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Inverse Nodal Problems for the Sturm-Liouville Operator with Some Nonlocal Integral Conditions

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Abstract

In this paper, we study three inverse nodal problems for the Sturm-Liouville operator with different nonlocal integral conditions. We get the conclusion that the potential function can be determined by a dense nodal subset uniquely. And we present some constructive procedures to solve the inverse nodal problems.

Keywords

Sturm-Liouville Operator, Inverse Nodal Problems, Nonlocal Integral Conditions

1. Introduction

Sturm-Liouville operators have an important role in physics, mathematics and other engineering fields [1] [2] [3] [4] [5]. Nonlocal problems are parts of differential equation theory. And the inverse nodal problems are aimed at reconstructing operator by the given nodes (zeros) of characteristic functions. Lots of results were given by scholars.

Generally, Sturm-Liouville problems with nonlocal conditions have three kinds of form: potential with integral delay or function delay; nonlocal potential; nonlocal boundary conditions.

In 1988, Mclaughlin raised the inverse nodal problem for the first time. Applying a dense nodal subset, he got the uniqueness theorem of potential function [6]. In 1989, by the similar methods, some general boundary conditions and corresponding reconstruction of the potential function were provided by Hald and Mclaughlin [7]. In 2001, Xuefeng Yang discussed a new inverse nodal problem. By using Gesztesy and Simmon's inverse spectrum theorem, they proved that more than half of nodal subset could be used to uniquely determine the potential function [8]. In 2008, Hikmet Koyunbakan considered the Sturm-Liouville problem with some special boundary conditions. By using the Xuefeng Yang's method in [9], he studied the potential function's reconstruction and obtained a uniqueness theorem [10]. Chuanfu Yang studied a lot of inverse nodal problems. For example, in 2012, Yang studied the discontinuous inverse nodal problems. Applying the Riemann-Lebesgue lemma and Fatou's lemma, he solved the uniqueness, reconstruction and stability problems [11]. Through similar methods, in 2014, Yang studied inverse nodal problems with a constant delay. The uniqueness theorem was proved. And a constructive procedure was provided to solve the inverse nodal problem [12]. In 2017, Yang considered inverse problems on a graph with cycles. By the dense nodal points, the potential on a graph with multi-cycles can be constructed [13]. Besides, other scholars also studied the inverse nodal problems in recent years. In 2015, the discontinuous Sturm-Liouville operators' inverse nodal problems were discussed by Yuping Wang. By using the Green formula and Riemann-Lebesgue lemma, he established several uniqueness theorems [14]. In 2015, Juan P. Pinasco and Cristian Scarola studied an inverse problem for weighted Sturm-Liouville operators. By using weaker hypotheses for positive weights, the weight and the parameters were determined by a dense nodal set [15]. In 2018, Guangsheng Wei and Yongxia Guo provided the sharp conditions of the uniqueness for inverse nodal Sturm-Liouville problems. Applying a dense nodal subset, the potential and boundary parameters can be uniquely determined [16].

Based on [17], discussing the inverse nodal problems with some nonlocal integral conditions is the main content of this article. We conclude that a dense nodal subset can determine the potential function uniquely. And three constructive procedures for the solution of corresponding problems are proposed. Our main results of this paper are uniqueness theorems (see Theorem 3.4, 4.4, 5.4 below). It makes sense for the theoretical integrity of the inverse nodal problem with nonlocal integral conditions.

This paper's composition is as follows. In Part 2, we introduce preliminaries. In Part 3, 4, 5, we establish several uniqueness theorems for different boundary conditions.

Now let

$$Ly = -y''(x) + q(x)y(x) = \lambda y(x), \quad x \in [0, \pi] \tag{1}$$

with nonlocal integral conditions

$$y(0) = \int_0^\pi y(x)\mu_0(x)\,\mathrm{d}x \tag{2}$$

$$y'(0) = \int_0^\pi y(x)\nu_0(x)\,\mathrm{d}x \tag{3}$$

$$y(\pi) = \int_0^\pi y(x)\mu_\pi(x)\,\mathrm{d}x \tag{4}$$

$$y'(\pi) = \int_0^\pi y(x)\nu_\pi(x)\,\mathrm{d}x \tag{5}$$

where $q(x)$, $\mu_0(x)$, $\mu_\pi(x)$, $v_0(x)$, $v_\pi(x)$ are real-valued, continuously differential functions on $[0,\pi]$.

