418	476	$2^2 \times 7 \times 17$	2	$2^4 7^4 17^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	
419	477	$3^2 \times 53$	1b	$5^2 3^4 53^4$	3	1	2	4	5	$(\times,-,-,-)$	ε	3×5
420	478	2×239	1b	$5^2 2^4 239^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$239, \mathcal{L}$
421	479		1b	$5^2 479^4$	1	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{L}$
422	481	13×37	1b	$5^2 13^4 37^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
423	*482	2×241	2	$2^4 241^4$	1	1	2	4	5	$(-,-,\otimes,-)$	β_1	$241, \mathfrak{K}_2$
424	483	$3 \times 7 \times 23$	1b	$5^2 3^4 7^4 23^4$	12	1	2	5	6	$(\times,-,-,-)$	γ	$7, 3 \times 23$
425	485	5×97	1a	$5^6 97^4$	4	0	0	0	5	$(\times,-,-,-)$	ε	
426	487		1b	$5^2 487^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
427	488	$2^3 \times 61$	1b	$5^2 2^4 61^4$	3	1	1	3	4	$(-,-,(\times),-)$	β_2	$2 \times 5^2, \mathcal{K}$
428	489	3×163	1b	$5^2 3^4 163^4$	3	1	2	4	5	$(\times,-,-,-)$	ε	3×5^2
429	490	$2 \times 5 \times 7^2$	1a	$5^6 2^4 7^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
430	491		1b	$5^2 491^4$	1	1	1	2	3	$(-,-,\otimes,-)$	α_2	$\mathcal{L}, \mathfrak{K}_1$
431	492	$2^2 \times 3 \times 41$	1b	$5^2 2^4 3^4 41^4$	13	1	2	5	6	$(-,-,\times,-)$	γ	$3, 2 \times 5^2$
432	493	17×29	2	$17^4 29^4$	1	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{K}$
433	494	$2 \times 13 \times 19$	1b	$5^2 2^4 13^4 19^4$	13	1	2	5	6	$(-,\times,-,-)$	γ	$2, 13 \times 5^2$
434	495	$3^2 \times 5 \times 11$	1a	$5^6 3^4 11^4$	16	1	1	3	4	$(-,-,(\times),-)$	β_2	$11, \mathcal{L}$
435	496	$2^4 \times 31$	1b	$5^2 2^4 31^4$	3	1	1	3	4	$(-,-,(\times),-)$	β_2	$5, \mathcal{K}$

Continued

436	497	7×71	1b	$5^2 7^4 71^4$	4	1	1	3	4	$(-,-,(\times),-)$	β_2	$7\times5^3,\mathcal{K}$
437	498	$2\times3\times83$	1b	$5^2 2^4 3^4 83^4$	13	1	2	5	6	$(\times,-,-,-)$	γ	$2^3\times3,2\times5^2$
438	499		2	499^4	1	1	1	2	3	$(-,\otimes,-,\times)$	δ_2	$\mathcal{L}$
439	501	3×167	2	$3^4 167^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
440	*502	2×251	1b	$5^2 2^4 251^4$	4	2*	4*	8*	5	$(-,-,\otimes,-)$	β_1	$251\times5,\mathfrak{K}_2$
441	503		1b	$5^2 503^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
442	504	$2^3\times3^2\times7$	1b	$5^2 2^4 3^4 7^4$	15	1	2	5	6	$(\times,-,-,-)$	γ	$7,2\times5^2$
443	*505	5×101	1a	$5^6 101^4$	4	1	1	3	4	$(-,-,(\times),\otimes)$	ζ_2	$\mathcal{K}$
444	506	$2\times11\times23$	1b	$5^2 2^4 11^4 23^4$	13	1	2	5	6	$(-,-,\times,-)$	γ	$23,2\times5$
445	508	$2^2\times127$	1b	$5^2 2^4 127^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	

Table 16. 45 pure metacyclic fields with normalized radicands $509\le D\le556$.

No.	D	Factors	S	f^4	m	V_L	V_M	V_N	E	$(1,2,4,5)$	T	P
446	509		1b	$5^2 509^4$	1	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{L}$
447	510	$2\times3\times5\times17$	1a	$5^6 2^4 3^4 17^4$	64	1	2	5	6	$(\times,-,-,-)$	γ	$2\times3,2\times17$
448	511	7×73	1b	$5^2 7^4 73^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	
449	513	$3^3\times19$	1b	$5^2 3^4 19^4$	3	1	2	4	5	$(-,\times,-,-)$	ε	3×5^2
450	514	2×257	1b	$5^2 2^4 257^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	
451	515	5×103	1a	$5^6 103^4$	4	0	0	0	5	$(\times,-,-,-)$	ε	
452	516	$2^2\times3\times43$	1b	$5^2 2^4 3^4 43^4$	12	1	2	5	6	$(\times,-,-,-)$	γ	$2\times3^2,2\times5^4$
453	*517	11×47	1b	$5^2 11^4 47^4$	3	2	3	6	3	$(-,-,\otimes,-)$	β_1	$11\times5^2,\mathfrak{K}_2$
454	518	$2\times7\times37$	2	$2^4 7^4 37^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	
455	519	3×173	1b	$5^2 3^4 173^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
456	520	$2^3\times5\times13$	1a	$5^6 2^4 13^4$	16	1	2	4	5	$(\times,-,-,-)$	ε	2
457	521		1b	$5^2 521^4$	1	1	2	3	4	$(-,-,\otimes,-)$	δ_1	$\mathfrak{K}_2$
458	522	$2\cdot\times3^2\times29$	1b	$5^2 2^4 3^4 29^4$	13	1	2	5	6	$(-,\times,-,-)$	γ	$2,29\times5$
459	523		1b	$5^2 523^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
460	524	$2^2\times131$	2	$2^4 131^4$	1	1	1	2	3	$(-,-,\otimes,-)$	α_2	$\mathcal{K},\mathfrak{K}_1$
461	525	$3\times5^2\times7$	1a	$5^6 3^4 7^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
462	526	2×263	2	$2^4 263^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
463	527	17×31	1b	$5^2 17^4 31^4$	3	1	1	3	4	$(-,-,(\times),-)$	β_2	$5,\mathcal{K}$
464	528	$2^4\times3\times11$	1b	$5^2 2^4 3^4 11^4$	13	1	2	5	6	$(-,-,\times,-)$	γ	$2\times5,3\times5^3$
465	530	$2\times5\times53$	1a	$5^6 2^4 53^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
466	531	$3^2\times59$	1b	$5^2 3^4 59^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$3,\mathcal{L}$
467	*532	$2^2\times7\times19$	2	$2^4 7^4 19^4$	4	1	2	4	5	$(-,\times,-,-)$	ε	2×7
468	533	13×41	1b	$5^2 13^4 41^4$	3	1	1	3	4	$(-,-,(\times),-)$	β_2	$13\times5^2,\mathcal{K}$
469	534	$2\times3\times89$	1b	$5^2 2^4 3^4 89^4$	13	1	2	5	6	$(-,\times,-,-)$	γ	$3,89\times5$

Continued

470	535	5×107	1a	$5^6 107^4$	4	0	0	1	6	$(-,-,-,\otimes)$	η	
471	536	$2^3\times 67$	1b	$5^2 2^4 67^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
472	537	3×179	1b	$5^2 3^4 179^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$3\times 5^2,\mathcal{K}$
473	538	2×269	1b	$5^2 2^4 269^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$2,\mathcal{L}$
474	539	$7^2\times 11$	1b	$5^2 7^4 11^4$	4	1	1	3	4	$(-,-,(\times),-)$	β_2	$11\times 5,\mathcal{K}$
475	540	$2^2\times 3^3\times 5$	1a	$5^6 2^4 3^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
476	541		1b	$5^2 541^4$	1	2	2	4	1	$(-,-,\otimes,-)$	α_2	$\mathcal{K},\mathfrak{K}_1$
477	542	2×271	1b	$5^2 2^4 271^4$	3	1	1	3	4	$(-,-,(\times),-)$	β_2	$5,\mathcal{K}$
478	543	3×181	2	$3^4 181^4$	1	1	1	2	3	$(-,-,\otimes,-)$	α_2	$\mathcal{K},\mathfrak{K}_1$
479	545	5×109	1a	$5^6 109^4$	4	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{K}$
480	*546	$2\times 3\times 7\times 13$	1b	$5^2 2^4 3^4 7^4 13^4$	52	2	4	9	6	$(\times,-,-,-)$	γ	$3\times 5\times 13$, $5^2\times 7\times 13$
481	547		1b	$5^2 547^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
482	548	$2^2\times 137$	1b	$5^2 2^4 137^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
483	549	$3^2\times 61$	2	$3^4 61^4$	3	2	2	4	1	$(-,-,\otimes,-)$	α_2	$\mathcal{K},\mathfrak{K}_1$
484	*550	$2\times 5^2\times 11$	1a	$5^6 2^4 11^4$	16	2	3	6	3	$(-,-,\otimes,-)$	α_2	$\mathcal{K},\mathfrak{K}_1$
485	*551	19×29	2	$19^4 29^4$	1	2	2	5	2	$(-,\otimes,-,-)$	α_3	$\mathcal{K}_{(19)},\mathcal{K}_{(29)}$
486	552	$2^3\times 3\times 23$	1b	$5^2 2^4 3^4 23^4$	13	1	2	5	6	$(\times,-,-,-)$	γ	$23,3\times 5$
487	553	7×79	1b	$5^2 7^4 79^4$	4	1	2	4	5	$(-,\times,-,-)$	ε	5
488	554	2×277	1b	$5^2 2^4 277^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
489	555	$3\times 5\times 37$	1a	$5^6 3^4 37^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
490	556	$2^2\times 139$	1b	$5^2 2^4 139^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$2\times 5^2,\mathcal{K}$

Table 17. 45 pure metacyclic fields with normalized radicands $557\le D\le 604$.

No.	D	Factors	S	f^4	m	V_L	V_M	V_N	E	$(1,2,4,5)$	T	P
491	557		2	557^4	1	0	0	0	5	$(-,-,-,\otimes)$	ϑ	
492	558	$2\times 3^2\times 31$	1b	$5^2 2^4 3^4 31^4$	13	1	2	5	6	$(-,-,\times,-)$	γ	$5,3\times 31$
493	559	13×43	1b	$5^2 13^4 43^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	
494	560	$2^4\times 5\times 7$	1a	$5^6 2^4 7^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
495	561	$3\times 11\times 17$	1b	$5^2 3^4 11^4 17^4$	13	1	2	5	6	$(-,-,\times,-)$	γ	$11\times 5^3,17\times 5^4$
496	562	2×281	1b	$5^2 2^4 281^4$	3	1	2	4	5	$(-,-,\times,-)$	ε	2×5
497	563		1b	$5^2 563^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
498	564	$2^2\times 3\times 47$	1b	$5^2 2^4 3^4 47^4$	13	1	2	5	6	$(\times,-,-,-)$	γ	$2\times 47,2\times 5^2$
499	565	5×113	1a	$5^6 113^4$	4	0	0	0	5	$(\times,-,-,-)$	ε	
500	566	2×283	1b	$5^2 2^4 283^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	

Continued

501	567	$3^4\times 7$	1b	$5^2 3^4 7^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	
502	568	$2^3\times 71$	2	$2^4 71^4$	1	1	1	2	3	$(-,-,\otimes,-)$	α_2	$\mathcal{K},\mathfrak{K}_1$
503	569		1b	$5^2 569^4$	1	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{L}$
504	*570	$2\times 3\times 5\times 19$	1a	$5^6 2^4 3^4 19^4$	64	1	2	5	6	$(-,\times,-,-)$	γ	$2^4\times 3, 2^4\times 5$
505	571		1b	$5^2 571^4$	1	2	2	4	1	$(-,-,\otimes,-)$	α_2	$\mathcal{L},\mathfrak{K}_1$
506	572	$2^2\times 11\times 13$	1b	$5^2 2^4 11^4 13^4$	13	2	3	6	3	$(-,-,\otimes,-)$	β_1	$2\times 13^2\times 5^3,\mathfrak{K}_2$
507	573	3×191	1b	$5^2 3^4 191^4$	3	1	1	3	4	$(-,-,(\times),-)$	β_2	$5,\mathcal{K}$
508	*574	$2\times 7\times 41$	2	$2^4 7^4 41^4$	4	1	1	3	4	$(-,-,(\times),-)$	β_2	$41,\mathcal{L}$
509	575	$5^2\times 23$	1a	$5^6 23^4$	4	0	0	0	5	$(\times,-,-,-)$	ε	
510	577		1b	$5^2 577^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
511	579	3×193	1b	$5^2 3^4 193^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	
512	580	$2^2\times 5\times 29$	1a	$5^6 2^4 29^4$	16	1	1	3	4	$(-,\otimes,-,-)$	β_2	$2^4\times 5,\mathcal{K}$
513	581	7×83	1b	$5^2 7^4 83^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	
514	582	$2\times 3\times 97$	2	$2^4 3^4 97^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
515	583	11×53	1b	$5^2 11^4 53^4$	3	1	1	3	4	$(-,-,(\times),-)$	β_2	$53\times 5,\mathcal{K}$
516	584	$2^3\times 73$	1b	$5^2 2^4 73^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
517	585	$3^2\times 5\times 13$	1a	$5^6 3^4 13^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
518	586	2×293	1b	$5^2 2^4 293^4$	4	1	2	4	5	$(\times,-,-,-)$	ε	2
519	587		1b	$5^2 587^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
520	589	19×31	1b	$5^2 19^4 31^4$	3	2	2	5	2	$(-,\otimes,(\times),-)$	α_3	$\mathcal{K}_{(19)},\mathcal{K}_{(31)}$
521	*590	$2\times 5\times 59$	1a	$5^6 2^4 59^4$	16	2	3	7*	4	$(-,\otimes,-,-)$	β_2	$5,\mathcal{K}$
522	591	3×197	1b	$5^2 3^4 197^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
523	592	$2^4\times 37$	1b	$5^2 2^4 37^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
524	593		2	593^4	1	0	0	0	5	$(-,-,-,\otimes)$	ϑ	
525	594	$2\times 3^3\times 11$	1b	$5^2 2^4 3^4 11^4$	13	1	2	5	6	$(-,-,\times,-)$	γ	$11, 3^2\times 5$
526	595	$5\times 7\times 17$	1a	$5^6 7^4 17^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
527	596	$2^2\times 149$	1b	$5^2 2^4 149^4$	4	1	2	4	5	$(-,\times,-,-)$	ε	2×5^2
528	597	3×199	1b	$5^2 3^4 199^4$	4	1	1	3	4	$(-,\otimes,-,-)$	β_2	$3^4\times 5,\mathcal{K}$
529	598	$2\times 13\times 23$	1b	$5^2 2^4 13^4 23^4$	13	1	2	5	6	$(\times,-,-,-)$	γ	$2^4\times 5, 13\times 5^3$
530	599		2	599^4	1	1	1	2	3	$(-,\otimes,-,\times)$	δ_2	$\mathcal{L}$
531	600	$2^3\times 3\times 5^2$	1a	$5^6 2^4 3^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
532	601		2	601^4	1	1	1	2	3	$(-,-,\otimes,\times)$	α_2	$\mathcal{L},\mathfrak{K}_2$
533	*062	$2\times 7\times 43$	1b	$5^2 2^4 7^4 43^4$	16	1	2	5	6	$(\times,-,-,-)$	γ	2, 7
534	603	$3^2\times 67$	1b	$5^2 3^4 67^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
535	604	$2^2\times 151$	1b	$5^2 2^4 151^4$	4	1	2	4	5	$(-,-,\otimes,-)$	β_1	$2,\mathfrak{K}_1$

Table 18. 45 pure metacyclic fields with normalized radicands $605 \le D \le 653$.

No.	D	Factors	S	f^4	m	V_L	V_M	V_N	E	$(1,2,4,5)$	T	P
536	605	5×11^2	1a	$5^6 11^4$	4	1	1	2	3	$(-,-,\otimes,-)$	α_2	$\mathcal{K},\mathfrak{K}_1$
537	*606	$2 \times 3 \times 101$	1b	$5^2 2^4 3^4 101^4$	12	1	2	5	6	$(-,-,\times,-)$	γ	$2 \times 5, 101$
538	607		2	607^4	1	0	0	0	5	$(-,-,-,\otimes)$	ϑ	
539	*609	$3 \times 7 \times 29$	1b	$5^2 3^4 7^4 29^4$	12	2	3	7	4	$(-,\otimes,-,-)$	β_2	$7^2 \times 5, \mathcal{K}$
540	610	$2 \times 5 \times 61$	1a	$5^6 2^4 61^4$	16	1	1	3	4	$(-,-,(\times),-)$	β_2	$2 \times 5^2, \mathcal{K}$
541	611	13×47	1b	$5^2 13^4 47^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
542	612	$2^2 \times 3^2 \times 17$	1b	$5^2 2^4 3^4 17^4$	13	1	2	5	6	$(\times,-,-,-)$	γ	$17, 3^2 \times 5$
543	613		1b	$5^2 613^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
544	614	2×307	1b	$5^2 2^4 307^4$	4	1	2	4	5	$(\times,-,-,-)$	ε	2
545	615	$3 \times 5 \times 41$	1a	$5^6 3^4 41^4$	16	1	1	3	4	$(-,-,(\times),-)$	β_2	$3, \mathcal{K}$
546	616	$2^3 \times 7 \times 11$	1b	$5^2 2^4 7^4 11^4$	12	2	3	7	4	$(-,-,(\times),-)$	β_2	$7 \times 5^2, \mathcal{K}$
547	617		1b	$5^2 617^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
548	618	$2 \times 3 \times 103$	2	$2^4 3^4 103^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
549	619		1b	$5^2 619^4$	1	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{L}$
550	*620	$2^2 \times 5 \times 31$	1a	$5^6 2^4 31^4$	16	2	4*	8*	5	$(-,-,\otimes,-)$	β_1	$5, \mathfrak{K}_1$
551	621	$3^3 \times 23$	1b	$5^2 3^4 23^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
552	622	2×311	1b	$5^2 2^4 311^4$	3	2	2	4	1	$(-,-,\otimes,-)$	α_2	$\mathcal{K},\mathfrak{K}_1$
553	623	7×89	1b	$5^2 7^4 89^4$	4	1	1	3	4	$(-,\otimes,-,-)$	β_2	$7 \times 5, \mathcal{K}$
554	624	$2^4 \times 3 \times 13$	2	$2^4 3^4 13^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
555	626	2×313	2	$2^4 313^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
556	*627	$3 \times 11 \times 19$	1b	$5^2 3^4 11^4 19^4$	13	3	4	9	2	$(-,\otimes,(\times),-)$	α_3	$\mathcal{K}_{(11)}, \mathcal{K}_{(19)}$
557	628	$2^2 \times 157$	1b	$5^2 2^4 157^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	
558	629	17×37	1b	$5^2 17^4 37^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
559	630	$2 \times 3^2 \times 5 \times 7$	1a	$5^6 2^4 3^4 7^4$	64	1	2	5	6	$(\times,-,-,-)$	γ	3, 5
560	631		1b	$5^2 631^4$	1	1	1	2	3	$(-,-,\otimes,-)$	α_2	$\mathcal{L},\mathfrak{K}_1$
561	632	$2^3 \times 79$	2	$2^4 79^4$	1	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{K}$
562	633	3×211	1b	$5^2 3^4 211^4$	3	1	2	4	5	$(-,-,\times,-)$	ε	3×5
563	634	2×317	1b	$5^2 2^4 317^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
564	635	5×127	1a	$5^6 127^4$	4	0	0	0	5	$(\times,-,-,-)$	ε	
565	636	$2^2 \times 3 \times 53$	1b	$5^2 2^4 3^4 53^4$	13	1	2	5	6	$(\times,-,-,-)$	γ	$3^2 \times 5, 53$
566	637	$7^2 \times 13$	1b	$5^2 7^4 13^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	
567	*638	$2 \times 11 \times 29$	1b	$5^2 2^4 11^4 29^4$	13	2	3	7	4	$(-,\otimes,(\times),-)$	β_2	$2^4 \times 11 \times 5,$ $\mathcal{K}_{(11)} \times \mathcal{K}_{(29)}^4$
568	639	$3^2 \times 71$	1b	$5^2 3^4 71^4$	3	1	1	3	4	$(-,-,(\times),-)$	β_2	$3^2 \times 5, \mathcal{K}$
569	641		1b	$5^2 641^4$	1	1	2	4	5	$(-,-,\otimes,-)$	β_1	$5, \mathfrak{K}_2$
570	642	$2 \times 3 \times 107$	1b	$5^2 2^4 3^4 107^4$	12	1	2	5	6	$(\times,-,-,-)$	γ	$107, 3 \times 5$

Continued

571	643		2	643^4	1	0	0	0	5	$(-,-,-,\otimes)$	ϑ	
572	644	$2^2 \times 7 \times 23$	1b	$5^2 2^4 7^4 23^4$	12	1	2	5	6	$(\times,-,-,-)$	γ	$2 \times 5, 7 \times 5$
573	645	$3 \times 5 \times 43$	1a	$5^6 3^4 43^4$	16	1	2	4	5	$(\times,-,-,-)$	ε	43
574	646	$2 \times 17 \times 19$	1b	$5^2 2^4 17^4 19^4$	13	1	2	5	6	$(-,\times,-,-)$	γ	$2 \times 5, 17$
575	647		1b	$5^2 647^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
576	*649	11×59	2	$11^4 59^4$	1	2	2	5	2	$(-,\otimes,(\times),-)$	α_3	$\mathcal{K}_{(11)}, \mathcal{K}_{(59)}$
577	650	$2 \times 5^2 \times 13$	1a	$5^6 2^4 13^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
578	651	$3 \times 7 \times 31$	2	$3^4 7^4 31^4$	4	1	1	3	4	$(-,-,(\times),-)$	β_2	$3^2 \times 7, \mathcal{K}$
579	652	$2^2 \times 163$	1b	$5^2 2^4 163^4$	3	1	2	4	5	$(\times,-,-,-)$	ε	5
580	653		1b	$5^2 653^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	

Table 19. 44 pure metacyclic fields with normalized radicands $654 \le D \le 701$.