2. Preliminaries

Firstly, we give two classical theorems, they are useful to proof theorems in this paper.

Lemma 2.1 (Rouche's theorem) For any two complex-valued functions $f(x)$ and $g(x)$,

1) If they are holomorphic inside some region D with closed contour C

2) If $|f(x)|>|g(x)|$ holds on C

Then $f(x)$ and $f(x)+g(x)$ have the same number of zeros inside C, where each zero is counted as many times as its multiplicity.

Lemma 2.2 (Taylor theorem) If the function $f(x)$ exists the *nth* order continuous derivative on $[a,b]$, and the function $f(x)$ exists the $(n+1)th$ order derivative on (a,b), then for any given $x, x_0 \in [a,b]$, at least one point $\xi \in (a,b)$ exists, such that

$$\begin{aligned} f(x) &= f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots \\ &\quad +\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1} \end{aligned} \tag{6}$$

Then, let ϕ and ψ be the solutions of (1), and they satisfy the following conditions:

$$\phi(0,\lambda)=1,\ \phi'(0,\lambda)=0,\ \psi(0,\lambda)=0,\ \psi'(0,\lambda)=1$$

By [5], we have

$$\phi(x,\lambda)=\cos(\rho x)+\frac{\sin(\rho x)}{2\rho}\int_0^x q(t)\,\mathrm{d}t+\frac{\cos(\rho x)}{\rho^2}q_1(x)+o\left(\frac{\mathrm{e}^{|\tau|x}}{|\rho|^3}\right) \tag{7}$$

$$\psi(x,\lambda)=\frac{\sin(\rho x)}{\rho}-\frac{\cos(\rho x)}{2\rho^2}\int_0^x q(t)\,\mathrm{d}t+o\left(\frac{\mathrm{e}^{|\tau|x}}{|\rho|^3}\right) \tag{8}$$

where $\rho=\sqrt{\lambda}=\sigma+i\tau$, $\sigma,\tau\in R$ and

$$q_1(x)=\frac{q(x)-q(0)}{4}-\frac{1}{8}\left(\int_0^x q(t)\,\mathrm{d}t\right)^2 \tag{9}$$

$$Q(x)=\int_0^x q(t)\,\mathrm{d}t \tag{10}$$

Let a nontrivial solution of (1) be

$$y(x,\lambda)=c_1(\lambda)\phi(x,\lambda)+c_2(\lambda)\psi(x,\lambda) \tag{11}$$

In the following, we will discuss the problem (1), (2), (5); (1), (3), (4) and (1), (3), (5) separately.

3. Uniqueness and Reconstruction Problem of (1), (2), (5)

By (2), (5) and (11), we have

$$c_1(\lambda)\left[\phi(0,\lambda)-\int_0^{\pi}\phi(x,\lambda)\mu_0(x)\mathrm{d}x\right]+c_2(\lambda)\left[\psi(0,\lambda)-\int_0^{\pi}\psi(x,\lambda)\mu_0(x)\mathrm{d}x\right]=0 \quad (12)$$

$$c_1(\lambda)\left[\phi'(\pi,\lambda)-\int_0^{\pi}\phi(x,\lambda)v_{\pi}(x)\mathrm{d}x\right]+c_2(\lambda)\left[\psi'(\pi,\lambda)-\int_0^{\pi}\psi(x,\lambda)v_{\pi}(x)\mathrm{d}x\right]=0 \quad (13)$$

So the eigenfunction of (1), (2), (5) is

$$\Delta(\lambda)=\begin{vmatrix}\phi(0,\lambda)-\int_0^{\pi}\phi(x,\lambda)\mu_0(x)\mathrm{d}x & \psi(0,\lambda)-\int_0^{\pi}\psi(x,\lambda)\mu_0(x)\mathrm{d}x \\ \phi'(\pi,\lambda)-\int_0^{\pi}\phi(x,\lambda)v_{\pi}(x)\mathrm{d}x & \psi'(\pi,\lambda)-\int_0^{\pi}\psi(x,\lambda)v_{\pi}(x)\mathrm{d}x\end{vmatrix} \quad (14)$$

Without losing generality, let's suppose

$$c_1(\lambda)=-\psi(0,\lambda)+\int_0^{\pi}\psi(x,\lambda)\mu_0(x)\mathrm{d}x \quad (15)$$

$$c_2(\lambda)=\phi(0,\lambda)-\int_0^{\pi}\phi(x,\lambda)\mu_0(x)\mathrm{d}x \quad (16)$$

Let λ_n is the eigenvalue of the problem, just then

$$y(x,\lambda_n)=c_1(\lambda_n)\phi(x,\lambda_n)+c_2(\lambda_n)\psi(x,\lambda_n) \quad (17)$$

is the corresponding characteristic function. Let x_n^j be nodes of the characteristic function.