No.	D	Factors	S	f	m	V_L	V_M	V_N	E	$(1,2,4,5)$	T	P
581	654	$2 \times 3 \times 109$	1b	$5^2 2^4 3^4 109^4$	13	1	2	5	6	$(\times,-,-,-)$	γ	$3 \times 5, 109$
582	655	5×131	1a	$5^6 131^4$	4	1	1	2	3	$(-,-,\otimes,-)$	α_2	$\mathcal{K}, \mathfrak{K}_1$
583	656	$2^4 \times 41$	1b	$5^2 2^4 41^4$	3	2	2	4	1	$(-,-,\otimes,-)$	α_2	$\mathcal{K}, \mathfrak{K}_1$
584	657	$3^2 \times 73$	2	$3^4 73^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
585	658	$2 \times 7 \times 47$	1b	$5^2 2^4 7^4 47^4$	12	1	2	5	6	$(\times,-,-,-)$	γ	$2 \times 5, 47 \times 5$
586	659		1b	$5^2 659^4$	1	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{L}$
587	*660	$2^2 \times 3 \times 5 \times 11$	1a	$5^6 2^4 3^4 11^4$	64	1	2	5	6	$(-,-,\times,-)$	γ	$2 \times 5, 5 \times 11$
588	661		1b	$5^2 661^4$	1	1	1	2	3	$(-,-,\otimes,-)$	α_2	$\mathcal{L}, \mathfrak{K}_1$
589	662	2×331	1b	$5^2 2^4 331^4$	3	1	1	3	4	$(-,-,(\times),-)$	β_2	$2^4 \times 5, \mathcal{K}$
590	663	$3 \times 13 \times 17$	1b	$5^2 3^4 13^4 17^4$	13	1	2	5	6	$(\times,-,-,-)$	γ	$3^2 \times 5, 13^2 \times 5$
591	664	$2^3 \times 83$	1b	$5^2 2^4 83^4$	3	1	2	4	5	$(\times,-,-,-)$	ε	$2^2 \times 5$
592	*665	$5 \times 7 \times 19$	1a	$5^6 7^4 19^4$	16	1	1	3	4	$(-,\otimes,-,-)$	β_2	$7, \mathcal{K}$
593	666	$2 \times 3^2 \times 37$	1b	$5^2 2^4 3^4 37^4$	13	1	2	5	6	$(\times,-,-,-)$	γ	$37, 2 \times 5^2$
594	667	23×29	1b	$5^2 23^4 29^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$23 \times 5^2, \mathcal{K}$
595	668	$2^2 \times 167$	2	$2^4 167^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
596	669	3×223	1b	$5^2 3^4 223^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
597	670	$2 \times 5 \times 67$	1a	$5^6 2^4 67^4$	16	1	2	4	5	$(\times,-,-,-)$	ε	2
598	*671	11×61	1b	$5^2 11^4 61^4$	3	3	4	8	1	$(-,-,\otimes,-)$	α_2	$\mathcal{K}_{(11)} \mathcal{K}_{(61)},$ $\mathfrak{K}_{(11),2} \mathfrak{K}^2_{(61),1}$
599	673		1b	$5^2 673^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
600	674	2×337	2	$2^4 337^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
601	677		1b	$5^2 677^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
602	678	$2 \times 3 \times 113$	1b	$5^2 2^4 3^4 113^4$	13	1	2	5	6	$(\times,-,-,-)$	γ	$2^2 \times 5, 3^4 \times 5$
603	679	7×97	1b	$5^2 7^4 97^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	

Continued

604	680	$2^3\times 5\times 17$	1a	$5^6 2^4 17^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
605	681	3×227	1b	$5^2 3^4 227^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
606	*682	$2\times 11\times 31$	2	$2^4 11^4 31^4$	3	2	3	6	3	$(-,-,\otimes,-)$	α_2	$\mathcal{K}_{(11)}\mathcal{K}_{(31)}^3$, $\mathfrak{K}_{(11),1}\mathfrak{K}_{(31),2}^3$
607	683		1b	$5^2 683^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
608	684	$2^2\times 3^2\times 19$	1b	$5^2 2^4 3^4 19^4$	13	1	2	5	6	$(-,\times,-,-)$	γ	$19\times 5, 2\times 5^3$
609	685	5×137	1a	$5^6 137^4$	4	0	0	0	5	$(\times,-,-,-)$	ε	
610	687	3×229	1b	$5^2 3^4 229^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$3\times 5^2, \mathcal{K}$
611	688	$2^4\times 43$	1b	$5^2 2^4 43^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	
612	689	13×53	1b	$5^2 13^4 53^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
613	690	$2\times 3\times 5\times 23$	1a	$5^6 2^4 3^4 23^4$	64	1	2	5	6	$(\times,-,-,-)$	γ	$2, 5\times 23^2$
614	*691		1b	$5^2 691^4$	1	3	4	8	1	$(-,-,\otimes,-)$	α_2	$\mathcal{L}, \mathfrak{K}_2$
615	692	$2^2\times 173$	1b	$5^2 2^4 173^4$	3	1	2	4	5	$(\times,-,-,-)$	ε	2×5
616	*693	$3^2\times 7\times 11$	2	$3^4 7^4 11^4$	4	2	3	6	3	$(-,-,\otimes,-)$	β_1	$3\times 7, \mathfrak{K}_2$
617	694	2×347	1b	$5^2 2^4 347^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
618	*695	5×139	1a	$5^6 139^4$	4	1	2	4	5	$(-,\times,-,-)$	ε	139
619	696	$2^3\times 3\times 29$	1b	$5^2 2^4 3^4 29^4$	13	1	2	5	6	$(-,\times,-,-)$	γ	$5, 2\times 29$
620	697	17×41	1b	$5^2 17^4 41^4$	3	1	1	3	4	$(-,-,(\times),-)$	β_2	$17\times 5, \mathcal{K}$
621	698	2×349	1b	$5^2 2^4 349^4$	4	1	1	3	4	$(-,\otimes,-,-)$	β_2	$2^4\times 5, \mathcal{K}$
622	699	3×233	2	$3^4 233^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
623	700	$2^2\times 5^2\times 7$	1a	$5^6 2^4 7^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
624	701		2	701^4	1	2	3	5	2	$(-,-,\otimes,\times)$	α_1	$\mathfrak{K}_1, \mathfrak{K}_2$

Table 20. 47 pure metacyclic fields with normalized radicands $702 \le D \le 753$.

No.	D	Factors	S	f^4	m	V_L	V_M	V_N	E	$(1,2,4,5)$	T	P
625	*702	$2\times 3^3\times 13$	1b	$5^2 2^4 3^4 13^4$	13	2	4	8	5	$(\times,-,-,-)$	ε	13×5^4
626	703	19×37	1b	$5^2 19^4 37^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$19\times 5^2, \mathcal{K}$
627	705	$3\times 5\times 47$	1a	$5^6 3^4 47^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
628	706	2×353	1b	$5^2 2^4 353^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
629	*707	7×101	2	$7^4 101^4$	4	1	1	3	4	$(-,-,(\times),\otimes)$	ζ_2	$\mathcal{K}$
630	708	$2^2\times 3\times 59$	1b	$5^2 2^4 3^4 59^4$	13	1	2	5	6	$(-,\times,-,-)$	γ	$3\times 5, 59\times 5$
631	709		1b	$5^2 709^4$	1	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{L}$
632	*710	$2\times 5\times 71$	1a	$5^6 2^4 71^4$	16	2	2	4	1	$(-,-,\otimes,-)$	α_2	$\mathcal{K}, \mathfrak{K}_2$
633	711	$3^2\times 79$	1b	$5^2 3^4 79^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$5, \mathcal{K}$
634	712	$2^3\times 89$	1b	$5^2 2^4 89^4$	3	1	2	4	5	$(-,\times,-,-)$	ε	$2^2\times 5$
635	713	23×31	1b	$5^2 23^4 31^4$	3	1	1	3	4	$(-,-,(\times),-)$	β_2	$5, \mathcal{K}$
636	714	$2\times 3\times 7\times 17$	1b	$5^2 2^4 3^4 7^4 17^4$	52	2	4	9	6	$(\times,-,-,-)$	γ	17×5^3, $2\times 7^2\times 5^3$

Continued

637	715	$5\times 11\times 13$	1a	$5^6 11^4 13^4$	16	1	1	3	4	$(-,-,(\times),-)$	β_2	$11,\mathcal{L}$
638	716	$2^2\times 179$	1b	$5^2 2^4 179^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$179,\mathcal{L}$
639	717	3×239	1b	$5^2 3^4 239^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$3^2\times 5,\mathcal{K}$
640	718	2×359	2	$2^4 359^4$	1	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{K}$
641	719		1b	$5^2 719^4$	1	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{L}$
642	720	$2^4\times 3^2\times 5$	1a	$5^6 2^4 3^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
643	721	7×103	1b	$5^2 7^4 103^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	
644	723	3×241	1b	$5^2 3^4 241^4$	3	1	1	3	4	$(-,-,(\times),-)$	β_2	$3\times 5,\mathcal{K}$
645	724	$2^2\times 181$	2	$2^4 181^4$	1	1	1	2	3	$(-,-,\otimes,-)$	α_2	$\mathcal{K},\mathfrak{K}_1$
646	725	$5^2\times 29$	1a	$5^6 29^4$	4	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{K}$
647	726	$2\times 3\times 11^2$	2	$2^4 3^4 11^4$	3	1	1	3	4	$(-,-,(\times),-)$	β_2	$2^2\times 3,\mathcal{K}$
648	727		1b	$5^2 727^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
649	728	$2^3\times 7\times 13$	1b	$5^2 2^4 7^4 13^4$	12	1	2	5	6	$(\times,-,-,-)$	γ	5, 7
650	730	$2\times 5\times 73$	1a	$5^6 2^4 73^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
651	731	17×43	1b	$5^2 17^4 43^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	
652	732	$2^2\times 3\times 61$	2	$2^4 3^4 61^4$	3	1	1	3	4	$(-,-,(\times),-)$	β_2	$2^3\times 3^2,\mathcal{K}$
653	733		1b	$5^2 733^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
654	734	2×367	1b	$5^2 2^4 367^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
655	735	$3\times 5\times 7^2$	1a	$5^6 3^4 7^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
656	737	11×67	1b	$5^2 11^4 67^4$	3	1	2	4	5	$(-,-,\otimes,-)$	β_1	$11,\mathfrak{K}_1$
657	738	$2\times 3^2\times 41$	1b	$5^2 2^4 3^4 41^4$	13	1	2	5	6	$(-,-,\times,-)$	γ	$3,2\times 5^2$
658	739		1b	$5^2 739^4$	1	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{L}$
659	740	$2^2\times 5\times 37$	1a	$5^6 2^4 37^4$	16	1	2	4	5	$(\times,-,-,-)$	ε	37
660	741	$3\times 13\times 19$	1b	$5^2 3^4 13^4 19^4$	13	1	2	5	6	$(-,\times,-,-)$	γ	$3\times 5^2,19\times 5$
661	742	$2\times 7\times 53$	1b	$5^2 2^4 7^4 53^4$	12	1	2	5	6	$(\times,-,-,-)$	γ	$2\times 5^2,7\times 5^2$
662	743		2	743^4	1	0	0	0	5	$(-,-,-,\otimes)$	ϑ	
663	744	$2^3\times 3\times 31$	1b	$5^2 2^4 3^4 31^4$	13	1	2	5	6	$(-,-,\times,-)$	γ	$5,3\times 31^2$
664	*745	5×149	1a	$5^6 149^4$	4	1	1	3	4	$(-,\otimes,-,\otimes)$	ζ_2	$\mathcal{K}$
665	746	2×373	1b	$5^2 2^4 373^4$	3	1	2	4	5	$(\times,-,-,-)$	ε	$2^4\times 5$
666	747	$3^2\times 83$	1b	$5^2 3^4 83^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
667	748	$2^2\times 11\times 17$	1b	$5^2 2^4 11^4 17^4$	13	1	2	5	6	$(-,-,\times,-)$	γ	$2\times 5,2\times 17$
668	*749	7×107	2	$7^4 107^4$	4	1	2	4	5	$(-,-,-,\times)$	ε	
669	*751		2	751^4	1	2	2	4	1	$(-,-,\otimes,\times)$	α_2	$\mathcal{L},\mathfrak{K}_1$
670	752	$2^4\times 47$	1b	$5^2 2^4 47^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
671	753	3×251	1b	$5^2 3^4 251^4$	4	1	1	3	4	$(-,-,(\times),-)$	β_2	$5,\mathcal{K}$

Table 21. 44 pure metacyclic fields with normalized radicands $754 \le D \le 799$.

No.	D	Factors	S	f^4	m	V_L	V_M	V_N	E	$(1,2,4,5)$	T	P
672	754	$2\times13\times29$	1b	$5^2 2^4 13^4 29^4$	13	1	2	5	6	$(-,\times,-,-)$	γ	$13\times5, 29\times5$
673	755	5×151	1a	$5^6 151^4$	4	1	1	3	4	$(-,-,(\times),\otimes)$	ζ_2	$\mathcal{K}$
674	756	$2^2\times3^3\times7$	1b	$5^2 2^4 3^4 7^4$	12	1	2	5	6	$(\times,-,-,-)$	γ	$7, 2^3\times5^2$
675	757		2	757^4	1	0	0	0	5	$(-,-,-,\otimes)$	ϑ	
676	758	2×379	1b	$5^2 2^4 379^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$2\times5^2, \mathcal{K}$
677	759	$3\times11\times23$	1b	$5^2 3^4 11^4 23^4$	13	2	3	6	3	$(-,-,\otimes,-)$	β_1	$3^2\times23\times5, \mathfrak{K}_2$
678	760	$2^3\times5\times19$	1a	$5^6 2^4 19^4$	16	1	1	3	4	$(-,\otimes,-,-)$	β_2	$2^4\times19, \mathcal{K}$
679	761		1b	$5^2 761^4$	1	2	3	5	2	$(-,-,\otimes,-)$	α_1	$\mathfrak{K}_1, \mathfrak{K}_2$
680	762	$2\times3\times127$	1b	$5^2 2^4 3^4 127^4$	13	1	2	5	6	$(\times,-,-,-)$	γ	$127\times5, 3\times5^3$
681	763	7×109	1b	$5^2 7^4 109^4$	4	1	1	3	4	$(-,\otimes,-,-)$	β_2	$7\times5^2, \mathcal{K}$
682	764	$2^2\times191$	1b	$5^2 2^4 191^4$	3	2	2	4	1	$(-,-,\otimes,-)$	α_2	$\mathcal{K}, \mathfrak{K}_2$
683	765	$3^2\times5\times17$	1a	$5^6 3^4 17^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
684	766	2×383	1b	$5^2 2^4 383^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
685	767	13×59	1b	$5^2 13^4 59^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$13\times5^2, \mathcal{K}$
686	769		1b	$5^2 769^4$	1	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{L}$
687	*770	$2\times5\times7\times11$	1a	$5^6 2^4 7^4 11^4$	64	1	2	5	6	$(-,-,\times,-)$	γ	$2\times5, 2^2\times11$
688	771	3×257	1b	$5^2 3^4 257^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	
689	772	$2^2\times193$	1b	$5^2 2^4 193^4$	4	1	2	4	5	$(\times,-,-,-)$	ε	2
690	773		1b	$5^2 773^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
691	774	$2\times3^2\times43$	2	$2^4 3^4 43^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	
692	775	$5^2\times31$	1a	$5^6 31^4$	4	2	3	5	2	$(-,-,\otimes,-)$	α_1	$\mathfrak{K}_1, \mathfrak{K}_2$
693	776	$2^3\times97$	2	$2^4 97^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
694	777	$3\times7\times37$	1b	$5^2 3^4 7^4 37^4$	12	1	2	5	6	$(\times,-,-,-)$	γ	$3, 7\times5$
695	778	2×389	1b	$5^2 2^4 389^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$5, \mathcal{K}$
696	*779	19×41	1b	$5^2 19^4 41^4$	3	3	4	8	1	$(-,\otimes,\otimes,-)$	α_2	$\mathcal{K}_{(19)}\mathcal{K}_{(41)}, \mathfrak{K}_{(41),1}$
697	780	$2^2\times3\times5\times13$	1a	$5^6 2^4 3^4 13^4$	64	1	2	5	6	$(\times,-,-,-)$	γ	$2\times5, 5\times13$
698	781	11×71	1b	$5^2 11^4 71^4$	3	3	5	9	2	$(-,-,\otimes,-)$	α_1	$\mathfrak{K}_{(11),1}\mathfrak{K}_{(71),1}^4,$ $\mathfrak{K}_{(11),2}\mathfrak{K}_{(71),2}^2$
699	782	$2\times17\times23$	2	$2^4 17^4 23^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
700	783	$3^3\times29$	1b	$5^2 3^4 29^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$29, \mathcal{L}$
701	*785	5×157	1a	$5^6 157^4$	4	1	2	4	5	$(-,-,-,\times)$	ε	
702	786	$2\times3\times131$	1b	$5^2 2^4 3^4 131^4$	13	2	3	7	4	$(-,-,(\times),-)$	β_2	$2\times3\times5^2, \mathcal{K}$
703	787		1b	$5^2 787^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
704	788	$2^2\times197$	1b	$5^2 2^4 197^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
705	789	3×263	1b	$5^2 3^4 263^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
706	790	$2\times5\times79$	1a	$5^6 2^4 79^4$	16	1	2	4	5	$(-,\times,-,-)$	ε	5
707	791	7×113	1b	$5^2 7^4 113^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	

Continued

708	792	$2^3\times 3^2\times 11$	1b	$5^2 2^4 3^4 11^4$	13	1	2	5	6	$(-,-,\times,-)$	γ	$2\times 5, 11\times 5$
709	793	13×61	2	$13^4 61^4$	1	1	2	4	5	$(-,-,\otimes,-)$	β_1	$61, \mathfrak{K}_2$
710	794	2×397	1b	$5^2 2^4 397^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
711	795	$3\times 5\times 53$	1a	$5^6 3^4 53^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
712	796	$2^2\times 199$	1b	$5^2 2^4 199^4$	4	1	1	3	4	$(-,\otimes,-,-)$	β_2	$2\times 5, \mathcal{K}$
713	797		1b	$5^2 797^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
714	*798	$2\times 3\times 7\times 19$	1b	$5^2 2^4 3^4 7^4 19^4$	52	2	4	9	6	$(-,\times,-,-)$	γ	$3\times 19, 7\times 5^4$
715	799	17×47	2	$17^4 47^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	

Table 22. 46 pure metacyclic fields with normalized radicands $801\le D\le 848$.

No.	D	Factors	S	$\mathfrak{f}^4$	m	V_L	V_M	V_N	E	$(1,2,4,5)$	T	P
716	801	$3^2\times 89$	2	$3^4 89^4$	1	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{K}$
717	802	2×401	1b	$5^2 2^4 401^4$	4	1	1	3	4	$(-,-,(\times),-)$	β_2	$2^2\times 5, \mathcal{K}$
718	803	11×73	1b	$5^2 11^4 73^4$	3	1	1	3	4	$(-,-,(\times),-)$	β_2	$73\times 5^2, \mathcal{K}$
719	804	$2^2\times 3\times 67$	1b	$5^2 2^4 3^4 67^4$	13	1	2	5	6	$(\times,-,-,-)$	γ	$3, 2\times 5^3$
720	805	$5\times 7\times 23$	1a	$5^6 7^4 23^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
721	806	$2\times 13\times 31$	1b	$5^2 2^4 13^4 31^4$	13	2	3	6	3	$(-,-,\otimes,-)$	β_1	$31\times 5^2, \mathfrak{K}_2$
722	807	3×269	2	$3^4 269^4$	1	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{K}$
723	*808	$2^3\times 101$	1b	$5^2 2^4 101^4$	4	2	2	4	1	$(-,-,\otimes,-)$	α_2	$\mathcal{K}, \mathfrak{K}_1$
724	809		1b	$5^2 809^4$	1	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{L}$
725	810	$2\times 3^4\times 5$	1a	$5^6 2^4 3^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
726	811		1b	$5^2 811^4$	1	1	1	2	3	$(-,-,\otimes,-)$	α_2	$\mathcal{L}, \mathfrak{K}_1$
727	812	$2^2\times 7\times 29$	1b	$5^2 2^4 7^4 29^4$	12	1	2	5	6	$(-,\times,-,-)$	γ	$29, 7^2\times 5$
728	813	3×271	1b	$5^2 3^4 271^4$	3	1	1	3	4	$(-,-,(\times),-)$	β_2	$5, \mathcal{K}$
729	814	$2\times 11\times 37$	1b	$5^2 2^4 11^4 37^4$	13	2	3	6	3	$(-,-,\otimes,-)$	β_1	$11\times 37^2\times 5, \mathfrak{K}_2$
730	815	5×163	1a	$5^6 163^4$	4	0	0	0	5	$(\times,-,-,-)$	ε	
731	816	$2^4\times 3\times 17$	1b	$5^2 2^4 3^4 17^4$	13	1	2	5	6	$(\times,-,-,-)$	γ	$2\times 5^2, 3\times 5^3$
732	817	19×43	1b	$5^2 19^4 43^4$	4	1	1	3	4	$(-,\otimes,-,-)$	β_2	$19^2\times 5, \mathcal{K}$
733	818	2×409	2	$2^4 409^4$	1	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{K}$
734	819	$3^2\times 7\times 13$	1b	$5^2 3^4 7^4 13^4$	12	1	2	5	6	$(\times,-,-,-)$	γ	$13, 7\times 5^2$
735	820	$2^2\times 5\times 41$	1a	$5^6 2^4 41^4$	16	2	2	4	1	$(-,-,\otimes,-)$	α_2	$\mathcal{K}, \mathfrak{K}_1$
736	821		1b	$5^2 821^4$	1	1	1	2	3	$(-,-,\otimes,-)$	α_2	$\mathcal{L}, \mathfrak{K}_2$
737	822	$2\times 3\times 137$	1b	$5^2 2^4 3^4 137^4$	13	2	4	8	5	$(\times,-,-,-)$	ε	$3^3\times 137\times 5$
738	823		1b	$5^2 823^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
739	824	$2^3\times 103$	2	$2^4 103^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
740	*825	$3\times 5^2\times 11$	1a	$5^6 3^4 11^4$	16	1	2	4	5	$(-,-,\otimes,-)$	β_1	$3^2\times 11, \mathfrak{K}_1$
741	826	$2\times 7\times 59$	2	$2^4 7^4 59^4$	4	1	1	3	4	$(-,\otimes,-,-)$	β_2	$59, \mathcal{L}$

Continued

742	827		1b	$5^2 827^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
743	828	$2^2 \times 3^2 \times 23$	1b	$5^2 2^4 3^4 23^4$	13	1	2	5	6	$(\times,-,-,-)$	γ	$23, 2\times 5^2$
744	829		1b	$5^2 829^4$	1	1	1	3	4	$(-,\otimes,-,-)$	β_2	$829, \mathcal{L}$
745	830	$2\times 5\times 83$	1a	$5^6 2^4 83^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
746	831	3×277	1b	$5^2 3^4 277^4$	3	1	2	4	5	$(\times,-,-,-)$	ε	$3^2\times 5$
747	833	$7^2\times 17$	1b	$5^2 7^4 17^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	
748	834	$2\times 3\times 139$	1b	$5^2 2^4 3^4 139^4$	13	1	2	5	6	$(-,\times,-,-)$	γ	$2\times 5, 3\times 5^3$
749	835	5×167	1a	$5^6 167^4$	4	0	0	0	5	$(\times,-,-,-)$	ε	
750	836	$2^2\times 11\times 19$	1b	$5^2 2^4 11^4 19^4$	13	2	3	7	4	$(-,\otimes,(\times),-)$	β_2	$2\times 11\times 5,$ $\mathcal{K}_{(11)}\times\mathcal{K}^3_{(19)}$
751	837	$3^3\times 31$	1b	$5^2 3^4 31^4$	3	1	1	3	4	$(-,-,(\times),-)$	β_2	$5, \mathcal{K}$
752	838	2×419	1b	$5^2 2^4 419^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$2\times 5, \mathcal{K}$
753	839		1b	$5^2 839^4$	1	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{L}$
754	840	$2^3\times 3\times 5\times 7$	1a	$5^6 2^4 3^4 7^4$	64	1	2	5	6	$(\times,-,-,-)$	γ	$2\times 7, 3\times 7$
755	842	2×421	1b	$5^2 2^4 421^4$	3	1	2	4	5	$(-,-,\times,-)$	ε	2×5^3
756	*843	3×281	2	$3^4 281^4$	1	1	2	3	4	$(-,-,\otimes,-)$	δ_1	$\mathfrak{K}_1$
757	844	$2^2\times 211$	1b	$5^2 2^4 211^4$	3	1	2	4	5	$(-,-,\times,-)$	ε	2×5^2
758	845	5×13^2	1a	$5^6 13^4$	4	0	0	0	5	$(\times,-,-,-)$	ε	
759	846	$2\times 3^2\times 47$	1b	$5^2 2^4 3^4 47^4$	13	1	2	5	6	$(\times,-,-,-)$	γ	2, 3
760	847	7×11^2	1b	$5^2 7^4 11^4$	4	1	1	3	4	$(-,-,(\times),-)$	β_2	$7\times 5^2, \mathcal{K}$
761	848	$2^4\times 53$	1b	$5^2 2^4 53^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	

Table 23. 47 pure metacyclic fields with normalized radicands $849 \le D \le 901$.

No.	D	Factors	S	f^4	m	V_L	V_M	V_N	E	$(1,2,4,5)$	T	P
762	849	3×283	2	$3^4 283^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
763	850	$2\times 5^2\times 17$	1a	$5^6 2^4 17^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
764	851	23×37	2	$23^4 37^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
765	852	$2^2\times 3\times 71$	1b	$5^2 2^4 3^4 71^4$	13	1	2	5	6	$(-,-,\times,-)$	γ	$2\times 5^3, 3\times 5^3$
766	853		1b	$5^2 853^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
767	854	$2\times 7\times 61$	1b	$5^2 2^4 7^4 61^4$	12	1	2	5	6	$(-,-,\times,-)$	γ	$2\times 5^2, 61$
768	855	$3^2\times 5\times 19$	1a	$5^6 3^4 19^4$	16	1	1	3	4	$(-,\otimes,-,-)$	β_2	$3, \mathcal{K}$
769	856	$2^3\times 107$	1b	$5^2 2^4 107^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	
770	857		2	857^4	1	0	0	0	5	$(-,-,-,\otimes)$	ϑ	
771	*858	$2\times 3\times 11\times 13$	1b	$5^2 2^4 3^4 11^4 13^4$	51	2	4	9	6	$(-,-,\times,-)$	γ	$2\times 5, 13\times 5$
772	859		1b	$5^2 859^4$	1	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{L}$
773	860	$2^2\times 5\times 43$	1a	$5^6 2^4 43^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
774	*861	$3\times 7\times 41$	1b	$5^2 3^4 7^4 41^4$	12	3	4	8	1	$(-,-,\otimes,-)$	α_2	$\mathcal{K}, \mathfrak{K}_1$

Continued

775	862	2×431	1b	$5^2 2^4 431^4$	3	1	2	4	5	$(-,-,\otimes,-)$	β_1	$431,\mathfrak{K}_1$
776	863		1b	$5^2 863^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
777	865	5×173	1a	$5^6 173^4$	4	0	0	0	5	$(\times,-,-,-)$	ε	
778	866	2×433	1b	$5^2 2^4 433^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
779	868	$2^2\times 7\times 31$	2	$2^4 7^4 31^4$	4	1	1	3	4	$(-,-,(\times),-)$	β_2	$2\times 7^2,\mathcal{K}$
780	869	11×79	1b	$5^2 11^4 79^4$	3	2	2	5	2	$(-,\otimes,(\times),-)$	α_3	$\mathcal{K}_{(11)},\mathcal{K}_{(79)}$
781	870	$2\times 3\times 5\times 29$	1a	$5^6 2^4 3^4 29^4$	64	1	2	5	6	$(-,\times,-,-)$	γ	$29,3^2\times 5$
782	871	13×67	1b	$5^2 13^4 67^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
783	872	$2^3\times 109$	1b	$5^2 2^4 109^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$109,\mathcal{L}$
784	873	$3^2\times 97$	1b	$5^2 3^4 97^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
785	874	$2\times 19\times 23$	2	$2^4 19^4 23^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$2\times 19^2,\mathcal{K}$
786	876	$2^2\times 3\times 73$	2	$2^4 3^4 73^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
787	877		1b	$5^2 877^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
788	878	2×439	1b	$5^2 2^4 439^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$5,\mathcal{K}$
789	879	3×293	1b	$5^2 3^4 293^4$	4	1	2	4	5	$(\times,-,-,-)$	ε	3
790	880	$2^4\times 5\times 11$	1a	$5^6 2^4 11^4$	16	2	2	4	1	$(-,-,\otimes,-)$	α_2	$\mathcal{K},\mathfrak{K}_2$
791	881		1b	$5^2 881^4$	1	1	2	3	4	$(-,-,\otimes,-)$	δ_1	$\mathfrak{K}_2$
792	883		1b	$5^2 883^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
793	884	$2^2\times 13\times 17$	1b	$5^2 2^4 13^4 17^4$	13	1	2	5	6	$(\times,-,-,-)$	γ	$17,13\times 5^2$
794	885	$3\times 5\times 59$	1a	$5^6 3^4 59^4$	16	1	1	3	4	$(-,\otimes,-,-)$	β_2	$3,\mathcal{K}$
795	886	2×443	1b	$5^2 2^4 443^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	
796	887		1b	$5^2 887^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
797	888	$2^3\times 3\times 37$	1b	$5^2 2^4 3^4 37^4$	13	1	2	5	6	$(\times,-,-,-)$	γ	2, 3
798	889	7×127	1b	$5^2 7^4 127^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	
799	890	$2\times 5\times 89$	1a	$5^6 2^4 89^4$	16	1	1	3	4	$(-,\otimes,-,-)$	β_2	$2^2\times 5,\mathcal{K}$
800	891	$3^4\times 11$	1b	$5^2 3^4 11^4$	3	1	1	3	4	$(-,-,(\times),-)$	β_2	$3\times 5^3,\mathcal{K}$
801	892	$2^2\times 223$	1b	$5^2 2^4 223^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
802	*893	19×47	2	$19^4 47^4$	1	1	1	3	4	$(-,\otimes,-,-)$	β_2	$19,\mathfrak{L}$
803	*894	$2\times 3\times 149$	1b	$5^2 2^4 3^4 149^4$	12	1	2	5	6	$(-,\times,-,-)$	γ	$5,2\times 3^2$
804	895	5×179	1a	$5^6 179^4$	4	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{K}$
805	897	$3\times 13\times 23$	1b	$5^2 3^4 13^4 23^4$	13	1	2	5	6	$(\times,-,-,-)$	γ	$13\times 5^2,3^3\times 5^2$
806	898	2×449	1b	$5^2 2^4 449^4$	4	1	1	3	4	$(-,\otimes,-,-)$	β_2	$2\times 5^2,\mathcal{K}$
807	899	29×31	2	$29^4 31^4$	1	2	2	5	2	$(-,\otimes,(\times),-)$	α_3	$\mathcal{K}_{(29)},\mathcal{K}_{(31)}$
808	901	17×53	2	$17^4 53^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	

Table 24. 47 pure metacyclic fields with normalized radicands $902 \le D \le 949$.

No.	D	Factors	S	f^4	m	V_L	V_M	V_N	E	$(1,2,4,5)$	T	P
809	*902	$2\times 11\times 41$	1b	$5^2 2^4 11^4 41^4$	13	2	3	7	4	$(-,-,(\times),-)$	β_2	$11\times 5,\mathcal{K}_{(11)}\times\mathcal{K}_{(41)}$
810	903	$3\times 7\times 43$	1b	$5^2 3^4 7^4 43^4$	16	1	2	5	6	$(\times,-,-,-)$	γ	$5,7^2\times 43$
811	904	$2^3\times 113$	1b	$5^2 2^4 113^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
812	905	5×181	1a	$5^6 181^4$	4	1	1	2	3	$(-,-,\otimes,-)$	α_2	$\mathcal{K},\mathfrak{K}_1$
813	906	$2\times 3\times 151$	1b	$5^2 2^4 3^4 151^4$	12	1	2	5	6	$(-,-,\times,-)$	γ	$2,3\times 5^2$
814	907		2	907^4	1	0	0	0	5	$(-,-,-,\otimes)$	ϑ	
815	908	$2^2\times 227$	1b	$5^2 2^4 227^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
816	909	$3^2\times 101$	1b	$5^2 3^4 101^4$	4	2	2	4	1	$(-,-,\otimes,-)$	α_2	$\mathcal{K},\mathfrak{K}_1$
817	910	$2\times 5\times 7\times 13$	1a	$5^6 2^4 7^4 13^4$	64	1	2	5	6	$(\times,-,-,-)$	γ	$2,13\times 5^2$
818	911		1b	$5^2 911^4$	1	1	1	2	3	$(-,-,\otimes,-)$	α_2	$\mathcal{L},\mathfrak{K}_2$
819	912	$2^4\times 3\times 19$	1b	$5^2 2^4 3^4 19^4$	13	1	2	5	6	$(-,\times,-,-)$	γ	$3,2\times 5^2$
820	913	11×83	1b	$5^2 11^4 83^4$	3	1	1	3	4	$(-,-,(\times),-)$	β_2	$11\times 5,\mathcal{K}$
821	914	2×457	1b	$5^2 2^4 457^4$	4	1	2	4	5	$(\times,-,-,-)$	ε	2
822	915	$3\times 5\times 61$	1a	$5^6 3^4 61^4$	16	1	1	3	4	$(-,-,(\times),-)$	β_2	$3\times 5^2,\mathcal{K}$
823	916	$2^2\times 229$	1b	$5^2 2^4 229^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$2^4\times 5,\mathcal{K}$
824	917	7×131	1b	$5^2 7^4 131^4$	4	1	1	3	4	$(-,-,(\times),-)$	β_2	$131\times 5,\mathcal{K}$
825	918	$2\times 3^3\times 17$	2	$2^4 3^4 17^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
826	919		1b	$5^2 919^4$	1	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{L}$
827	920	$2^3\times 5\times 23$	1a	$5^6 2^4 23^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
828	921	3×307	1b	$5^2 3^4 307^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	
829	922	2×461	1b	$5^2 2^4 461^4$	3	1	2	4	5	$(-,-,\times,-)$	ε	$2^2\times 5$
830	923	13×71	1b	$5^2 13^4 71^4$	3	2	2	4	1	$(-,-,\otimes,-)$	α_2	$\mathcal{K},\mathfrak{K}_2$
831	*924	$2^2\times 3\times 7\times 11$	2	$2^4 3^4 7^4 11^4$	12	1	2	5	6	$(-,-,\times,-)$	γ	$3\times 7,7\times 11$
832	925	$5^2\times 37$	1a	$5^6 37^4$	4	0	0	0	5	$(\times,-,-,-)$	ε	
833	926	2×463	2	$2^4 463^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
834	927	$3^2\times 103$	1b	$5^2 3^4 103^4$	3	1	2	4	5	$(\times,-,-,-)$	ε	$3^3\times 5^2$
835	929		1b	$5^2 929^4$	1	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{L}$
836	930	$2\times 3\times 5\times 31$	1a	$5^6 2^4 3^4 31^4$	64	2	4	9	6	$(-,-,\times,-)$	γ	5, 31
837	931	$7^2\times 19$	1b	$5^2 7^4 19^4$	4	1	1	3	4	$(-,\otimes,-,-)$	β_2	$7,\mathcal{L}$
838	932	$2^2\times 233$	2	$2^4 233^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
839	933	3×311	1b	$5^2 3^4 311^4$	3	1	1	3	4	$(-,-,(\times),-)$	β_2	$3\times 5,\mathcal{K}$
840	934	2×467	1b	$5^2 2^4 467^4$	3	0	0	1	6	$(\times,-,-,-)$	γ	
841	935	$5\times 11\times 17$	1a	$5^6 11^4 17^4$	16	2	2	4	1	$(-,-,\otimes,-)$	α_2	$\mathcal{K},\mathfrak{K}_2$
842	936	$2^3\times 3^2\times 13$	1b	$5^2 2^4 3^4 13^4$	13	1	2	5	6	$(\times,-,-,-)$	γ	3, 5
843	937		1b	$5^2 937^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
844	938	$2\times 7\times 67$	1b	$5^2 2^4 7^4 67^4$	12	1	2	5	6	$(\times,-,-,-)$	γ	$7,67\times 5$

Continued

845	939	3×313	1b	$5^2 3^4 313^4$	3	1	2	4	5	$(\times,-,-,-)$	ε	5
846	940	$2^2\times5\times47$	1a	$5^6 2^4 47^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
847	941		1b	$5^2 941^4$	1	2	2	4	1	$(-,-,\otimes,-)$	α_2	$\mathcal{L},\mathfrak{K}_1$
848	942	$2\times3\times157$	1b	$5^2 2^4 3^4 157^4$	12	2	4	8	5	$(\times,-,-,-)$	ε	157
849	943	23×41	2	$23^4 41^4$	1	1	1	2	3	$(-,-,\otimes,-)$	α_2	$\mathcal{K},\mathfrak{K}_2$
850	944	$2^4\times59$	1b	$5^2 2^4 59^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$2\times5^4,\mathcal{K}$
851	945	$3^3\times5\times7$	1a	$5^6 3^4 7^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
852	946	$2\times11\times43$	1b	$5^2 2^4 11^4 43^4$	12	1	2	5	6	$(-,-,\times,-)$	γ	$2\times5,43$
853	947		1b	$5^2 947^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
854	948	$2^2\times3\times79$	1b	$5^2 2^4 3^4 79^4$	13	1	2	5	6	$(-,\times,-,-)$	γ	$2\times5^2,3\times5^2$
855	949	13×73	2	$13^4 73^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	

Table 25. 45 pure metacyclic fields with normalized radicands $950\le D\le 999$.