Lemma 3.1 As n is large enough, the property of eigenvalues λ_n for (1), (2), (5) is listed:

$$\rho_n=\frac{2n+1}{2}, n\in Z \quad (18)$$

Proof. Using (7), (8) and (14), as $|\rho|\to\infty$, we have

$$\begin{aligned}\Delta(\lambda)=&\frac{\sin(\rho\pi)}{\rho}\left[\frac{1}{2}Q(\pi)-\mu_0(0)\right]+\frac{\cos(\rho\pi)}{\rho^2}\left[\mu_0'(0)-\frac{1}{2}q(\pi)+v_{\pi}(\pi)\right.\\&\left.+\frac{1}{2}Q(\pi)\mu_0(0)\right]+\frac{\cos^2(\rho\pi)}{\rho^2}\left[\frac{1}{2}Q(\pi)\mu_0(\pi)-\mu_0'(\pi)\right]\\&-\frac{1}{\rho^2}\left[\frac{1}{2}\mu_0(\pi)Q(\pi)+v_{\pi}(0)\right]+\cos(\rho\pi)+o\left(\frac{e^{2|\tau|\pi}}{|\rho|^3}\right)\end{aligned} \quad (19)$$

If $\lambda=\rho^2$ is eigenvalue of Sturm-Liouville operator (1), (2), (5), as $|\rho|\to\infty$, we have

$$\cos(\rho\pi)=0 \quad (20)$$

so we obtain (18).

Lemma 3.2 As n is large enough, the characteristic function $y(x,\lambda_n)$ of (1), (2), (5) has $n-1$ nodal points on $[0,\pi]$:

$$x_n^j=\frac{2j\pi}{2n+1}-\frac{4\mu_0(0)}{(2n+1)^2}+\frac{2}{(2n+1)^2}Q(x_n^j)+o\left(\frac{1}{n^3}\right) \quad (21)$$

Proof. From (7) and (8), we can get the following formula:

$$y(x,\lambda_n)=\frac{\sin(\rho_n x)}{\rho_n}+\frac{\cos(\rho_n x)}{\rho_n^2}\left[\mu_0(0)-\frac{1}{2}Q(x)\right]+o\left(\frac{1}{|\rho_n|^3}\right) \quad (22)$$

for large enough *n*, $n \ge N_0$, we obtain that *nth* characteristic function $y(x,\lambda_n)$ has $n-1$ zeros x_n^j on $[0,\pi]$, $(j=1,2,\cdots,n-1)$. By $y(x_n^j,\lambda_n)=0$ we have

$$\tan\left(\rho_n x_n^j\right)=\frac{Q\left(x_n^j\right)}{2\rho_n}-\frac{\mu_0(0)}{\rho_n}+o\left(\frac{1}{|\rho_n|^2}\right) \tag{23}$$

By Lemma 2.2, as $n\to\infty$, we get

$$\rho_n x_n^j = j\pi+\frac{Q\left(x_n^j\right)}{2\rho_n}-\frac{\mu_0(0)}{\rho_n}+o\left(\frac{1}{|\rho_n|^2}\right) \tag{24}$$

which implies

$$x_n^j = \frac{j\pi}{\rho_n}+\frac{Q\left(x_n^j\right)}{2\rho_n^2}-\frac{\mu_0(0)}{\rho_n^2}+o\left(\frac{1}{|\rho_n|^3}\right) \tag{25}$$