No.	D	Factors	S	f^4	m	V_L	V_M	V_N	E	$(1,2,4,5)$	T	P
856	950	$2\times5^2\times19$	1a	$5^6 2^4 19^4$	16	1	1	3	4	$(-,\otimes,-,-)$	β_2	$5,\mathcal{K}$
857	951	3×317	2	$3^4 317^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
858	952	$2^3\times7\times17$	1b	$5^2 2^4 7^4 17^4$	12	1	2	5	6	$(\times,-,-,-)$	γ	$2\times5^4,7\times5^4$
859	953		1b	$5^2 953^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
860	954	$2\times3^2\times53$	1b	$5^2 2^4 3^4 53^4$	13	1	2	5	6	$(\times,-,-,-)$	γ	$3\times5,2\times5^2$
861	*955	5×191	1a	$5^6 191^4$	4	2*	3*	6*	3	$(-,-,\otimes,-)$	α_2	$\mathcal{K},\mathfrak{K}_1$
862	956	$2^2\times239$	1b	$5^2 2^4 239^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$239,\mathcal{L}$
863	*957	$3\times11\times29$	2	$3^4 11^4 29^4$	3	2	2	5	2	$(-,\otimes,(\times),-)$	α_3	$\mathcal{K}_{(11)},\mathcal{K}_{(29)}$
864	958	2×479	1b	$5^2 2^4 479^4$	3	1	2	4	5	$(-,\times,-,-)$	ε	$2^3\times5$
865	959	7×137	1b	$5^2 7^4 137^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	
866	962	$2\times13\times37$	1b	$5^2 2^4 13^4 37^4$	13	1	2	5	6	$(\times,-,-,-)$	γ	$2\times5,37\times5$
867	963	$3^2\times107$	1b	$5^2 3^4 107^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	
868	964	$2^2\times241$	1b	$5^2 2^4 241^4$	3	1	2	4	5	$(-,-,\otimes,-)$	β_1	$241,\mathfrak{K}_2$
869	965	5×193	1a	$5^6 193^4$	4	0	0	1	6	$(-,-,-,\otimes)$	η	
870	966	$2\times3\times7\times23$	1b	$5^2 2^4 3^4 7^4 23^4$	52	2	4	9	6	$(\times,-,-,-)$	γ	$3\times5^3,2\times23^2$
871	967		1b	$5^2 967^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
872	969	$3\times17\times19$	1b	$5^2 3^4 17^4 19^4$	13	1	2	5	6	$(-,\times,-,-)$	γ	$19,3\times5^2$
873	970	$2\times5\times97$	1a	$5^6 2^4 97^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
874	971		1b	$5^2 971^4$	1	2	2	4	1	$(-,-,\otimes,-)$	α_2	$\mathcal{L},\mathfrak{K}_1$
875	973	7×139	1b	$5^2 7^4 139^4$	4	1	1	3	4	$(-,\otimes,-,-)$	β_2	$7\times5^2,\mathcal{K}$
876	974	2×487	2	$2^4 487^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
877	975	$3\times5^2\times13$	1a	$5^6 3^4 13^4$	16	0	0	1	6	$(\times,-,-,-)$	γ	
878	976	$2^4\times61$	2	$2^4 61^4$	1	1	1	2	3	$(-,-,\otimes,-)$	α_2	$\mathcal{K},\mathfrak{K}_2$

Continued

879	977		1b	$5^2 977^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
880	978	$2\times3\times163$	1b	$5^2 2^4 3^4 163^4$	13	1	2	5	6	$(\times,-,-,-)$	γ	$3\times5^2, 2\times5^4$
881	979	11×89	1b	$5^2 11^4 89^4$	3	2	3	7	4	$(-,\otimes,(\times),-)$	β_2	$11, \mathcal{K}_{(11)}\cdot\mathcal{K}_{(89)}^2$
882	980	$2^2\times5\times7^2$	1a	$5^6 2^4 7^4$	16	1	2	4	5	$(\times,-,-,-)$	ε	7
883	981	$3^2\times109$	1b	$5^2 3^4 109^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$109, \mathcal{L}$
884	*982	2×491	2	$2^4 491^4$	1	1	1	2	3	$(-,-,\otimes,-)$	α_2	$\mathcal{K}, \mathfrak{K}_1$
885	983		1b	$5^2 983^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
886	984	$2^3\times3\times41$	1b	$5^2 2^4 3^4 41^4$	13	1	2	5	6	$(-,-,\times,-)$	γ	$3, 2\times5^2$
887	985	5×197	1a	$5^6 197^4$	4	0	0	0	5	$(\times,-,-,-)$	ε	
888	986	$2\times17\times29$	1b	$5^2 2^4 17^4 29^4$	13	2	4	8	5	$(-,\times,-,-)$	ε	2×5
889	987	$3\times7\times47$	1b	$5^2 3^4 7^4 47^4$	12	1	2	5	6	$(\times,-,-,-)$	γ	$7\times5^2, 3\times5^4$
890	988	$2^2\times13\times19$	1b	$5^2 2^4 13^4 19^4$	13	1	2	5	6	$(-,\times,-,-)$	γ	$2, 13\times5^2$
891	989	23×43	1b	$5^2 23^4 43^4$	4	0	0	1	6	$(\times,-,-,-)$	γ	
892	990	$2\times3^2\times5\times11$	1a	$5^6 2^4 3^4 11^4$	64	1	2	5	6	$(-,-,\times,-)$	γ	$2\times5, 3\times5^2$
893	991		1b	$5^2 991^4$	1	1	2	3	4	$(-,-,\otimes,-)$	δ_1	$\mathfrak{K}_1$
894	993	3×331	2	$3^4 331^4$	1	1	1	2	3	$(-,-,\otimes,-)$	α_2	$\mathcal{K}, \mathfrak{K}_2$
895	994	$2\times7\times71$	1b	$5^2 2^4 7^4 71^4$	12	2	3	7	4	$(-,-,(\times),-)$	β_2	$2\times5^3, \mathcal{K}$
896	995	5×199	1a	$5^6 199^4$	4	1	1	3	4	$(-,\otimes,-,\otimes)$	ζ_2	$\mathcal{K}$
897	996	$2^2\times3\times83$	1b	$5^2 2^4 3^4 83^4$	13	1	2	5	6	$(\times,-,-,-)$	γ	$2\times5^2, 3\times5^4$
898	997		1b	$5^2 997^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	
899	998	2×499	1b	$5^2 2^4 499^4$	4	1	1	3	4	$(-,\otimes,-,-)$	β_2	$499, \mathcal{L}$
900	999	$3^3\times37$	2	$3^4 37^4$	1	0	0	0	5	$(\times,-,-,-)$	ε	

Table 26. Absolute frequencies of differential principal factorization types.

Type	100	200	300	400	500	600	700	800	900	1000	%
α_1	1	2	3	4	5	5	5	9	9	9	
α_2	10	17	23	30	35	42	52	57	63	75	8.3
α_3	0	0	0	1	1	3	5	5	7	8	
β_1	0	2	4	7	8	11	15	18	22	23	
β_2	7	24	40	54	80	94	108	126	146	161	17.9
γ	25	55	88	117	148	187	222	259	290	324	36.0
δ_1	0	0	1	1	3	4	4	4	6	7	
δ_2	8	14	19	23	31	35	38	44	51	53	5.9
ε	26	45	67	95	110	128	150	165	184	208	23.4
ζ_1	0	1	1	1	1	1	1	1	1	1	
ζ_2	0	0	0	0	0	1	1	4	4	5	
η	1	2	4	5	5	6	6	6	6	7	
ϑ	3	6	8	9	11	13	15	17	18	19	
Total	81	168	258	347	438	530	622	715	807	900	100.0

It is striking that type α_1 with 2-dimensional relative principal factorization, $R=2$, and type α_3 with 2-dimensional intermediate principal factorization, $I=2$, are populated rather sparsely, in favour of a remarkable contribution by type α_2 with mixed intermediate and relative principal factorization, $I=R=1$ (place four with 8%).

The appearance of the four types $\zeta_1,\zeta_2,\eta,\vartheta$ with norm representation $N_{N/K}(Z)=\zeta$, $Z\in U_N$, of the primitive fifth root of unity $\zeta=\zeta_5$ is marginal ([6] Thm. 8.2), in spite of the parametrized contribution by all prime conductors $f=q\equiv\pm 7\bmod(25)$ to type ϑ, as we shall prove in Theorem 4.1 (1) in § 4.3.

4.2. Similarity Classes and Prototypes

In [7], we came to the conviction that for deeper insight into the arithmetical structure of the fields under investigation, the prime factorization of the class field theoretic conductor f of the abelian extension N/K over the cyclotomic field $K=\mathbb{Q}(\zeta)$ and the primary invariants of all involved 5-class groups must be taken in consideration. These ideas were inspired by [16] [17] and have lead to the concept of *similarity classes* and representative *prototypes*, which refines the differential principal factorization (DPF) types

Let t be the number of primes $q_1,\cdots,q_t\in\mathbb{P}$ distinct from 5 which divide the conductor f. Among these prime numbers, we separately count
$u:=\#\{1\le i\le t\mid q_i\equiv\pm 1,\pm 7\bmod(25)\}$ free primes, $v:=t-u$ restrictive primes,
$s_2:=\#\{1\le i\le t\mid q_i\equiv -1\bmod(5)\}$ 2-split primes, and
$s_4:=\#\{1\le i\le t\mid q_i\equiv +1\bmod(5)\}$ 4-split primes. The multiplicity $m=m(f)$ is given in terms of t,u,v, according to § 2, and the dimensions of various spaces of primitive ambiguous ideals over the finite field $\mathbb{F}_5$ are given in terms of t,s_2,s_4, according to [6] § 4. By $\eta=\frac{1}{2}\left(1+\sqrt{5}\right)$ we denote the fundamental unit of $K^+=\mathbb{Q}\left(\sqrt{5}\right)$. The dimensions of the spaces of absolute, intermediate and relative DPF over $\mathbb{F}_5$ are denoted by A, I and R, identical with the additive (logarithmic) version in ([6] Thm. 6.1). Further, let $M=\mathbb{Q}\left(\sqrt{5},\sqrt[5]{D}\right)$ be the maximal real intermediate field of N/L, and denote by U_0 the subgroup of the unit group U_N of $N=\mathbb{Q}\left(\zeta,\sqrt[5]{D}\right)$ generated by the units of all conjugate fields of $L=\mathbb{Q}\left(\sqrt[5]{D}\right)$ and of $K=\mathbb{Q}(\zeta)$, where $\zeta=\zeta_5$ is a primitive fifth root of unity. For a number field F, let $V_F:=v_5(\#\mathrm{Cl}(F))$ be the 5-valuation of the class number of F.

Definition 4.1 A set of normalized fifth power free radicands $D>1$ is called *a similarity class* if the associated pure quintic fields $L=\mathbb{Q}\left(\sqrt[5]{D}\right)$ share the following common multiplets of invariants:

- the refined Dedekind species $(e_0;t,u,v,m;n,s_2,s_4)$, where

$$f^4=5^{e_0}\cdot q_1^4\cdots q_t^4 \text{ with } e_0\in\{0,2,6\},\, t\ge 0,\, n=t-s_2-s_4, \tag{4.1}$$

- the differential principal factorization type $(U,\eta,\zeta;A,I,R)$, where

$$\left(U_K:N_{N/K}(U_N)\right)=5^U \text{ and } U+1=A+I+R, \tag{4.2}$$

- the structure of the 5-class groups $(V_L, V_M, V_N; E)$, where

$$\left(U_N : U_0\right) = 5^E \quad \text{and} \quad V_N = 4 \cdot V_L + E - 5. \tag{4.3}$$

Warning 4.1 To reduce the number of invariants, we abstain from defining additional counters $s_2' := \#\{1 \le i \le t \mid q_i \equiv -1 \bmod (25)\}$ and $s_4' := \#\{1 \le i \le t \mid q_i \equiv +1 \bmod (25)\}$ for free splitting prime divisors of the conductor f. However, we point out that occasionally a similarity class in the sense of Definition 4.1 will be split in two separate classes, having the same invariants, but distinct contributions to the counters u and s_4, resp. s_2. For instance, the similarity classes [**77**] and [**202**] with $77 = 7 \times 11$ and $202 = 2 \times 101$ share identical multiplets of invariants $(e_0; t, u, v, m; n, s_2, s_4) = (2; 2, 1, 1, 4; 1, 0, 1)$ (species 1b), $(U, \eta, \zeta; A, I, R) = (2, -, -; 2, 1, 0)$ (type β_2), and $(V_L, V_M, V_N; E) = (1, 1, 3; 4)$. But $u = 1$ and $n = 1$ are due to 7, $v = 1$ and $s_4 = 1$ are due to 11, in the former case, whereas $v = 1$ and $n = 1$ are due to 2, $u = 1$ and $s_4 = 1$ are due to 101, in the latter case. Therefore, the contributions by primes congruent to $\pm 1 \bmod (25)$ will be indicated by writing $u = 1'$ and $s_4 = 1'$, resp. $s_2 = 1'$.

We also emphasize that in the rare cases of non-elementary 5-class groups, the actual structures (abelian type invariants) of the 5-class groups will be taken into account, and not only the 5-valuations V_L, V_M, V_N.

Definition 4.2 The minimal element $\mathbf{M}$ of a similarity class (with respect to the natural order of positive integers $\mathbb{N}$) is called the representative *prototype* of the class, which is denoted by writing its prototype in square brackets $[\mathbf{M}]$.

The remaining elements of a similarity class, which are bigger than the prototype, only reproduce the arithmetical invariants of the prototype and do not provide any additional information, exept possibly about *other primary components* of the class groups, that is the structure of ℓ-class groups $\mathrm{Cl}_\ell(F)$ of the fields $F \in \{L, M, N\}$ for $\ell \in \mathbb{P} \setminus \{5\}$.

Whereas there are only 13 DPF types of pure quintic fields, the *number of similarity classes* is obviously infinite, since firstly the number t of primes dividing the conductor is unbounded and secondly the number of *states*, defined by the triplet (V_L, V_M, V_N) of 5-valuations of class numbers, is also unlimited.

Given a fixed refined Dedekind species $(e_0; t, u, v, m; n, s_2, s_4)$, the set of all associated normalized fifth power free radicands D usually splits into several similarity classes defined by distinct DPF types (*type splitting*). Occasionally it even splits further into different structures of 5-class groups, called *states*, with increasing complexity of abelian type invariants (*state splitting*).

The 134 prototypes $2 \le \mathbf{M} < 10^3$ of pure quintic fields are listed in the Tables 27-30. By $|\mathbf{M}| := \#\{D \in [\mathbf{M}] \mid D < B\}$ we denote the number of elements of the similarity class $[\mathbf{M}]$ defined by the prototype $\mathbf{M}$, truncated at the upper bound $B := 10^3$ of our systematic investigations.

4.3. General Theorems on DPF Types and Polya Fields

There is only a single finite similarity class [5] = {5}, characterized by the

Table 27. 46 prototypes $2 \le \mathbf{M} < 200$ of pure metacyclic fields.

No.	$\mathbf{M}$	Factors	S	f^4	m	V_L	V_M	V_N	E	$(1,2,4,5)$	T	P	$\lvert\mathbf{M}\rvert$
1	2		1b	$5^2 2^4$	1	0	0	0	5	$(\times,-,-,-)$	ε		71
2	5		1a	5^6	1	0	0	0	5	$(-,-,-,\otimes)$	ϑ		1
3	6	2×3	1b	$5^2 2^4 3^4$	3	0	0	1	6	$(\times,-,-,-)$	γ		77
4	7		2	7^4	1	0	0	0	5	$(-,-,-,\otimes)$	ϑ		18
5	10	2×5	1a	$5^6 2^4$	4	0	0	0	5	$(\times,-,-,-)$	ε		31
6	11		1b	$5^2 11^4$	1	1	1	2	3	$(-,-,\otimes,-)$	α_2	$\mathcal{L},\mathfrak{K}_1$	14
7	14	2×7	1b	$5^2 2^4 7^4$	4	0	0	1	6	$(\times,-,-,-)$	γ		44
8	18	2×3^2	2	$2^4 3^4$	1	0	0	0	5	$(\times,-,-,-)$	ε		37
9	19		1b	$5^2 19^4$	1	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{L}$	27
10	22	2×11	1b	$5^2 2^4 11^4$	3	1	1	3	4	$(-,-,(\times),-)$	β_2	$2\times 5,\mathcal{K}$	35
11	30	$2\times 3\times 5$	1a	$5^6 2^4 3^4$	16	0	0	1	6	$(\times,-,-,-)$	γ		37
12	31		1b	$5^2 31^4$	1	2	3	5	2	$(-,-,\otimes,-)$	α_1	$\mathfrak{K}_1,\mathfrak{K}_2$	2
13	33	3×11	1b	$5^2 3^4 11^4$	3	2	2	4	1	$(-,-,\otimes,-)$	α_2	$\mathcal{K},\mathfrak{K}_2$	8
14	35	5×7	1a	$5^6 7^4$	4	0	0	1	6	$(-,-,-,\otimes)$	η		6
15	38	2×19	1b	$5^2 2^4 19^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$5,\mathcal{K}$	44
16	42	$2\times 3\times 7$	1b	$5^2 2^4 3^4 7^4$	12	1	2	5	6	$(\times,-,-,-)$	γ	$2\times 5,3\times 5^2$	22
17	55	5×11	1a	$5^6 11^4$	4	1	1	2	3	$(-,-,\otimes,-)$	α_2	$\mathcal{K},\mathfrak{K}_1$	6
18	57	3×19	2	$3^4 19^4$	1	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{K}$	10
19	66	$2\times 3\times 11$	1b	$5^2 2^4 3^4 11^4$	13	1	2	5	6	$(-,-,\times,-)$	γ	$2\times 5,3\times 5^3$	17
20	70	$2\times 5\times 7$	1a	$5^6 2^4 7^4$	16	0	0	1	6	$(\times,-,-,-)$	γ		14
21	77	7×11	1b	$5^2 7^4 11^4$	4	1	1	3	4	$(-,-,(\times),-)$	β_2	$11\times 5^3,\mathcal{K}$	7
22	78	$2\times 3\times 13$	1b	$5^2 2^4 3^4 13^4$	13	1	2	5	6	$(\times,-,-,-)$	γ	$3,2\times 5^3$	37
23	82	2×41	2	$2^4 41^4$	1	1	1	2	3	$(-,-,\otimes,-)$	α_2	$\mathcal{K},\mathfrak{K}_2$	15
24	95	5×19	1a	$5^6 19^4$	4	1	1	2	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{K}$	9
25	101		2	101^4	1	1	2	4	5	$(-,-,\otimes,\otimes)$	ζ_1	$\mathfrak{K}_2$	1
26	110	$2\times 5\times 11$	1a	$5^6 2^4 11^4$	16	1	1	3	4	$(-,-,(\times),-)$	β_2	$11,\mathcal{L}$	11
27	114	$2\times 3\times 19$	1b	$5^2 2^4 3^4 19^4$	13	1	2	5	6	$(-,\times,-,-)$	γ	$2\times 5^3,3\times 5^3$	20
28	123	3×41	1b	$5^2 3^4 41^4$	3	2	3	6	3	$(-,-,\otimes,-)$	α_2	$\mathcal{K},\mathfrak{K}_2$	2
29	126	$2\times 3^2\times 7$	2	$2^4 3^4 7^4$	4	0	0	1	6	$(\times,-,-,-)$	γ		6
30	131		1b	$5^2 131^4$	1	2	2	4	1	$(-,-,\otimes,-)$	α_2	$\mathcal{L},\mathfrak{K}_2$	6
31	132	$2^2\times 3\times 11$	2	$2^4 3^4 11^4$	3	1	1	3	4	$(-,-,(\times),-)$	β_2	$11,\mathcal{L}$	5
32	133	7×19	1b	$5^2 7^4 19^4$	4	1	1	3	4	$(-,\otimes,-,-)$	β_2	$7,\mathcal{L}$	7
33	139		1b	$5^2 139^4$	1	1	1	3	4	$(-,\otimes,-,-)$	β_2	$5,\mathcal{L}$	4
34	140	$2^2\times 5\times 7$	1a	$5^6 2^4 7^4$	16	1	2	4	5	$(\times,-,-,-)$	ε	7	5
35	141	3×47	1b	$5^2 3^4 47^4$	3	1	2	4	5	$(\times,-,-,-)$	ε	47×5	19
36	149		2	149^4	1	1	1	2	3	$(-,\otimes,-,\times)$	δ_2	$\mathcal{L}$	6