Theorem 3.3 For $x\in[0,\pi]$, let $\{x_n^j\}\subset X$, and $\lim\limits_{n\to\infty}x_n^j=x$. Then the following finite limit exists and the corresponding equality holds:

$$\lim_{n\to\infty}\left[\left(\frac{2n+1}{2}\right)^2 x_n^j-\frac{2n+1}{2}j\pi\right]:=f(x) \tag{26}$$

and

$$f(x)=\frac{1}{2}Q(x)-\mu_0(0) \tag{27}$$

Proof. By (21) and $\lim\limits_{n\to\infty}x_n^j=x$. For any integer *n*, we have

$$\begin{aligned}&\lim_{n\to\infty}\left[\left(\frac{2n+1}{2}\right)^2 x_n^j-\frac{2n+1}{2}j\pi\right]\\&=\lim_{n\to\infty}\left[\frac{1}{2}Q\left(x_n^j\right)-\mu_0(0)\right]=\frac{1}{2}Q(x)-\mu_0(0):=f(x)\end{aligned} \tag{28}$$

Theorem 3.4 Let $X_0\subset X$ be a dense subset in $(0,\pi)$, then X_0 can uniquely determine the potential function $q(x)$. And we have the algorithm below:

1) For a subset X_0, and for each $x\in[0,\pi]$, $\{x_n^j\}\subset X_0$, then $\lim\limits_{n\to\infty}x_n^j=x$.

2) By (28) we get

$$q(x)=2f'(x) \tag{29}$$

Proof. For a given X_0, The conclusion is obvious.

Next, let us continue to consider the inverse nodal problems for (1), (3), (4).

4. Uniqueness and Reconstruction Problem of (1), (3), (4)

In similar approach to Part 3, we get the eigenfunction of (1), (3), (4) is

$$\Delta(\lambda)=\begin{vmatrix}\phi'(0,\lambda)-\int_0^\pi\phi(x,\lambda)\nu_0(x)\mathrm{d}x & \psi'(0,\lambda)-\int_0^\pi\psi(x,\lambda)\nu_0(x)\mathrm{d}x\\ \phi(\pi,\lambda)-\int_0^\pi\phi(x,\lambda)\mu_\pi(x)\mathrm{d}x & \psi(\pi,\lambda)-\int_0^\pi\psi(x,\lambda)\mu_\pi(x)\mathrm{d}x\end{vmatrix} \tag{30}$$

Now, let's suppose

$$c_1(\lambda) = -\frac{1}{\rho}\psi'(0,\lambda) + \frac{1}{\rho}\int_0^{\pi}\psi(x,\lambda)v_0(x)\mathrm{d}x \tag{31}$$

$$c_2(\lambda) = \frac{1}{\rho}\phi'(0,\lambda) - \frac{1}{\rho}\int_0^{\pi}\phi(x,\lambda)v_0(x)\mathrm{d}x \tag{32}$$

Using (7), (8) and (30), we have

$$\begin{aligned}\Delta(\lambda) = {} & \frac{\cos(\rho\pi)}{\rho^2}\left[\mu'_{\pi}(\pi) + v_0(0) - q_1(\pi) - \frac{1}{2}Q(\pi)\mu_{\pi}(\pi)\right] \\ & + \frac{\sin(\rho\pi)}{\rho}\left[\mu_{\pi}(\pi) - \frac{1}{2}Q(\pi)\right] - \frac{1}{\rho^2}\left[\mu'_{\pi}(0) + v_0(\pi)\right] \\ & - \cos(\rho\pi) + o\left(\frac{\mathrm{e}^{2|\tau|\pi}}{|\rho|^3}\right)\end{aligned} \tag{33}$$

For the characteristic function (17), as $n \to \infty$, we get

$$y(x,\lambda_n) = -\frac{\cos(\rho_n x)}{\rho_n} - \frac{\sin(\rho_n x)}{2\rho_n^2}Q(x) + o\left(\frac{1}{|\rho_n|^3}\right) \tag{34}$$

Then, we can get some conclusions about the problem (1), (3), (4), we will list them without proof.

Lemma 4.1 As n is large enough, the property of eigenvalues λ_n for (1), (3), (4) is the same as (18).