Continued

37	151		2	151^4	1	1	1	2	3	$(-,-,\otimes,\times)$	α_2	$\mathcal{L},\mathfrak{K}_1$	3
38	154	$2\times7\times11$	1b	$5^2 2^4 7^4 11^4$	12	2	3	7	4	$(-,-,(\times),-)$	β_2	$2\times11^2,\mathcal{K}$	3
39	155	5×31	1a	$5^6 31^4$	4	2	3	5	2	$(-,-,\otimes,-)$	α_1	$\mathfrak{K}_1,\mathfrak{K}_2$	2
40	171	$3^2\times19$	1b	$5^2 3^4 19^4$	3	1	2	4	5	$(-,\times,-,-)$	ε	19×5^2	6
41	174	$2\times3\times29$	2	$2^4 3^4 29^4$	3	1	1	3	4	$(-,\otimes,-,-)$	β_2	$3^2\times29,\mathcal{K}$	3
42	180	$2^2\times3^2\times5$	1a	$5^6 2^4 3^4$	16	1	2	4	5	$(\times,-,-,-)$	ε	3	5
43	182	$2\times7\times13$	2	$2^4 7^4 13^4$	4	1	2	4	5	$(\times,-,-,-)$	ε	7	1
44	186	$2\times3\times31$	1b	$5^2 2^4 3^4 31^4$	13	2	3	6	3	$(-,-,\otimes,-)$	β_1	$5,\mathfrak{K}_2$	7
45	190	$2\times5\times19$	1a	$5^6 2^4 19^4$	16	1	1	3	4	$(-,\otimes,-,-)$	β_2	$5,\mathcal{K}$	9
46	191		1b	$5^2 191^4$	1	1	2	4	5	$(-,-,\otimes,-)$	β_1	$5,\mathfrak{K}_1$	3

Table 28. 44 prototypes $200<\mathbf{M}<510$ of pure metacyclic fields.

No.	$\mathbf{M}$	Factors	S	f^4	m	V_L	V_M	V_N	E	$(1,2,4,5)$	T	P	$\lvert\mathbf{M}\rvert$
47	202	2×101	1b	$5^2 2^4 101^4$	4	1	1	3	4	$(-,\otimes,-,-)$	β_2	$2\times5,\mathcal{K}$	6
48	203	7×29	1b	$5^2 7^4 29^4$	4	1	2	4	5	$(-,\times,-,-)$	ε	29×5	2
49	209	11×19	1b	$5^2 11^4 19^4$	3	2	3	7	4	$(-,\otimes,\times,-)$	β_2	$11\times5^2,\mathcal{K}_{(19)}$	2
50	210	$2\times3\times5\times7$	1a	$5^6 2^4 3^4 7^4$	64	1	2	5	6	$(\times,-,-,-)$	γ	$7,3^2\times5$	5
51	211		1b	$5^2 211^4$	1	3	5	9	2	$(-,-,\otimes,-)$	δ_1	$\mathfrak{K}_2$	1
52	218	2×109	2	$2^4 109^4$	1	1	2	4	5	$(-,\times,-,-)$	ε	2	1
53	231	$3\times7\times11$	1b	$5^2 3^4 7^4 11^4$	12	1	2	5	6	$(-,-,\times,-)$	γ	$11,7\times5^2$	5
54	247	13×19	1b	$5^2 13^4 19^4$	3	1	2	5	6	$(-,\times,-,-)$	γ		2
55	253	11×23	1b	$5^2 11^4 23^4$	3	1	2	4	5	$(-,-,\otimes,-)$	β_1	$11,\mathfrak{K}_1$	4
56	259	7×37	1b	$5^2 7^4 37^4$	4	1	2	5*	6	$(\times,-,-,-)$	γ		1
57	266	$2\times7\times19$	1b	$5^2 2^4 7^4 19^4$	12	1	2	5	6	$(-,\times,-,-)$	γ	$5,2^3\times7$	3
58	273	$3\times7\times13$	1b	$5^2 3^4 7^4 13^4$	12	2	4	8	5	$(\times,-,-,-)$	ε	$7\times13^2\times5^4$	4
59	275	$5^2\times11$	1a	$5^6 11^4$	4	2	2	4	1	$(-,-,\otimes,-)$	α_2	$\mathcal{K},\mathfrak{K}_2$	2
60	276	$2^2\times3\times23$	2	$2^4 3^4 23^4$	3	0	0	1	6	$(\times,-,-,-)$	γ		10
61	281		1b	$5^2 281^4$	1	3	5*	9*	2	$(-,-,\otimes,-)$	α_1	$\mathfrak{K}_1,\mathfrak{K}_2$	1
62	285	$3\times5\times19$	1a	$5^6 3^4 19^4$	16	1	2	5	6	$(-,\times,-,-)$	γ		1
63	286	$2\times11\times13$	1b	$5^2 2^4 11^4 13^4$	13	2	3	7	4	$(-,-,(\times),-)$	β_2	$11^2\times5,\mathcal{K}$	3
64	287	7×41	1b	$5^2 7^4 41^4$	4	2	2	4	1	$(-,-,\otimes,-)$	α_2	$\mathcal{K},\mathfrak{K}_1$	1
65	290	$2\times5\times29$	1a	$5^6 2^4 29^4$	16	1	2	4	5	$(-,\times,-,-)$	ε	29	2
66	298	2×149	1b	$5^2 2^4 149^4$	4	1	2	4	5	$(-,\times,-,-)$	ε	5	2
67	301	7×43	2	$7^4 43^4$	4	0	0	1	6	$(-,-,-,\otimes)$	η		1
68	302	2×151	1b	$5^2 2^4 151^4$	4	1	2	4	5	$(-,-,\otimes,-)$	β_1	$2,\mathfrak{K}_1$	2
69	319	11×29	1b	$5^2 11^4 29^4$	3	2	2	5	2	$(-,\otimes,(\times),-)$	α_3	$\mathcal{K}_{(11)},\mathcal{K}_{(29)}$	3
70	329	7×47	1b	$5^2 7^4 47^4$	4	1	2	4	5	$(\times,-,-,-)$	ε	47	7

Continued

71	330	$2\times3\times5\times11$	1a	$5^6 2^4 3^4 11^4$	64	2	4	9	6	$(-,-,\times,-)$	γ	$3\times11, 5\times11^3$	2
72	341	11×31	1b	$5^2 11^4 31^4$	3	3	5	9	2	$(-,-,\otimes,-)$	α_1	$\mathfrak{K}_{(11),1}\mathfrak{K}_{(31),2}^3$, $\mathfrak{K}_{(11),2}\mathfrak{K}_{(31),1}$	2
73	348	$2^2\times3\times29$	1b	$5^2 2^4 3^4 29^4$	13	2	4	8	5	$(-,\times,-,-)$	ε	$2\times3\times5$	2
74	377	13×29	1b	$5^2 13^4 29^4$	3	2	3	6	3	$(-,\otimes,-,-)$	δ_2	$\mathcal{K}$	1
75	379		1b	$5^2 379^4$	1	1	2	4	5	$(-,\times,-,-)$	ε	5	1
76	385	$5\times7\times11$	1a	$5^6 7^4 11^4$	16	1	1	3	4	$(-,-,(\times),-)$	β_2	$7\times11^2, \mathcal{K}$	1
77	390	$2\times3\times5\times13$	1a	$5^6 2^4 3^4 13^4$	64	1	2	5	6	$(\times,-,-,-)$	γ	2, 5	4
78	398	2×199	1b	$5^2 2^4 199^4$	4	1	1	3	4	$(-,\otimes,-,-)$	β_2	$5, \mathcal{K}$	7
79	399	$3\times7\times19$	2	$3^4 7^4 19^4$	4	1	1	3	4	$(-,\otimes,-,-)$	β_2	$3\times19^2, \mathcal{K}$	2
80	401		2	401^4	1	2	3	5	2	$(-,-,\otimes,\times)$	α_1	$\mathfrak{K}_1, \mathfrak{K}_2$	2
81	418	$2\times11\times19$	2	$2^4 11^4 19^4$	3	2	3	6	3	$(-,\otimes,\otimes,-)$	α_2	$\mathcal{K}_{(19)}, \mathfrak{K}_{(11),1}$	1
82	421		1b	$5^2 421^4$	1	1	2	3	4	$(-,-,\otimes,-)$	δ_1	$\mathfrak{K}_2$	5
83	422	2×211	1b	$5^2 2^4 211^4$	3	1	2	4	5	$(-,-,\times,-)$	ε	2×5^2	6
84	451	11×41	2	$11^4 41^4$	1	2	3	6	3	$(-,-,\otimes,-)$	α_2	$\mathcal{K}_{(11)}\mathcal{K}_{(41)}^4$, $\mathfrak{K}_{(11),1}\mathfrak{K}_{(41),2}^3$	1
85	462	$2\times3\times7\times11$	1b	$5^2 2^4 3^4 7^4 11^4$	52	2	4	9	6	$(-,-,\times,-)$	γ	$5^2\times7$, $3\times5\times11^2$	1
86	465	$3\times5\times31$	1a	$5^6 3^4 31^4$	16	2	3	7*	4	$(-,-,(\times),-)$	β_2	$5, \mathcal{K}$	1
87	473	11×43	1b	$5^2 11^4 43^4$	4	2	3	7*	4	$(-,-,(\times),-)$	β_2	$11, \mathcal{L}$	1
88	482	2×241	2	$2^4 241^4$	1	1	2	4	5	$(-,-,\otimes,-)$	β_1	$241, \mathfrak{K}_2$	2
89	502	2×251	1b	$5^2 2^4 251^4$	4	*2	*4	*8	5	$(-,-,\otimes,-)$	β_1	$251\times5, \mathfrak{K}_2$	1
90	505	5×101	1a	$5^6 101^4$	4	1	1	3	4	$(-,-,(\times),\otimes)$	ζ_2	$\mathcal{K}$	2

Table 29. 39 prototypes $510 < \mathbf{M} < 900$ of pure metacyclic fields.

No.	$\mathbf{M}$	Factors	S	f^4	m	V_L	V_M	V_N	E	$(1,2,4,5)$	T	P	$\lvert\mathbf{M}\rvert$
91	517	11×47	1b	$5^2 11^4 47^4$	3	2	3	6	3	$(-,-,\otimes,-)$	β_1	$11\times5^2, \mathfrak{K}_2$	1
92	532	$2^2\times7\times19$	2	$2^4 7^4 19^4$	4	1	2	4	5	$(-,\times,-,-)$	ε	2×7	1
93	546	$2\times3\times7\times13$	1b	$5^2 2^4 3^4 7^4 13^4$	52	2	4	9	6	$(\times,-,-,-)$	γ	$3\times5\times13$, $5^2\times7\times13$	3
94	550	$2\times5^2\times11$	1a	$5^6 2^4 11^4$	16	2	3	6	3	$(-,-,\otimes,-)$	α_2	$\mathcal{K}, \mathfrak{K}_1$	1
95	551	19×29	2	$19^4 29^4$	1	2	2	5	2	$(-,\otimes,-,-)$	α_3	$\mathcal{K}_{(19)}, \mathcal{K}_{(29)}$	1
96	570	$2\times3\times5\times19$	1a	$5^6 2^4 3^4 19^4$	64	1	2	5	6	$(-,\times,-,-)$	γ	$2^4\times3, 2^4\times5$	2
97	574	$2\times7\times41$	2	$2^4 7^4 41^4$	4	1	1	3	4	$(-,-,(\times),-)$	β_2	$41, \mathcal{L}$	3
98	590	$2\times5\times59$	1a	$5^6 2^4 59^4$	16	2	3	*7	4	$(-,\otimes,-,-)$	β_2	$5, \mathcal{K}$	1
99	602	$2\times7\times43$	1b	$5^2 2^4 7^4 43^4$	16	1	2	5	6	$(\times,-,-,-)$	γ	2, 7	2
100	606	$2\times3\times101$	1b	$5^2 2^4 3^4 101^4$	12	1	2	5	6	$(-,-,\times,-)$	γ	$2\times5, 101$	2
101	609	$3\times7\times29$	1b	$5^2 3^4 7^4 29^4$	12	2	3	7	4	$(-,\otimes,-,-)$	β_2	$7^2\times5, \mathcal{K}$	1
102	620	$2^2\times5\times31$	1a	$5^6 2^4 31^4$	16	2	4*	8*	5	$(-,-,\otimes,-)$	β_1	$5, \mathfrak{K}_1$	1

Continued

103	627	$3\times11\times19$	1b	$5^2 3^4 11^4 19^4$	13	3	4	9	2	$(-,\otimes,(\times),-)$	α_3	$\mathcal{K}_{(11)},\mathcal{K}_{(19)}$	1
104	638	$2\times11\times29$	1b	$5^2 2^4 11^4 29^4$	13	2	3	7	4	$(-,\otimes,(\times),-)$	β_2	$2^4\times11\times5$, $\mathcal{K}_{(11)}\times\mathcal{K}_{(29)}^4$	2
105	649	11×59	2	$11^4 59^4$	1	2	2	5	2	$(-,\otimes,(\times),-)$	α_3	$\mathcal{K}_{(11)},\mathcal{K}_{(59)}$	2
106	660	$2^2\times3\times5\times11$	1a	$5^6 2^4 3^4 11^4$	64	1	2	5	6	$(-,-,\times,-)$	γ	$2\times5,5\times11$	2
107	665	$5\times7\times19$	1a	$5^6 7^4 19^4$	16	1	1	3	4	$(-,\otimes,-,-)$	β_2	$7,\mathcal{K}$	1
108	671	11×61	1b	$5^2 11^4 61^4$	3	3	4	8	1	$(-,-,\otimes,-)$	α_2	$\mathcal{K}_{(11)}\mathcal{K}_{(61)}$, $\mathfrak{K}_{(11),2}\mathfrak{K}_{(61),1}^2$	1
109	682	$2\times11\times31$	2	$2^4 11^4 31^4$	3	2	3	6	3	$(-,-,\otimes,-)$	α_2	$\mathcal{K}_{(11)}\mathcal{K}_{(31)}^3$, $\mathfrak{K}_{(11),1}\mathfrak{K}_{(31),2}^3$	1
110	691		1b	$5^2 691^4$	1	3	4	8	1	$(-,-,\otimes,-)$	α_2	$\mathcal{L},\mathfrak{K}_2$	1
111	693	$3^2\times7\times11$	2	$3^4 7^4 11^4$	4	2	3	6	3	$(-,-,\otimes,-)$	β_1	$3\times7,\mathfrak{K}_2$	1
112	695	5×139	1a	$5^6 139^4$	4	1	2	4	5	$(-,\times,-,-)$	ε	139	1
113	702	$2\times3^3\times13$	1b	$5^2 2^4 3^4 13^4$	13	2	4	8	5	$(\times,-,-,-)$	ε	13×5^4	2
114	707	7×101	2	$7^4 101^4$	4	1	1	3	4	$(-,-,(\times),\otimes)$	ζ_2	$\mathcal{K}$	1
115	710	$2\times5\times71$	1a	$5^6 2^4 71^4$	16	2	2	4	1	$(-,-,\otimes,-)$	α_2	$\mathcal{K},\mathfrak{K}_2$	4
116	745	5×149	1a	$5^6 149^4$	4	1	1	3	4	$(-,\otimes,-,\otimes)$	ζ_2	$\mathcal{K}$	2
117	749	7×107	2	$7^4 107^4$	4	1	2	4	5	$(-,-,-,\times)$	ε		1
118	751		2	751^4	1	2	2	4	1	$(-,-,\otimes,\times)$	α_2	$\mathcal{L},\mathfrak{K}_1$	1
119	770	$2\times5\times7\times11$	1a	$5^6 2^4 7^4 11^4$	64	1	2	5	6	$(-,-,\times,-)$	γ	$2\times5,2^2\times11$	1
120	779	19×41	1b	$5^2 19^4 41^4$	3	3	4	8	1	$(-,\otimes,\otimes,-)$	α_2	$\mathcal{K}_{(19)}\mathcal{K}_{(41)}$, $\mathfrak{K}_{(41),1}$	1
121	785	5×157	1a	$5^6 157^4$	4	1	2	4	5	$(-,-,-,\times)$	ε		1
122	798	$2\times3\times7\cdot19$	1b	$5^2 2^4 3^4 7^4 19^4$	52	2	4	9	6	$(-,\times,-,-)$	γ	$3\times19,7\times5^4$	1
123	808	$2^3\times101$	1b	$5^2 2^4 101^4$	4	2	2	4	1	$(-,-,\otimes,-)$	α_2	$\mathcal{K},\mathfrak{K}_1$	2
124	825	$3\times5^2\times11$	1a	$5^6 3^4 11^4$	16	1	2	4	5	$(-,-,\otimes,-)$	β_1	$3^2\times11,\mathfrak{K}_1$	1
125	843	3×281	2	$3^4 281^4$	1	1	2	3	4	$(-,-,\otimes,-)$	δ_1	$\mathfrak{K}_1$	1
126	858	$2\times3\times11\times13$	1b	$5^2 2^4 3^4 11^4 13^4$	51	2	4	9	6	$(-,-,\times,-)$	γ	$2\times5,13\times5$	1
127	861	$3\times7\times41$	1b	$5^2 3^4 7^4 41^4$	13	3	4	8	1	$(-,-,\otimes,-)$	α_2	$\mathcal{K},\mathfrak{K}_1$	1
128	893	19×47	2	$19^4 47^4$	1	1	1	3	4	$(-,\otimes,-,-)$	β_2	$19,\mathfrak{L}$	1
129	894	$2\times3\times149$	1b	$5^2 2^4 3^4 149^4$	12	1	2	5	6	$(-,\times,-,-)$	γ	$5,2\times3^2$	1

exceptional number $t=0$ of primes $q\neq5$ dividing the conductor f (here $f^4=5^6$). The invariants of this unique metacyclic Polya field N are given by [5], species 1a, $(e_0;t,u,v,m;n,s_2,s_4)=(6;0,0,0,1;0,0,0)$,

type ϑ, $(U,\eta,\zeta;A,I,R)=(0,\times,\times;1,0,0)$, and $(V_L,V_M,V_N;E)=(0,0,0;5)$. (4.4)

We conjecture that all the other similarity classes are infinite. Precisely four of them can actually be given by parametrized infinite sequences in a deterministic way aside from the intrinsic probabilistic nature of the occurrence of primes in

Table 30. 5 prototypes $900 < \mathbf{M} < 1000$ of pure metacyclic fields.