Lemma 4.2 As n is large enough, the characteristic function $y(x,\lambda_n)$ of (1), (3), (4) has $n-1$ nodal points in $[0,\pi]$:

$$x_n^j = \frac{\pi}{2n+1} + \frac{2j\pi}{2n+1} - \frac{2}{Q(x)} + o\left(\frac{1}{n^3}\right) \tag{35}$$

Theorem 4.3 For $x \in [0,\pi]$, let $\{x_n^j\} \subset X$ and $\lim_{n\to\infty} x_n^j = x$. Then the following finite limit exists and the corresponding equality holds:

$$\lim_{n\to\infty}\left[(2n+1)Q(x)x_n^j - 2j\pi Q(x) + 2(2n+1)\right] := f(x) \tag{36}$$

and

$$f(x) = \pi Q(x) \tag{37}$$

Theorem 4.4 Let $X_0 \subset X$ be a dense subset in $(0,\pi)$, then X_0 can uniquely determine the potential function $q(x)$. And we have the algorithm below:

1) For a subset X_0, and for each $x \in [0,\pi]$, $\{x_n^j\} \subset X_0$, then $\lim_{n\to\infty} x_n^j = x$.

2) By (37) we get

$$q(x) = \frac{1}{\pi}f'(x) \tag{38}$$

5. Uniqueness and Reconstruction Problem of (1), (3), (5)

Similarly, the eigenfunction of (1), (3), (5) is

$$\Delta(\lambda)=\begin{vmatrix}\phi'(0,\lambda)-\int_0^\pi\phi(x,\lambda)v_0(x)\mathrm{d}x & \psi'(0,\lambda)-\int_0^\pi\psi(x,\lambda)v_0(x)\mathrm{d}x\\ \phi'(\pi,\lambda)-\int_0^\pi\phi(x,\lambda)v_\pi(x)\mathrm{d}x & \psi'(\pi,\lambda)-\int_0^\pi\psi(x,\lambda)v_\pi(x)\mathrm{d}x\end{vmatrix} \tag{39}$$

Without losing generality, we still assume (31) and (32) hold in this case.

Lemma 5.1 As n is large enough, the property of eigenvalues λ_n for (1), (3), (5) is listed:

$$\begin{aligned}\rho_n=n&+\frac{1}{n\pi}\frac{(-1)^n\left[\frac{1}{2}v_0(\pi)Q(\pi)+v_\pi'(0)\right]-(-1)^n\cos^2(\varepsilon_n\pi)\left[\frac{1}{2}v_0(\pi)Q(\pi)-v_0'(\pi)\right]}{q_1(\pi)-\frac{1}{2}q(\pi)-v_0(0)+v_\pi(\pi)}\\&-\frac{1}{n\pi}\frac{\cos(\varepsilon_n\pi)\left[\frac{1}{2}v_0(0)Q(\pi)+v_\pi'(\pi)-\frac{1}{2}v_\pi(\pi)Q(\pi)-q_1'(\pi)+v_0'(0)\right]}{q_1(\pi)-\frac{1}{2}q(\pi)-v_0(0)+v_\pi(\pi)}\\&+o\left(\frac{1}{n^2}\right),n\in Z\end{aligned} \tag{40}$$

where $\rho\sin(\rho\pi)=\frac{1}{2}\cos(\rho\pi)Q(\pi)$.

Proof. Using (7), (8) and (39), as $|\rho|\to\infty$, we have

$$\begin{aligned}\Delta(\lambda)=&\frac{\sin(\rho\pi)}{\rho}\left[q_1(\pi)-\frac{1}{2}q(\pi)-v_0(0)+v_\pi(\pi)\right]\\&+\frac{\cos(\rho\pi)}{\rho^2}\left[\frac{1}{2}Q(\pi)v_0(0)+v_\pi'(\pi)-\frac{1}{2}Q(\pi)v_\pi(\pi)-q_1'(\pi)+v_0'(0)\right]\\&+\frac{\cos^2(\rho\pi)}{\rho^2}\left[\frac{1}{2}v_0(\pi)Q(\pi)-v_0'(\pi)\right]\\&-\frac{1}{\rho^2}\left[\frac{1}{2}v_0(\pi)Q(\pi)+v_\pi'(0)\right]+o\left(\frac{\mathrm{e}^{2|\tau|\pi}}{|\rho|^3}\right)\end{aligned} \tag{41}$$