No.	$\mathbf{M}$	Factors	S	f^4	m	V_L	V_M	V_N	E	$(1,2,4,5)$	T	P	$\lvert\mathbf{M}\rvert$
130	902	$2\times 11\times 41$	1b	$5^2 2^4 11^4 41^4$	13	2	3	7	4	$(-,-,(\times),-)$	β_2	$11\times 5, \mathcal{K}_{(11)}\times\mathcal{K}_{(41)}$	1
131	924	$2^2\times 3\times 7\times 11$	2	$2^4 3^4 7^4 11^4$	12	1	2	5	6	$(-,-,\times,-)$	γ	$3\times 7, 7\times 11$	1
132	955	5×191	1a	$5^6 191^4$	4	2*	3*	6*	3	$(-,-,\otimes,-)$	α_2	$\mathcal{K}, \mathfrak{K}_1$	1
133	957	$3\times 11\times 29$	2	$3^4 11^4 29^4$	3	2	2	5	2	$(-,\otimes,(\times),-)$	α_3	$\mathcal{K}_{(11)}, \mathcal{K}_{(29)}$	1
134	982	2×491	2	$2^4 491^4$	1	1	1	2	3	$(-,-,\otimes,-)$	α_2	$\mathcal{K}, \mathfrak{K}_1$	2

residue classes and of composite integers with assigned shape of prime decomposition. This was proved in ([6] Thm. 10.1)and ([7] Thm. 2.1).

Theorem 4.1 *Each of the following infinite sequences of conductors* $f = f_{N/K}$ *unambiguously determines the DPF type of the pure metacyclic fields* N *in the associated multiplet with* $m = m(f)$ *members.*

1) $f = q$ *with* $q \in \mathbb{P}$, $q \equiv \pm 7 \bmod (25)$ *gives rise to a singulet,* $m = 1$, *with DPF type* ϑ,

2) $f^4 = 5^2 \cdot q^4$ *with* $q \in \mathbb{P}$, $q \equiv \pm 2 \bmod (5)$, $q \not\equiv \pm 7 \bmod (25)$ *gives rise to a singulet,* $m = 1$, *with DPF type* ε,

3) $f^4 = 5^6 \cdot q^4$ *with* $q \in \mathbb{P}$, $q \equiv \pm 2 \bmod (5)$, $q \not\equiv \pm 7 \bmod (25)$ *gives rise to a quartet,* $m = 4$, *with homogeneous DPF type* $(\varepsilon, \varepsilon, \varepsilon, \varepsilon)$,

4) $f = q_1 \cdot q_2$ *with* $q_i \in \mathbb{P}$, $q_i \equiv \pm 2 \bmod (5)$, $q_i \not\equiv \pm 7 \bmod (25)$ *gives rise to a singulet,* $m = 1$, *with DPF type* ε.

In fact, the shape of the conductors in Theorem 4.1 does not only determine the refined Dedekind species and the DPF type, but also the structure of the 5-class groups of the fields *L*, *M* and *N*.

Corollary 4.1 The invariants of the similarity classes defined by the four infinite sequences of conductors in Theorem 4.1 are given as follows, in the same order:

$$[7], \textit{species}\, 2,\ (e_0; t, u, v, m; n, s_2, s_4) = (0;1,1,0,1;1,0,0),$$
$$\textit{type}\ \vartheta,\ (U, \eta, \zeta; A, I, R) = (0, \times, \times; 1, 0, 0),\ \textit{and}\ (V_L, V_M, V_N; E) = (0,0,0;5); \quad (4.5)$$

$$[2], \textit{species}\, 1b,\ (e_0; t, u, v, m; n, s_2, s_4) = (2;1,0,1,1;1,0,0),$$
$$\textit{type}\ \varepsilon,\ (U, \eta, \zeta; A, I, R) = (1, \times, -; 2, 0, 0),\ \textit{and}\ (V_L, V_M, V_N; E) = (0,0,0;5); \quad (4.6)$$

$$[10], \textit{species}\, 1a,\ (e_0; t, u, v, m; n, s_2, s_4) = (6;1,0,1,4;1,0,0),$$
$$\textit{type}\ \varepsilon,\ (U, \eta, \zeta; A, I, R) = (1, \times, -; 2, 0, 0),\ \textit{and}\ (V_L, V_M, V_N; E) = (0,0,0;5); \quad (4.7)$$

$$[18], \textit{species}\, 2,\ (e_0; t, u, v, m; n, s_2, s_4) = (0;2,0,2,1;2,0,0),$$
$$\textit{type}\ \varepsilon,\ (U, \eta, \zeta; A, I, R) = (1, \times, -; 2, 0, 0),\ \textit{and}\ (V_L, V_M, V_N; E) = (0,0,0;5). \quad (4.8)$$

The pure metacyclic fields *N* associated with these four similarity classes are Polya fields.

Remark 4.1 The statements concerning 5-class groups in Corollary 4.1 were proved by Parry in ([1] Thm. IV, p. 481), where Formula (10) gives the shape of radicands associated with the conductors in our Theorem 4.1.

Proof. (of Theorem 4.1 and Corollary 4.1) It only remains to show the claims for the composite radicands associated with conductors $f^4 = 5^6 \cdot q^4$ and $f = q_2 \cdot q_2$. See ([6] Thm. 10.6).

For similarity classes distinct from the four infinite classes in Theorem 4.1 we cannot provide deterministic criteria for the DPF type and for the homogeneity of multiplets with $m > 1$. In general, the members of a multiplet belong to distinct similarity classes, thus giving rise to *heterogeneous* DPF types. We explain these phenomena with the simplest cases where only two or three DPF types are involved (type splitting).

Theorem 4.2 *Each of the following infinite sequences of conductors* $f = f_{N/K}$ *admits precisely three DPF types of the pure metacyclic fields N in the associated quartet with* $m = 4$ *members.*

1) $f^4 = 5^6 \cdot q^4$ *with* $q \in \mathbb{P}$, $q \equiv \pm 7 \bmod (25)$ *gives rise to a quartet with possibly heterogeneous DPF type* $(\varepsilon^x, \eta^y, \vartheta^z)$, $x + y + z = 4$, *conjecturally always* $z = 0$,

2) $f = q_2 \cdot q_2$ *with* $q_i \in \mathbb{P}$, $q_i \equiv \pm 7 \bmod (25)$ *gives rise to a quartet with possibly heterogeneous DPF type* $(\varepsilon^x, \eta^y, \vartheta^z)$, $x + y + z = 4$, *conjecturally always* $z = 0$.

Example 4.1 It is quite easy to find complete quartets, whose members are spread rather widely. The smallest quartet $(35, 175, 245, 4375) = (5 \times 7, 5^2 \times 7, 5 \times 7^2, 5^4 \times 7)$ belonging to the first infinite sequence contains the member $D = 4375$ outside of the range of our systematic computations. We have determined its DPF type separately and thus discovered a homogeneous quartet of type (η, η, η, η). However, we cannot generally exclude the occurrence of heterogeneous quartets.

Corollary 4.2 The invariants of the similarity classes defined by the two infinite sequences of conductors in Theorem 4.2 are given as follows, in the same order. The statements concerning 5-class groups are only conjectural. Each sequence splits in two similarity classes.

The classes for $f^4 = 5^6 \cdot q^4$ are:

$$\begin{aligned} &[\mathbf{35}], \textit{species } 1a, \ (e_0; t, u, v, m; n, s_2, s_4) = (6; 1, 1, 0, 4; 1, 0, 0), \\ &\textit{type } \eta, \ (U, \eta, \zeta; A, I, R) = (1, -, \times; 2, 0, 0), \textit{ and } (V_L, V_M, V_N; E) = (0, 0, 1; 6); \end{aligned} \tag{4.9}$$

$$\begin{aligned} &[\mathbf{785}], \textit{species } 1a, \ (e_0; t, u, v, m; n, s_2, s_4) = (6; 1, 1, 0, 4; 1, 0, 0), \\ &\textit{type } \varepsilon, \ (U, \eta, \zeta; A, I, R) = (1, \times, -; 2, 0, 0), \textit{ and } (V_L, V_M, V_N; E) = (1, 2, 4; 5). \end{aligned} \tag{4.10}$$

The classes for $f = q_1 \cdot q_2$ *are:*

$$\begin{aligned} &[\mathbf{301}], \textit{species } 2, \ (e_0; t, u, v, m; n, s_2, s_4) = (0; 2, 2, 0, 4; 2, 0, 0), \\ &\textit{type } \eta, \ (U, \eta, \zeta; A, I, R) = (1, -, \times; 2, 0, 0), \textit{ and } (V_L, V_M, V_N; E) = (0, 0, 1; 6); \end{aligned} \tag{4.11}$$

$$[\mathbf{749}], \textit{species } 2, \ (e_0; t, u, v, m; n, s_2, s_4) = (0; 2, 2, 0, 4; 2, 0, 0),$$

type ε, $(U,\eta,\zeta;A,I,R)=(1,\times,-;2,0,0)$, *and* $(V_L,V_M,V_N;E)=(1,2,4;5)$. (4.12)

All pure metacyclic fields N associated with these four similarity classes are Polya fields.

Proof. (of Theorem 4.2 and Corollary 4.2) We use $T=2$, $v=s_2=s_4=0$ and ([6] Thm. 6.1).

Remark 4.2. The statements on 5-class groups in Corollary 4.2 have been verified for all examples with $2\le D<1000$ by our computations. In particular, the occurrence of the radicands $D=749=7\times 107$ and $D=785=5\times 157$, both with $V_L=1$, proves the impossibility of the general claim $5\nmid h(L)$ for the two situations mentioned in ([18] Lem. 3.3 (ii) and (iv), p. 204) and ([15] Thm. 5 (ii) and (iv), p. 5), partially also indicated in ([1] Thm. IV (11), p. 481).

Theorem 4.3 Each of the following infinite sequences of conductors $f=f_{N/K}$ admits precisely two DPF types of the pure metacyclic fields N in the associated hexadecuplet with $m=16$ members.

1) $f^4=5^6\cdot q_1^4q_2^4$ *with* $q_i\in\mathbb{P}$, $q_i\equiv\pm 2\bmod(5)$, *both* $q_i\not\equiv\pm 7\bmod(25)$ *gives rise to a hexadecuplet with possibly heterogeneous DPF type* (ε^x,γ^y), $x+y=16$,

2) $f^4=5^6\cdot q_1^4q_2^4$ *with* $q_i\in\mathbb{P}$, $q_i\equiv\pm 2\bmod(5)$, *only one* $q_i\equiv\pm 7\bmod(25)$ *gives rise to a hexadecuplet with possibly heterogeneous DPF type* (ε^x,γ^y), $x+y=16$.

Example 4.2 It is not difficult to find complete hexadecuplets, whose members are spread rather widely. The smallest hexadecuplet

$$\begin{aligned}&(30,60,90,120,150,180,240,270,360,540,600,720,810,1350,1620,3750)\\&=(2\times 3\times 5,2^2\times 3\times 5,2\times 3^2\times 5,2^3\times 3\times 5,2\times 3\times 5^2,2^2\times 3^2\times 5,\\&\quad 2^4\times 3\times 5,2\times 3^3\times 5,2^3\times 3^2\times 5,2^2\times 3^3\times 5,2^3\times 3\times 5^2,2^4\times 3^2\times 5,\\&\quad 2\times 3^4\times 5,2\times 3^3\times 5^2,2^2\times 3^4\times 5,2\times 3\times 5^4)\end{aligned}$$

belonging to the first infinite sequence contains the members $D=1350,1620,3750$ outside of the range of our systematic computations. We have determined their DPF type separately and thus discovered a heterogeneous hexadecuplet (in the same order) of type

$$(\varepsilon^3,\gamma^{13})=(\gamma,\gamma,\gamma,\gamma,\gamma,\varepsilon,\varepsilon,\gamma,\gamma,\gamma,\gamma,\gamma,\gamma,\varepsilon,\gamma,\gamma).$$

Corollary 4.3 The invariants of the similarity classes defined by the two infinite sequences of conductors in Theorem 4.3 are given as follows, in the same order. The statements concerning 5-class groups are only conjectural. Each sequence splits into two similarity classes.

The classes for $f^4=5^6\cdot q_1^4q_2^4$, both $q_i\not\equiv\pm 7\bmod(25)$ are:

[**30**], *species* 1*a*, $(e_0;t,u,v,m;n,s_2,s_4)=(6;2,0,2,16;2,0,0)$,

type γ, $(U,\eta,\zeta;A,I,R)=(2,-,-;3,0,0)$, *and* $(V_L,V_M,V_N;E)=(0,0,1;6)$; (4.13)

[**180**], *species* 1*a*, $(e_0;t,u,v,m;n,s_2,s_4)=(6;2,0,2,16;2,0,0)$,

type ε, $(U,\eta,\zeta;A,I,R)=(1,\times,-;2,0,0)$, *and* $(V_L,V_M,V_N;E)=(1,2,4;5)$. (4.14)

The classes for $f^4=5^6\cdot q_1^4q_2^4$, *only one* $q_i\equiv\pm 7\bmod(25)$ *are*:

[**70**], *species* 1*a*, $(e_0;t,u,v,m;n,s_2,s_4)=(6;2,1,1,16;2,0,0)$,

type γ, $(U,\eta,\zeta;A,I,R)=(2,-,-;3,0,0)$, *and* $(V_L,V_M,V_N;E)=(0,0,1;6)$; (4.15)

[**140**], *species* 1*a*, $(e_0;t,u,v,m;n,s_2,s_4)=(6;2,1,1,16;2,0,0)$,

type ε, $(U,\eta,\zeta;A,I,R)=(1,\times,-;2,0,0)$, *and* $(V_L,V_M,V_N;E)=(1,2,4;5)$. (4.16)

Only the pure metacyclic fields *N* of type γ associated with (14) and (16) are Polya fields.

Proof. (of Theorem 4.3 and Corollary 4.3) See ([6] Thm. 10.7).

Theorem 4.4 *A pure metacyclic field* $N=\mathbb{Q}\left(\zeta_5,\sqrt[5]{\ell}\right)$ *with prime radicand* $\ell\equiv\pm 1\bmod(25)$ *has a prime conductor* $f=\ell$, *and possesses the Polya property, regardless of its DPF type and the complexity of its* 5*-class group structure.*

Proof. This is an immediate consequence of ([6] Thm. 10.5 and Thm. 6.1), taking into account that we have the value $t=1$ for the number of primes dividing the conductor in the present situation, and thus the estimate in ([6] Cor. 4.1) yields $1\le A\le\min(3,t)=\min(3,1)=1$. For the Polya property we must have $A=t=1$, according to ([6] Thm. 10.5), which admits the DPF types $\alpha_1,\alpha_2,\alpha_3,\delta_1,\delta_2,\zeta_1,\zeta_2$ or ϑ ([6] Thm. 1.3 and Tbl. 1). However, DPF type α_3 is excluded by ([6] Cor. 4.2), since the requirement $s_2+s_4\ge 2$ cannot be fulfilled in our situation where either $s_2=0$ and $s_4=1$ for $\ell\equiv+1\bmod(25)$ or $s_2=1$ and $s_4=0$ for $\ell\equiv-1\bmod(25)$.

Theorem 4.5 *A pure metacyclic field* $N=\mathbb{Q}\left(\zeta_5,\sqrt[5]{\ell}\right)$ *with prime radicand* $\ell\equiv\pm 1\bmod(5)$ *but* $\ell\not\equiv\pm 1\bmod(25)$ *has a composite conductor* $f^4=5^2\cdot\ell^4$, *and the following conditions are equivalent:*

1) *N possesses the Polya property.*
2) $\left(\exists\alpha\in L=\mathbb{Q}\left(\sqrt[5]{\ell}\right)\right)N_{L/\mathbb{Q}}(\alpha)=5$.
3) *The prime ideal* $\mathfrak{p}\in\mathbb{P}_L$ *with* $5\mathcal{O}_L=\mathfrak{p}^5$ *is principal.*
4) *N is of DPF type either* β_1 *or* β_2 *or* ε.

Proof. This is a consequence of ([6] Thm. 10.5 and Thm. 6.1), taking into account that the prime 5 is not included in the current definition of the counter t (with value $t=1$ in the present situation), and thus the estimate in ([6] Cor. 4.1) must be replaced by $1\le A\le\min(3,t+1)=\min(3,2)=2$. For the Polya property we must have $A=t+1=2$ ([6] Thm. 10.5), which determines the DPF types $\beta_1,\beta_2,\varepsilon$ or η ([6] Thm. 1.3 and Tbl. 1). However, DPF type η is excluded by the prime $\ell\not\equiv\pm 1,\pm 7\bmod(25)$ dividing the conductor (([6] Thm. 8.1)).

Inspired by the last two theorems, it is worth ones while to summarize, for each kind of prime radicands, what is known about the possibilities for differential principal factorizations.

Theorem 4.6 *Let* $N = \mathbb{Q}\left(\zeta_5, \sqrt[5]{D}\right)$ *be a pure metacyclic field with prime radicand* $D \in \mathbb{P}$.

1) *If* $D = q$ *with* $q \equiv \pm 7 \bmod (25)$ *or* $q = 5$, *then N is of type* ϑ.

2) *If* $D = \ell$ *with* $\ell \equiv -1 \bmod (25)$, *then N is of one of the types* $\delta_2, \zeta_2, \vartheta$.

3) *If* $D = \ell$ *with* $\ell \equiv +1 \bmod (25)$, *then N is of one of the types* $\alpha_1, \alpha_2, \delta_1, \delta_2, \zeta_1, \zeta_2, \vartheta$.

4) *If* $D = q$ *with* $q \equiv \pm 2 \bmod (5)$ *but* $q \not\equiv \pm 7 \bmod (25)$, *then N is of type* ε.

5) *If* $D = \ell$ *with* $\ell \equiv -1 \bmod (5)$ *but* $\ell \not\equiv -1 \bmod (25)$, *then N is of one of the types* $\beta_2, \delta_2, \varepsilon$.

6) *If* $D = \ell$ *with* $\ell \equiv +1 \bmod (5)$ *but* $\ell \not\equiv +1 \bmod (25)$, *then N is of one of the types* $\alpha_1, \alpha_2, \beta_1, \beta_2, \delta_1, \delta_2, \varepsilon$.

A pure metacyclic field with prime radicand can never be of any of the types α_3, γ, η.

Proof. By making use of the bounds [6] §4 for $\mathbb{F}_5$-dimensions of spaces of differential principal factors (DPF),

$$\begin{aligned} &1 \le A \le \min(3, t), \\ &0 \le I \le \min(2, 2(s_2 + s_4)), \\ &0 \le R \le \min(2, 4s_4), \end{aligned} \tag{4.17}$$

we can determine the possible DPF types of pure metacyclic fields $N = \mathbb{Q}\left(\zeta_5, \sqrt[5]{D}\right)$ with prime radicands $D \in \mathbb{P}$. We start with a few general observations.

Firstly, if $D \equiv \pm 1, \pm 7 \bmod (25)$, resp. $D = 5$, is prime, then *N* is of Dedekind species 2, resp. 1a, with prime power conductor $f = D$, resp. $f^4 = 5^6$, and $t = 1$, whence $A = 1$ and the types $\beta_1, \beta_2, \gamma, \varepsilon, \eta$ with $A \ge 2$ are forbidden. However, if $D \not\equiv \pm 1, \pm 7 \bmod (25)$ and $D \ne 5$ is prime, then the congruence requirement eliminates the types $\zeta_1, \zeta_2, \eta, \vartheta$, the field *N* is of Dedekind species 1b with composite conductor $f^4 = 5^2 \cdot D^4$, and $t = 2$, whence $1 \le A \le 2$ and type γ with $A = 3$ is discouraged. So, the types γ and η are generally forbidden for prime radicands.