Let

$$\rho^2\Delta(\lambda)=h_1(\lambda)+h_2(\lambda) \tag{42}$$

where

$$h_1(\lambda)=\rho\sin(\rho\pi)\left[q_1(\pi)-\frac{1}{2}q(\pi)-v_0(0)+v_\pi(\pi)\right] \tag{43}$$

$$\begin{aligned}h_2(\lambda)=&\cos(\rho\pi)\left[\frac{1}{2}v_0(0)Q(\pi)+v_\pi'(\pi)-\frac{1}{2}v_\pi(\pi)Q(\pi)-q_1'(\pi)+v_0'(0)\right]\\&+\cos^2(\rho\pi)\left[\frac{1}{2}Q(\pi)v_0(\pi)-v_0'(\pi)\right]\\&-\left[\frac{1}{2}Q(\pi)v_0(\pi)+v_\pi'(0)\right]+o\left(\frac{\mathrm{e}^{2|\tau|\pi}}{|\rho|}\right)\end{aligned} \tag{44}$$

In order to accurately estimate eigenvalues λ_n, for given region $r_n(\delta)=\{\rho:|\rho-n|=\delta\}$, as radius δ is small enough and $\delta>0$, we get

$$|h_1(\lambda)|>|h_2(\lambda)| \tag{45}$$

So by Lemma 2.1, as n is large enough, we have that $\Delta(\lambda)$ has one zero inside $r_n(\delta)$, denote it ρ_n. $\delta > 0$ is arbitrary, so

$$\rho_n = n + \varepsilon_n, \quad \varepsilon_n = o(1) \quad \text{as} \quad n \to \infty \tag{46}$$

Substituting (46) into (41), we get

$$\begin{aligned} 0 = \Delta(\lambda_n) &= (n+\varepsilon_n)\sin\left[(n+\varepsilon_n)\pi\right]\left[q_1(\pi) - \frac{1}{2}q(\pi) - v_0(0) + v_\pi(\pi)\right] \\ &+ \cos\left[(n+\varepsilon_n)\pi\right]\left[\frac{1}{2}Q(\pi)v_0(0) + v_\pi'(\pi) - \frac{1}{2}Q(\pi)v_\pi(\pi) - q_1'(\pi) + v_0'(0)\right] \\ &+ \cos^2\left[(n+\varepsilon_n)\pi\right]\left[\frac{1}{2}Q(\pi)v_0(\pi) - v_0'(\pi)\right] \\ &- \left[\frac{1}{2}Q(\pi)v_0(\pi) + v_\pi'(0)\right] + o\left(\frac{1}{n}\right) \end{aligned} \tag{47}$$

$$\begin{aligned} 0 = n\sin(\varepsilon_n\pi)&\left[q_1(\pi) - \frac{1}{2}q(\pi) - v_0(0) + v_\pi(\pi)\right] \\ &+ \cos(\varepsilon_n\pi)\left[\frac{1}{2}v_0(0)Q(\pi) + v_\pi'(\pi) - \frac{1}{2}v_\pi(\pi)Q(\pi) - q_1'(\pi) + v_0'(0)\right] \\ &+ (-1)^n\cos^2(\varepsilon_n\pi)\left[\frac{1}{2}v_0(\pi)Q(\pi) - v_0'(\pi)\right] \\ &- (-1)^n\left[\frac{1}{2}v_0(\pi)Q(\pi) + v_\pi'(0)\right] + o(1) \end{aligned} \tag{48}$$

Using (48), we obtain

$$\begin{aligned} \varepsilon_n = \frac{1}{n\pi}&\frac{(-1)^n\left[\frac{1}{2}v_0(\pi)Q(\pi) + v_\pi'(0)\right] - (-1)^n\cos^2(\varepsilon_n\pi)\left[\frac{1}{2}v_0(\pi)Q(\pi) - v_0'(\pi)\right]}{q_1(\pi) - \frac{1}{2}q(\pi) - v_0(0) + v_\pi(\pi)} \\ &- \frac{1}{n\pi}\frac{\cos(\varepsilon_n\pi)\left[\frac{1}{2}v_0(0)Q(\pi) + v_\pi'(\pi) - \frac{1}{2}v_\pi(\pi)Q(\pi) - q_1'(\pi) + v_0'(0)\right]}{q_1(\pi) - \frac{1}{2}q(\pi) - v_0(0) + v_\pi(\pi)} \\ &+ o\left(\frac{1}{n^2}\right), n \in Z \end{aligned} \tag{49}$$

It can be seen from (46) and (49), the Lemma holds.