Secondly, for a prime radicand $D \equiv \pm 1 \bmod (5)$ which splits in *M*, the space of radicals $\Delta = \left\langle \sqrt[5]{D} \right\rangle$ is a 1-dimensional subspace of absolute DPF contained in the 2-dimensional space $\Delta \oplus \Delta'$ of differential factors generated by the two prime ideals of *M* over *D*. Consequently, in this special situation there arises an additional constraint $I \le 1$ for the dimension of the space of intermediate DPF, which must be contained in the 1-dimensional complement Δ'. This generally excludes type α_3 with $I = 2$ for prime radicands.

1) If $D = q$ with $q \equiv \pm 7 \bmod (25)$, then $t = 1$, $s_2 = s_4 = 0$, and thus $A = 1$, $I = R = 0$. These conditions eliminate the types $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \gamma, \delta_1, \delta_2, \varepsilon, \zeta_1, \zeta_2, \eta$ with either $A \ge 2$ or $I \ge 1$ or $R \ge 1$, and only type ϑ remains admissible.

2) If $D = \ell$ with $\ell \equiv -1 \bmod (25)$, then $t = s_2 = 1$, $s_4 = 0$, and thus $A = 1$,

$0 \le I \le 1$, $R = 0$, whence the types $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \gamma, \delta_1, \varepsilon, \zeta_1, \eta$ with either $A \ge 2$ or $I = 2$ or $R \ge 1$ are excluded, and only the types $\delta_2, \zeta_2, \vartheta$ remain admissible.

3) If $D = \ell$ with $\ell \equiv +1 \bmod (25)$, then $t = s_4 = 1$, $s_2 = 0$, and thus $A = 1$, $0 \le I \le 1$, $0 \le R \le 2$, whence the types $\alpha_3, \beta_1, \beta_2, \gamma, \varepsilon, \eta$ with either $A \ge 2$ or $I = 2$ are excluded, and only the types $\alpha_1, \alpha_2, \delta_1, \delta_2, \zeta_1, \zeta_2, \vartheta$ remain admissible.

4) If $D = q$ with $q \equiv \pm 2 \bmod (5)$ but $q \not\equiv \pm 7 \bmod (25)$, then $t = 2$, $s_2 = s_4 = 0$, and thus $1 \le A \le 2$, $I = R = 0$. These conditions eliminate the types $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \gamma, \delta_1, \delta_2, \zeta_1, \zeta_2$ with either $A = 3$ or $I \ge 1$ or $R \ge 1$, and only the types $\varepsilon, \eta, \vartheta$ remain admissible. However, the congruence requirement modulo 25 discourages the types η, ϑ, and only type ε is possible.

5) If $D = \ell$ with $\ell \equiv -1 \bmod (5)$ but $\ell \not\equiv -1 \bmod (25)$, then $t = 2$, $s_2 = 1$, $s_4 = 0$, and thus $1 \le A \le 2$, $0 \le I \le 1$, $R = 0$, whence the types $\alpha_1, \alpha_2, \alpha_3, \beta_1, \gamma, \delta_1, \zeta_1$ with either $A = 3$ or $I = 2$ or $R \ge 1$ are forbidden. The types ζ_2, η, ϑ are excluded by congruence conditions, and only the types $\beta_2, \delta_2, \varepsilon$ remain admissible.

6) If $D = \ell$ with $\ell \equiv +1 \bmod (5)$ but $\ell \not\equiv +1 \bmod (25)$ then $t = 2$, $s_2 = 0$, $s_4 = 1$, and thus $1 \le A \le 2$, $0 \le I \le 1$, $0 \le R \le 2$, whence the types α_3, γ with either $A = 3$ or $I = 2$ are forbidden. The types $\zeta_1, \zeta_2, \eta, \vartheta$ are excluded by congruence conditions, and only the types $\alpha_1, \alpha_2, \beta_1, \beta_2, \delta_1, \delta_2, \varepsilon$ remain admissible.

Example 4.3 Concerning numerical realizations of Theorem 4.6, we refer to Corollary 4.1 for the parametrized infinite sequences [**7**] and [**2**] which realize item (1) and (4). (See also Tables 23 and 19 for the types ϑ and ε.) In all the other cases, there occurs type splitting:

The similarity class [**149**] partially realizes item (2). (See Table 38 for the type δ_2.) Outside the range of our systematic investigations, we found that the similarity class [**1049**] realizes type ζ_2. Realizations of the type ϑ are unknown up to now.

The similarity classes [**401**], [**151**] and [**101**] partially realize item (3). (See Table 31, Table 32 and Table 40 for the types α_1, α_2 and ζ_1.) Outside the range of our systematic investigations, we found that the similarity class [**1151**], resp. [**3251**], realizes type δ_1, resp. δ_2. Realizations of the types ζ_2 and ϑ are unknown up to now.

The similarity classes [**139**], [**19**] and [**379**] completely realize item (5). (See Table 35, Table 38 and Table 39 for the types β_2, δ_2 and ε.)

The similarity classes [**31**], [**11**], [**191**] and [**211**] partially realize item (6). (See Table 31, Table 32, Table 34 and Table 37 for the types α_1, α_2, β_1 and δ_1.) Realizations of the types β_2, δ_2 and ε are unknown up to now.

4.4. Non-Elementary 5-Class Groups

Although most of the 5-class groups of pure metacyclic fields N, maximal real subfields M and pure quintic subfields L are elementary abelian, there occur sparse examples with non-elementary structure. For instance, we have only 8 occurrences within the range $2 \le D < 10^3$ of our computations:

1) $\mathrm{Cl}_5(N) \simeq C_{25} \times C_5^3$, $(V_L, V_M, V_N; E) = (1, 2, 5*; 6)$ for $D = 259 = 7 \times 37$ (type γ),

2) $\mathrm{Cl}_5(N) \simeq C_{25} \times C_5^7$, $\mathrm{Cl}_5(M) \simeq C_{25} \times C_5^3$, $(V_L, V_M, V_N; E) = (3, 5*, 9*; 2)$ for $D = 281$ prime (type α_1),

3) $\mathrm{Cl}_5(N) \simeq C_{25} \times C_5^5$, $(V_L, V_M, V_N; E) = (2, 3, 7*; 4)$ for $D = 465 = 3 \times 5 \times 31$ (type β_2),

4) $\mathrm{Cl}_5(N) \simeq C_{25} \times C_5^5$, $(V_L, V_M, V_N; E) = (2, 3, 7*; 4)$ for $D = 473 = 11 \times 43$ (type β_2),

5) $\mathrm{Cl}_5(N) \simeq C_{25} \times C_5^6$, $\mathrm{Cl}_5(M) \simeq C_{25} \times C_5^2$, $\mathrm{Cl}_5(L) \simeq C_{25}$, $(V_L, V_M, V_N; E) = (2*, 4*, 8*; 5)$ for $D = 502 = 2 \times 251$ (type β_1),

6) $\mathrm{Cl}_5(N) \simeq C_{25} \times C_5^5$, $(V_L, V_M, V_N; E) = (2, 3, 7*; 4)$ for $D = 590 = 2 \times 5 \times 59$ (type β_2),

7) $\mathrm{Cl}_5(N) \simeq C_{25}^2 \times C_5^4$, $\mathrm{Cl}_5(M) \simeq C_{25} \times C_5^2$, $(V_L, V_M, V_N; E) = (2, 4*, 8*; 5)$ for $D = 620 = 2^2 \times 5 \times 31$ (type β_1),

8) $\mathrm{Cl}_5(N) \simeq C_{25} \times C_5^4$, $\mathrm{Cl}_5(M) \simeq C_{25} \times C_5$, $\mathrm{Cl}_5(L) \simeq C_{25}$, $(V_L, V_M, V_N; E) = (2*, 3*, 6*; 3)$ for $D = 955 = 5 \times 191$ (type α_2).

However, outside the range of systematic computations, we additionally found:

a) $\mathrm{Cl}_5(N) \simeq C_{25}^3 \times C_5$, $\mathrm{Cl}_5(M) \simeq C_{25} \times C_5$, $\mathrm{Cl}_5(L) \simeq C_{25}$, $(V_L, V_M, V_N; E) = (2*, 3*, 7*; 4)$ for $D = 1049$ prime (type ζ_2),

b) $\mathrm{Cl}_5(N) \simeq C_{25}^2 \times C_5^6$, $\mathrm{Cl}_5(M) \simeq C_{25} \times C_5^3$, $(V_L, V_M, V_N; E) = (3, 5*, 10*; 3)$ for $D = 3001$ prime (type α_2),

c) $\mathrm{Cl}_5(N) \simeq C_{25}^5 \times C_5^4$, $\mathrm{Cl}_5(M) \simeq C_{25}^2 \times C_5^3$, $\mathrm{Cl}_5(L) \simeq C_{25} \times C_5^2$, $(V_L, V_M, V_N; E) = (4*, 7*, 14*; 3)$ for $D = 3251$ prime (type δ_2),

d) $\mathrm{Cl}_5(N) \simeq C_{25}^2 \times C_5^2$, $\mathrm{Cl}_5(M) \simeq C_{25} \times C_5$, $\mathrm{Cl}_5(L) \simeq C_{25}$, $(V_L, V_M, V_N; E) = (2*, 3*, 6*; 3)$ for $D = 5849$ prime (type δ_2).

We point out that in all of the last four examples, the normal field N is a Polya field, since the radicands D are primes $\ell \equiv \pm 1 \bmod (25)$, the conductors are primes $f = \ell$, and thus all primitive ambiguous ideals are principal, generated by the radical $\delta = \sqrt[5]{D}$ and its powers. Consequently, there seems to be no upper bound for the complexity of 5-class groups $\mathrm{Cl}_5(N)$ of pure metacyclic Polya fields N in Theorem 4.4.

4.5. Refinement of DPF Types by Similarity Classes

Based on the definition of similarity classes and prototypes in § 4.2, on the explicit listing of all prototypes in the range between 2 and 10^3 in Tables 27-30, and on theoretical foundations in § 4.3, we are now in the position to establish the intended refinement of our 13 differential principal factorization types into similarity classes in Tables 31-43, as far as the range of our computations for normalized radicands $2 \le D < 10^3$ is concerned. The cardinalities $|\mathbf{M}|$ refine the statistical evaluation in Table 26.

DPF types are characterized by the multiplet $(U,\eta,\zeta;A,I,R)$, refined Dedekind species, S, by the multiplet $(e_0;t,u,v,m;n,s_2,s_4)$, and 5-class groups by the multiplet $(V_L,V_M,V_N;E)$.

DPF type α_1 splits into 3 similarity classes in the ground state $(V_L,V_M,V_N)=(2,3,5)$ and 2 similarity classes in the first excited state $(V_L,V_M,V_N)=(3,5,9)$. Summing up the partial frequencies 6+3 of these states in Table 31 yields the modest absolute frequency 9 of type α_1 in the range $2\le D<10^3$, as given in Table 26. The logarithmic subfield unit index of type α_1 is restricted to the single value $E=2$. Type α_1 is the unique type with 2-dimensional relative principal factorization, $R=2$.

The logarithmic subfield unit index of DPF type α_2 can take two values, either $E=3$ or $E=1$. Type α_2 with $E=3$ splits into 5 similarity classes in the ground state $(V_L,V_M,V_N)=(1,1,2)$ and 6 similarity classes in the first excited state $(V_L,V_M,V_N)=(2,3,6)$. Type α_2 with $E=1$ splits into 7 similarity classes in the ground state $(V_L,V_M,V_N)=(2,2,4)$ and 4 similarity classes in the first excited state $(V_L,V_M,V_N)=(3,4,8)$. Summing up the partial frequencies 40 + 7, resp. 24 + 4, of these states in Table 32 yields the considerable absolute frequency 75 of type α_2 in the range $2\le D<10^3$, as given in Table 26. Type α_2 is the unique type with mixed intermediate and relative principal factorization, $I=R=1$.

DPF type α_3 splits into 4 similarity classes in the ground state $(V_L,V_M,V_N)=(2,2,5)$ and 1 similarity class in the first excited state $(V_L,V_M,V_N)=(3,4,9)$. Summing up the partial frequencies 7 + 1 of these states in Table 33 yields the modest absolute frequency 8 of type α_3 in the range $2\le D<10^3$, as given in Table 26. The logarithmic subfield unit index of type α_3 is restricted to the unique value $E=2$. Type α_3 is the unique type with 2-dimensional intermediate principal factorization, $I=2$.

The logarithmic subfield unit index of DPF type β_1 can take two values, either $E=3$ or $E=5$. Type β_1 with $E=3$ consists of 3 similarity classes in the ground state $(V_L,V_M,V_N)=(2,3,6)$. Type β_1 with $E=5$ splits into 5 similarity classes in the ground state $(V_L,V_M,V_N)=(1,2,4)$ and 2 similarity classes in the first excited state $(V_L,V_M,V_N)=(2,4,8)$. Summing up the partial frequencies 9, resp. 12 + 2, of these states in Table 34 yields the modest absolute frequency 23 of type β_1 in the range $2\le D<10^3$, as given in Table 26. Type β_1 is the unique type with mixed absolute and relative principal factorization, $A=2$ and $R=1$.

DPF type β_2 splits into 16 similarity classes in the ground state $(V_L,V_M,V_N)=(1,1,3)$ and 9 similarity classes in the first excited state $(V_L,V_M,V_N)=(2,3,7)$. Summing up the partial frequencies 146 + 15 of these states in Table 35 yields the high absolute frequency 161 of type β_2 in the range $2\le D<10^3$, as given in Table 26. The logarithmic subfield unit index of type β_2 is restricted to the unique value $E=4$. Type β_2 is the unique type with mixed absolute and intermediate principal factorization, $A=2$ and $I=1$.

Table 31. Splitting of type α_1, $(U,\eta,\zeta;A,I,R)=(2,-,-;1,0,2)$: 5 similarity classes.

No.	S	e_0	t	u	v	m	n	s_2	s_4	V_L	V_M	V_N	E	**M**	|**M**|
1	1b	2	1	0	1	1	0	0	1	2	3	5	2	**31**	2
2	1a	6	1	0	1	4	0	0	1	2	3	5	2	**155**	2
3	1b	2	1	0	1	1	0	0	1	3	5*	9*	2	**281**	1
4	1b	2	2	0	2	3	0	0	2	3	5	9	2	**341**	2
5	2	0	1	1′	0	1	0	0	1′	2	3	5	2	**401**	2

Table 32. Splitting of type α_2, $(U,\eta,\zeta;A,I,R)=(2,-,-;1,1,1)$: 22 similarity classes.

No.	S	e_0	t	u	v	m	n	s_2	s_4	V_L	V_M	V_N	E	**M**	|**M**|
1	1b	2	1	0	1	1	0	0	1	1	1	2	3	**11**	14
2	1b	2	2	0	2	3	1	0	1	2	2	4	1	**33**	8
3	1a	6	1	0	1	4	0	0	1	1	1	2	3	**55**	6
4	1b	2	2	0	2	3	1	0	1	1	1	2	3	**82**	15
5	1b	2	2	0	2	3	1	0	1	2	3	6	3	**123**	2
6	1b	2	1	0	1	1	0	0	1	2	2	4	1	**131**	6
7	2	0	1	1′	0	1	0	0	1′	1	1	2	3	**151**	3
8	1a	6	1	0	1	4	0	0	1	2	2	4	1	**275**	2
9	1b	2	2	1	1	4	1	0	1	2	2	4	1	**287**	1
10	2	0	3	0	3	3	1	1	1	2	3	6	3	**418**	1
11	2	0	2	0	2	1	0	0	2	2	3	6	3	**451**	1
12	1a	6	2	0	2	16	1	0	1	2	3	6	3	**550**	1
13	1b	2	2	0	2	3	0	0	2	3	4	8	1	**671**	1
14	2	0	3	0	3	3	1	0	2	2	3	6	3	**682**	1
15	1b	2	1	0	1	1	0	0	1	3	4	8	1	**691**	1
16	1a	6	2	0	2	16	1	0	1	2	2	4	1	**710**	4
17	2	0	1	1′	0	1	0	0	1′	2	2	4	1	**751**	1
18	1b	2	2	0	2	3	0	1	1	3	4	8	1	**779**	1
19	1b	2	2	1′	1	4	1	0	1′	2	2	4	1	**808**	2
20	1b	2	3	1	2	12	2	0	1	3	4	8	1	**861**	1
21	1a	6	1	0	1	4	0	0	1	2*	3*	6*	3	**955**	1
22	2	0	2	0	2	1	1	0	1	1	1	2	3	**982**	2

Table 33. Splitting of type α_3, $(U,\eta,\zeta;A,I,R)=(2,-,-;1,2,0)$: 5 similarity classes.

No.	S	e_0	t	u	v	m	n	s_2	s_4	V_L	V_M	V_N	E	**M**	|**M**|
1	1b	2	2	0	2	3	0	1	1	2	2	5	2	**319**	3
2	2	0	2	0	2	1	0	2	0	2	2	5	2	**551**	1
3	1b	2	3	0	3	13	1	1	1	3	4	9	2	**627**	1
4	2	0	2	0	2	1	0	1	1	2	2	5	2	**649**	2
5	2	0	3	0	3	3	1	1	1	2	2	5	2	**957**	1

Table 34. Splitting of type β_1, $(U,\eta,\zeta;A,I,R)=(2,-,-;2,0,1)$: 10 similarity classes.

No.	S	e_0	t	u	v	m	n	s_2	s_4	V_L	V_M	V_N	E	M	|M|
1	1b	2	3	0	3	13	2	0	1	2	3	6	3	**186**	7
2	1b	2	1	0	1	1	0	0	1	1	2	4	5	**191**	3
3	1b	2	2	0	2	3	1	0	1	1	2	4	5	**253**	4
4	1b	2	2	1′	1	4	1	0	1′	1	2	4	5	**302**	2
5	2	0	2	0	2	1	1	0	1	1	2	4	5	**482**	2
6	1b	2	2	1′	1	4	1	0	1′	2*	4*	8*	5	**502**	1
7	1b	2	2	0	2	3	1	0	1	2	3	6	3	**517**	1
8	1a	6	2	0	2	16	1	0	1	2	4*	8*	5	**620**	1
9	2	0	3	1	2	4	2	0	1	2	3	6	3	**693**	1
10	1a	6	2	0	2	16	1	0	1	1	2	4	5	**825**	1

Table 35. Splitting of type β_2, $(U,\eta,\zeta;A,I,R)=(2,-,-;2,1,0)$: 25 similarity classes.

No.	S	e_0	t	u	v	m	n	s_2	s_4	V_L	V_M	V_N	E	M	|M|
1	1b	2	2	0	2	3	1	0	1	1	1	3	4	**22**	35
2	1b	2	2	0	2	3	1	1	0	1	1	3	4	**38**	44
3	1b	2	2	1	1	4	1	0	1	1	1	3	4	**77**	7
4	1a	6	2	0	2	16	1	0	1	1	1	3	4	**110**	11
5	2	0	3	0	3	3	2	0	1	1	1	3	4	**132**	5
6	1b	2	2	1	1	4	1	1	0	1	1	3	4	**133**	7
7	1b	2	1	0	1	1	0	1	0	1	1	3	4	**139**	4
8	1b	2	3	1	2	12	2	0	1	2	3	7	4	**154**	3
9	2	0	3	0	3	3	2	1	0	1	1	3	4	**174**	3
10	1a	6	2	0	2	16	1	1	0	1	1	3	4	**190**	9
11	1b	2	2	1′	1	4	1	0	1′	1	1	3	4	**202**	6
12	1b	2	2	0	2	3	0	1	1	2	3	7	4	**209**	2
13	1b	2	3	0	3	13	2	0	1	2	3	7	4	**286**	3
14	1a	6	2	1	1	16	1	0	1	1	1	3	4	**385**	1
15	1b	2	2	1′	1	4	1	1′	0	1	1	3	4	**398**	7
16	2	0	3	1	2	4	2	1	0	1	1	3	4	**399**	2
17	1a	6	2	0	2	16	1	0	1	2	3	7*	4	**465**	1
18	1b	2	2	1	1	4	1	0	1	2	3	7*	4	**473**	1
19	2	0	3	1	2	4	2	0	1	1	1	3	4	**574**	3
20	1a	6	2	0	2	16	1	1	0	2	3	7*	4	**590**	1
21	1b	2	3	1	2	12	2	1	0	2	3	7	4	**609**	1
22	1b	2	3	0	3	13	1	1	1	2	3	7	4	**638**	2
23	1a	6	2	1	1	16	1	1	0	1	1	3	4	**665**	1
24	2	0	2	0	2	1	1	1	0	1	1	3	4	**893**	1
25	1b	2	3	0	3	1.	1	0	2	2	3	7	4	**902**	1

Table 36. Splitting of type γ, $(U,\eta,\zeta;A,I,R)=(2,-,-;3,0,0)$: 29 similarity classes.