Lemma 5.2 As n is large enough, the characteristic function $y(x,\lambda_n)$ of (1), (3), (5) has $n-1$ nodal points in $[0,\pi]$:

$$\begin{aligned} x_n^j = &\frac{\pi}{2n} + \frac{j\pi}{n} - \frac{2}{Q(x)} \\ &- \frac{j}{n^3}\left\{\frac{(-1)^n\left[\frac{1}{2}v_0(\pi)Q(\pi) + v_\pi'(0)\right] - (-1)^n\cos^2(\varepsilon_n\pi)\left[\frac{1}{2}v_0(\pi)Q(\pi) - v_0'(\pi)\right]}{q_1(\pi) - \frac{1}{2}q(\pi) - v_0(0) + v_\pi(\pi)}\right\} \\ &+ \frac{j}{n^3}\left\{\frac{\cos(\varepsilon_n\pi)\left[\frac{1}{2}v_0(0)Q(\pi) + v_\pi'(\pi) - \frac{1}{2}v_\pi(\pi)Q(\pi) - q_1'(\pi) + v_0'(0)\right]}{q_1(\pi) - \frac{1}{2}q(\pi) - v_0(0) + v_\pi(\pi)}\right\} \\ &+ o\left(\frac{1}{n^3}\right), n \in Z \end{aligned} \tag{50}$$

Proof. From (7) and (8), as $n\to\infty$, we have

$$y(x,\lambda_n)=-\frac{\cos(\rho_n x)}{\rho_n}-\frac{\sin(\rho_n x)}{2\rho_n^2}Q(x)+o\left(\frac{1}{|\rho_n|^3}\right) \tag{51}$$

Similar to Part 3 and Part 4, as n is large enough and $n\geq N_0$, we have that nth characteristic function $y(x,\lambda_n)$ has $n-1$ zeros x_n^j in $[0,\pi]$. From $y(x_n^j,\lambda_n)=0$, we have

$$\tan(\rho_n x_n^j)=-\frac{2\rho_n}{Q(x_n^j)}+o\left(\frac{1}{|\rho_n|^2}\right) \tag{52}$$

Applying Lemma 2.2, we have

$$x_n^j=\frac{\pi}{2\rho_n}+\frac{j\pi}{\rho_n}-\frac{2}{Q(x_n^j)}+o\left(\frac{1}{|\rho_n|^3}\right) \tag{53}$$

From (40), we can get

$$\rho_n^{-1}=\frac{1}{n}-\frac{C_\mu}{n^3}+o\left(\frac{1}{n^4}\right),\quad \rho_n^{-2}=\frac{1}{n^2}-\frac{2C_\mu}{n^4}+o\left(\frac{1}{n^5}\right) \tag{54}$$

where

$$\begin{aligned}C_\mu=&\frac{1}{\pi}\left\{\frac{(-1)^n\left[\frac{1}{2}\nu_0(\pi)Q(\pi)+\nu_\pi'(0)\right]-(-1)^n\cos^2(\varepsilon_n\pi)\left[\frac{1}{2}\nu_0(\pi)Q(\pi)-\nu_0'(\pi)\right]}{q_1(\pi)-\frac{1}{2}q(\pi)-\nu_0(0)+\nu_\pi(\pi)}\right\}\\&-\frac{1}{\pi}\left\{\frac{\cos(\varepsilon_n\pi)\left[\frac{1}{2}\nu_0(0)Q(\pi)+\nu_\pi'(\pi)-\frac{1}{2}\nu_\pi(\pi)Q(\pi)-q_1'(\pi)+\nu_0'(0)\right]}{q_1(\pi)-\frac{1}{2}q(\pi)-\nu_0(0)+\nu_\pi(\pi)}\right\}\end{aligned} \tag{55}$$

Thus we can obtain (50).