No.	S	e_0	t	u	v	m	n	s_2	s_4	V_L	V_M	V_N	E	**M**	$\lvert\mathbf{M}\rvert$
1	1b	2	2	0	2	3	2	0	0	0	0	1	6	**6**	77
2	1b	2	2	1	1	4	2	0	0	0	0	1	6	**14**	44
3	1a	6	2	0	2	16	2	0	0	0	0	1	6	**30**	37
4	1b	2	3	1	2	12	3	0	0	1	2	5	6	**42**	22
5	1b	2	3	0	3	13	2	0	1	1	2	5	6	**66**	17
6	1a	6	2	1	1	16	2	0	0	0	0	1	6	**70**	14
7	1b	2	3	0	3	13	3	0	0	1	2	5	6	**78**	37
8	1b	2	3	0	3	13	2	1	0	1	2	5	6	**114**	20
9	2	0	3	1	2	4	3	0	0	0	0	1	6	**126**	6
10	1a	6	3	1	2	64	3	0	0	1	2	5	6	**210**	5
11	1b	2	3	1	2	12	2	0	1	1	2	5	6	**231**	5
12	1b	2	2	0	2	3	1	1	0	1	2	5	6	**247**	2
13	1b	2	2	1	1	4	2	0	0	1	2	5	6	**259**	1
14	1b	2	3	1	2	12	2	1	0	1	2	5	6	**266**	3
15	2	0	3	0	3	3	3	0	0	0	0	1	6	**276**	10
16	1a	6	2	0	2	16	1	1	0	1	2	5	6	**285**	1
17	1a	6	3	0	3	64	2	0	1	2	4	9	6	**330**	2
18	1a	6	3	0	3	64	3	0	0	1	2	5	6	**390**	4
19	1b	2	4	1	3	52	3	0	1	2	4	9	6	**462**	1
20	1b	2	4	1	3	25	4	0	0	2	4	9	6	**546**	3
21	1a	6	3	0	3	64	2	1	0	1	2	5	6	**570**	2
22	1b	2	3	2	1	16	3	0	0	1	2	5	6	**602**	2
23	1b	2	3	$1'$	2	12	2	0	$1'$	1	2	5	6	**606**	2
24	1a	6	3	0	3	64	2	0	1	1	2	5	6	**660**	2
25	1a	6	3	1	2	64	2	0	1	1	2	5	6	**770**	1
26	1b	2	4	1	3	52	3	1	0	2	4	9	6	**798**	1
27	1b	2	4	0	4	51	3	0	1	2	4	9	6	**858**	1
28	1b	2	3	$1'$	2	12	2	$1'$	0	1	2	5	6	**894**	1
29	2	0	4	1	3	12	3	0	1	1	2	5	6	**924**	1

DPF type γ splits into 6 similarity classes in the ground state $(V_L,V_M,V_N)=(0,0,1)$, 16 similarity classes in the first excited state $(V_L,V_M,V_N)=(1,2,5)$, and 5 similarity classes in the second excited state $(V_L,V_M,V_N)=(2,4,9)$. Summing up the partial frequencies 188+128+8 of these states in **Table 36** yields the maximal absolute frequency 324 of type γ among all 13 types in the range $2 \le D < 10^3$, as given in **Table 26**. The logarithmic subfield subfield unit index of type γ is restricted to the unique value $E=6$. Type γ is the unique type with 3-dimensional absolute principal factorization, $A=3$. However, the 1-dimensional subspace Δ is formed by radicals, and only the complementary 2-dimensional subspace is non-trivial.

The logarithmic subfield unit index of DPF type δ_1 can take two values, either $E=4$ or $E=2$. Type δ_1 with $E=4$ splits into 2 similarity classes in the ground state $(V_L,V_M,V_N)=(1,2,3)$. Type δ_1 with $E=2$ consists of 1 similarity class in the ground state $(V_L,V_M,V_N)=(3,5,9)$. Summing up the partial frequencies 6+1 of these states in Table 37 yields the modest absolute frequency 7 of type δ_1 in the range $2 \le D < 10^3$, as given in Table 26. Type δ_1 is a type with 1-dimensional relative principal factorization, $R=1$.

DPF type δ_2 splits into 4 similarity classes in the ground state $(V_L,V_M,V_N)=(1,1,2)$ and 1 similarity class in the first excited state $(V_L,V_M,V_N)=(2,3,6)$. Summing up the partial frequencies 52+1 of these states in Table 38 yields the considerable absolute frequency 53 of type δ_2 in the range $2 \le D < 10^3$, as given in Table 26. The logarithmic subfield unit index of type δ_2 is restricted to the unique value $E=3$. Type δ_2 is a type with 1-dimensional intermediate principal factorization, $I=1$.

DPF type ε splits into 3 similarity classes in the ground state $(V_L,V_M,V_N)=(0,0,0)$, 16 similarity classes in the first excited state $(V_L,V_M,V_N)=(1,2,4)$, and 3 similarity classes in the second excited state $(V_L,V_M,V_N)=(2,4,8)$. Summing up the partial frequencies 139+61+8 of these states in Table 39 yields the high absolute frequency 208 of type ε in the range $2 \le D < 10^3$, as given in Table 26. The logarithmic subfield unit index of type ε is restricted to the unique value $E=5$. Type ε is a type with 2-dimensional absolute principal factorization, $A=2$.

The logarithmic subfield unit index of DPF type ζ_1 is restricted to the unique value $E=5$. Type ζ_1 consists of 1 similarity class in the ground state $(V_L,V_M,V_N)=(1,2,4)$. The frequency 1 of this state in Table 40 coincides with the negligible absolute frequency 1 of type ζ_1 in the range $2 \le D < 10^3$, as given in Table 26. Type ζ_1 is a type with 1-dimensional relative principal factorization, $R=1$.

DPF type ζ_2 consists of 3 similarity classes. The modest absolute frequency 5 of type ζ_2 in the range $2 \le D < 10^3$, given in Table 26, is the sum 2+1+2 of partial frequencies in Table 41. Type ζ_2 only occurs with logarithmic subfield unit index $E=4$. It is a type with 1-dimensional intermediate principal factorization, $I=1$.

DPF type η splits in 2 similarity classes, [**35**] and [**301**]. The modest absolute frequency 7 of type η in the range $2 \le D < 10^3$, given in Table 26, is the sum 6+1 of partial frequencies in Table 42. Type η only occurs with logarithmic subfield unit index $E=6$. It is a type with 2-dimensional absolute principal factorization, $A=2$. However, it should be pointed out that outside of the range of our systematic investigations we found an excited state $(V_L,V_M,V_N)=(1,2,5)$ for the similarity class [**1505**], where $1505=5\times 7\times 43$ has three prime divisors, additionally to the ground state $(V_L,V_M,V_N)=(0,0,1)$.

Table 37. Splitting of type δ_1, $(U,\eta,\zeta;A,I,R)=(1,\times,-;1,0,1)$: 3 similarity classes.

No.	S	e_0	t	u	v	m	n	s_2	s_4	V_L	V_M	V_N	E	M	|M|
1	1b	2	1	0	1	1	0	0	1	3	5	9	2	**211**	1
2	1b	2	1	0	1	1	0	0	1	1	2	3	4	**421**	5
3	2	0	2	0	2	1	1	0	1	1	2	3	4	**843**	1

Table 38. Splitting of type δ_2, $(U,\eta,\zeta;A,I,R)=(1,\times,-;1,1,0)$: 5 similarity classes.

No.	S	e_0	t	u	v	m	n	s_2	s_4	V_L	V_M	V_N	E	M	|M|
1	1b	2	1	0	1	1	0	1	0	1	1	2	3	**19**	27
2	2	0	2	0	2	1	1	1	0	1	1	2	3	**57**	10
3	1a	6	1	0	1	4	0	1	0	1	1	2	3	**95**	9
4	2	0	1	1′	0	1	0	1′	0	1	1	2	3	**149**	6
5	1b	2	2	0	2	3	1	1	0	2	3	6	3	**377**	1

Table 39. Splitting of type ε, $(U,\eta,\zeta;A,I,R)=(1,\times,-;2,0,0)$: 22 similarity classes.

No.	S	e_0	t	u	v	m	n	s_2	s_4	V_L	V_M	V_N	E	M	|M|
1	1b	2	1	0	1	1	1	0	0	0	0	0	5	**2**	71
2	1a	6	1	0	1	1	1	0	0	0	0	0	5	**10**	31
3	2	0	2	0	2	1	2	0	0	0	0	0	5	**18**	37
4	1a	6	2	1	1	16	2	0	0	1	2	4	5	**140**	5
5	1b	2	2	0	2	3	2	0	0	1	2	4	5	**141**	19
6	1b	2	2	0	2	3	1	1	0	1	2	4	5	**171**	6
7	1a	6	2	0	2	16	2	0	0	1	2	4	5	**180**	5
8	2	0	3	1	2	4	2	0	0	1	2	4	5	**182**	1
9	1b	2	2	1	1	4	1	1	0	1	2	4	5	**203**	2
10	2	0	2	0	2	1	1	1	0	1	2	4	5	**218**	1
11	1b	2	3	1	2	12	3	0	0	2	4	8	5	**273**	4
12	1a	6	2	0	2	16	1	1	0	1	2	4	5	**290**	2
13	1b	2	2	0	2	3	1	1	0	1	2	4	5	**298**	2
14	1b	2	2	1	1	4	2	0	0	1	2	4	5	**329**	7
15	1b	2	3	0	3	13	2	1	0	2	4	8	5	**348**	2
16	1b	2	1	0	1	1	0	1	0	1	2	4	5	**379**	1
17	1b	2	2	0	2	3	1	0	1	1	2	4	5	**422**	6
18	2	0	3	1	2	4	2	1	0	1	2	4	5	**532**	1
19	1a	6	1	0	1	4	0	1	0	1	2	4	5	**695**	1
20	1b	2	3	0	3	13	3	0	0	2	4	8	5	**702**	2
21	2	0	2	2	0	4	2	0	0	1	2	4	5	**749**	1
22	1a	6	1	1	0	4	1	0	0	1	2	4	5	**785**	1

Table 40. No splitting of type ζ_1, $(U,\eta,\zeta;A,I,R)=(1,-,\times;1,0,1)$: 1 similarity class.

No.	S	e_0	t	u	v	m	n	s_2	s_4	V_L	V_M	V_N	E	M	|M|
1	2	0	1	1′	0	1	0	0	1′	1	2	4	5	**101**	1

Table 41. Splitting of type ζ_2, $(U,\eta,\zeta;A,I,R)=(1,-,\times;1,1,0)$: 3 similarity classes.

No.	S	e_0	t	u	v	m	n	s_2	s_4	V_L	V_M	V_N	E	M	|M|
1	1a	6	1	$1'$	0	4	0	0	$1'$	1	1	3	4	**505**	2
2	2	0	2	$2'$	0	4	1	0	$1'$	1	1	3	4	**707**	1
3	1a	6	1	1	0	4	0	1	0	1	1	3	4	**745**	2

Table 42. Splitting of type η, $(U,\eta,\zeta;A,I,R)=(1,-,\times;2,0,0)$: 2 similarity classes.

No.	S	e_0	t	u	v	m	n	s_2	s_4	V_L	V_M	V_N	E	M	|M|
1	1a	6	1	1	0	4	1	0	0	0	0	1	6	**35**	6
2	2	0	2	2	0	4	2	0	0	0	0	1	6	**301**	1

DPF type ϑ splits into the unique finite similarity class [**5**] with only a single element and the infinite parametrized sequence [**7**] consisting of all prime radicands $D=q$ congruent to $\pm 7 \bmod (25)$. The small absolute frequency 19 of type ϑ in the range $2 \le D < 10^3$, given in **Table 26**, is the sum $|\mathbf{5}|+|\mathbf{7}|=1+18$ in **Table 43**. Since no theoretical argument disables the occurrence of type ϑ for composite radicands D with prime factors 5 and $q \equiv \pm 7 \bmod (25)$, we conjecture that such cases will appear in bigger ranges with $D > 10^3$. Type ϑ only occurs with logarithmic subfield unit index $E=5$, and is the unique type where every unit of K occurs as norm of a unit of N, that is $U=0$.

4.6. Increasing Dominance of DPF Type γ for $T \to \infty$

In this final section, we want to show that the careful book keeping of similarity classes with representative prototypes in **Tables 31-43** is useful for the quantitative illumination of many other phenomena. For an explanation, we select the phenomenon of *absolute* principal factorizations.

The statistical distribution of DPF types in **Table 26** has proved that *type* γ with 324 occurrences, that is 36%, among all 900 fields $N=\mathbb{Q}\left(\zeta_5,\sqrt[5]{D}\right)$ with normalized radicands in the range $2 \le D < 10^3$ is doubtlessly the *high champion* of all DPF types. This means that there is a clear trend towards the maximal possible extent of 3-dimensional spaces of *absolute* principal factorizations, $A=3$, in spite of the disadvantage that the estimate $1 \le A \le \min(3,T)$ in the formulas (4.3) and (4.4) of ([6] Cor. 4.1) prohibits type γ for conductors f with $T \le 2$ prime divisors.

For the following investigation, we have to recall that the number T of all prime factors of $f^4 = 5^{e_0} \cdot q_1^4 \cdots q_t^4$ is given by $T=t+1$ for fields of Dedekind's species 1, where $e_0 \in \{2,6\}$, and by $T=t$ for fields of Dedekind's species 2, where $e_0=0$.

Conductors f with $T=4$ prime factors occur in six tables,

Table 43. Splitting of type ϑ, $(U,\eta,\zeta;A,I,R)=(0,\times,\times;1,0,0)$: 2 similarity classes.

No.	S	e_0	t	u	v	m	n	s_2	s_4	V_L	V_M	V_N	E	**M**	|**M**|
1	1a	6	0	0	0	1	0	0	0	0	0	0	5	**5**	1
2	2	0	1	1	0	1	1	0	0	0	0	0	5	**7**	18

1 case of type α_2 in a single similarity class of Table 32,
1 case of type α_3 in a single similarity class of Table 33,
7 cases of type β_1 in a single similarity class of Table 34,
10 cases of type β_2 in 5 similarity classes of Table 35,
126 cases of type γ in 16 similarity classes of Table 36,
8 cases of type ε in 3 similarity classes of Table 39,
that is, a total of 153 cases, with respect to the complete range $2 \le D < 10^3$ of our computations. Consequently, we have an increase of type γ from 36.0%, with respect to the entire database, to $\frac{126}{153} = 82.4\%$, with respect to $T = 4$.

The feature is even aggravated for conductors f with $T = 5$ prime factors, which exclusively occur in Table 36. There are 4 similarity classes with $T = 5$, namely [**462**], [**546**], [**798**], [**858**], with a total of 6 elements, all (100%) with associated fields of type γ.

5. Conclusions

In this paper, it was our intention to realize Parry's suggestion ([1] p. 484) concerning a numerical investigation of pure quintic number fields $L = \mathbb{Q}\left(\sqrt[5]{D}\right)$. For this purpose, we first developed theoretical foundations in a series of preparatory papers [6] [7] [8] [9] which expand the original germs in [12]. Since the non-Galois fields L do not contain the full wealth of arithmetical structures, we had to consider their pure metacyclic normal closures $N = \mathbb{Q}\left(\zeta_5, \sqrt[5]{D}\right)$ of degree 20.

On the one hand, this enabled us to use the *Galois cohomology* of the unit group U_N with respect to the relative automorphism group $G = \mathrm{Gal}(N/K) \simeq C_5$ over the cyclotomic field $K = \mathbb{Q}(\zeta_5)$ for defining 13 exhaustive and mutually exclusive *differential principal factorization* (DPF) *types* (Table 4 in § 3.1), based on the *unit norm index* $\#H^0(G,U_N) = (U_K : N_{N/K}(U_N)) = 5^U$, where $0 \le U \le 2$, which is connected with the order of the group of *primitive ambiguous principal ideals* $\#H^1(G,U_N) = (\mathcal{P}_{N/K} : \mathcal{P}_K) = 5^{U+1}$ via the Takagi/Hasse/Iwasawa-Theorem on the Herbrand quotient of U_N, and on a natural decomposition $U + 1 = A + I + R$ into the dimensions of the $\mathbb{F}_5$-vector spaces of *absolute*, *intermediate* and *relative* differential principal factors ([6], eqn. (6.3)).

On the other hand, our theory of the relatively cyclic quintic Kummer extension N/K as a 5-*ring class field* modulo the conductor $f = f_{N/K}$ over K admitted the calculation of the *multiplicity* $m = m(f)$ of the conductor f(§ 2), which is the number of non-isomorphic pure metacyclic fields N sharing the common conductor f and forming a *multiplet* $(N_1,\cdots,N_m)$ [8].

Equipped with this theoretical background, we were able to develop our Classification Algorithm 3.1 in § 3.2, and to prove that it determines the DPF type of L and N in finitely many steps. It also decides whether the normal field N is a Polya field or not. (It is known that L is a (trivial!) Polya field if and only if it possesses class number $h_L = 1$ [6].)

The algorithm was implemented as a Magma program script [3] [4] [5] and applied to the 900 fields N with normalized fifth power free radicands in the range $2 \le D < 10^3$, after some preliminary experiments with Pari/GP [2]. The result is documented in the twenty **Tables 6-25** of § 3.4. It is in perfect accordance with our theoretical predictions in 1991 [12]. Actually, each type occurs indeed, the most hardboiled type ζ_2 not earlier than for the radicand $D = 505$, and it is interesting to study the statistical distribution of the types in § 4.1, **Table 6**. Concerning the reliability of our extensive database, and for understanding the degree of precision contained in our paper, we point out that all tables have been thoroughly double checked with respect to misprints and copy-paste errors for *at least three times*.

Nevertheless, after the completion of the statistics in § 4, we came to the conviction that for deeper insight into the arithmetical structure of pure metacyclic fields N, the prime factorization of the class field theoretic conductor f of the abelian extension N/K (invariants $e_0;t,u,v;n,s_2,s_4$) and the primary invariants of all involved 5-class groups, partially given by the 5-valuations $(V_L,V_M,V_N;E)$, should be taken in consideration. In the last section, we present a corresponding refinement of the DPF types in *similarity classes* (§ 4.2), which is very useful for various applications. It reduces the number 900 of investigated radicands to 134 representative *prototypes* (§ 4.5). The remaining 766 radicands, which are bigger than the prototype of their similarity class, only reproduce the arithmetical invariants of the prototype and do not provide any additional relevant information. The rare phenomenon of non-elementary 5-class groups is documented in § 4.4. Theorems on DPF-types of members of multiplets are proved in § 4.3, and the striking dominance of type γ with 3-dimensional (maximal) absolute principal factorization for radicands with three and more prime divisors is presented and discussed in § 4.6.

Acknowledgements

We gratefully acknowledge that our research was supported by the Austrian Science Fund (FWF): project P 26008-N25. This work is dedicated to the memory of Charles J. Parry († 25 December 2010) who suggested a numerical investigation of pure quintic number fields. We are indebted to the anonymous referees for valuable suggestions concerning an improvement of the paper's layout.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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