Theorem 5.3 For $x\in[0,\pi]$, let $\{x_n^j\}\subset X$, and $\lim\limits_{n\to\infty}x_n^j=x$. Then the following finite limit exists and the corresponding equality holds:

$$\lim_{n\to\infty}\left[n^2Q(x)x_n^j-\frac{1}{2}j\pi nQ(x)-nj\pi Q(x)+2n^2\right]:=f(x) \tag{56}$$

and

$$\begin{aligned}f(x)=&\frac{x}{\pi}Q(x)\left\{\frac{\cos(\varepsilon_n\pi)\left[\frac{1}{2}\nu_0(0)Q(\pi)+\nu_\pi'(\pi)-\frac{1}{2}\nu_\pi(\pi)Q(\pi)-q_1'(\pi)+\nu_0'(0)\right]}{q_1(\pi)-\frac{1}{2}q(\pi)-\nu_0(0)+\nu_\pi(\pi)}\right\}\\&-\frac{x}{\pi}Q(x)\left\{-\frac{(-1)^n\left[\frac{1}{2}\nu_0(\pi)Q(\pi)+\nu_\pi'(0)\right]-(-1)^n\cos^2(\varepsilon_n\pi)\left[\frac{1}{2}\nu_0(\pi)Q(\pi)-\nu_0'(\pi)\right]}{q_1(\pi)-\frac{1}{2}q(\pi)-\nu_0(0)+\nu_\pi(\pi)}\right\}\end{aligned} \tag{57}$$

Proof. Using (50) and $\lim\limits_{n\to\infty}x_n^j=x$, we have

$$
\begin{aligned}
&\lim_{n\to\infty}\left[n^2Q(x)x_n^j-\frac{1}{2}j\pi nQ(x)-nj\pi Q(x)+2n^2\right]\\
&=\lim_{n\to\infty}\left\{\frac{x}{\pi}Q\left(x_n^j\right)\frac{\cos(\varepsilon_n\pi)\left[\frac{1}{2}v_0(0)Q(\pi)+v_\pi'(\pi)-\frac{1}{2}v_\pi(\pi)Q(\pi)-q_1'(\pi)+v_0'(0)\right]}{q_1(\pi)-\frac{1}{2}q(\pi)-v_0(0)+v_\pi(\pi)}\right.\\
&\left.\quad-\frac{x}{\pi}Q\left(x_n^j\right)\frac{(-1)^n\left[\frac{1}{2}v_0(\pi)Q(\pi)+v_\pi'(0)\right]-(-1)^n\cos^2(\varepsilon_n\pi)\left[\frac{1}{2}v_0(\pi)Q(\pi)-v_0'(\pi)\right]}{q_1(\pi)-\frac{1}{2}q(\pi)-v_0(0)+v_\pi(\pi)}\right\}\\
&=\frac{x}{\pi}Q(x)\left\{\frac{\cos(\varepsilon_n\pi)\left[\frac{1}{2}v_0(0)Q(\pi)+v_\pi'(\pi)-\frac{1}{2}v_\pi(\pi)Q(\pi)-q_1'(\pi)+v_0'(0)\right]}{q_1(\pi)-\frac{1}{2}q(\pi)-v_0(0)+v_\pi(\pi)}\right\}\\
&\quad-\frac{x}{\pi}Q(x)\left\{\frac{(-1)^n\left[\frac{1}{2}v_0(\pi)Q(\pi)+v_\pi'(0)\right]-(-1)^n\cos^2(\varepsilon_n\pi)\left[\frac{1}{2}v_0(\pi)Q(\pi)-v_0'(\pi)\right]}{q_1(\pi)-\frac{1}{2}q(\pi)-v_0(0)+v_\pi(\pi)}\right\}\\
&=f(x)
\end{aligned}
$$

So we obtain (57).

Theorem 5.4 Let $X_0\subset X$ be a dense subset in $(0,\pi)$, then X_0 can uniquely determine the potential function $q(x)$. And we have the algorithm below:

1) For a subset X_0, and for each $x\in[0,\pi]$, $\{x_n^j\}\subset X_0$, then $\lim_{n\to\infty}x_n^j=x$.

2) By (57) we get

$$
\frac{2q(x)+xq'(x)}{[q]}=\frac{f''(x)}{f(\pi)} \tag{58}
$$

where $[q]=\pi Q(\pi)$.

Proof. For a given X_0, by (57), we have

$$
\begin{aligned}
f''(x)=\left[\frac{2}{\pi}q(x)+\frac{x}{\pi}q'(x)\right]&\left\{\frac{\cos(\varepsilon_n\pi)\left[\frac{1}{2}v_0(0)Q(\pi)+v_\pi'(\pi)-\frac{1}{2}v_\pi(\pi)Q(\pi)-q_1'(\pi)+v_0'(0)\right]}{q_1(\pi)-\frac{1}{2}q(\pi)-v_0(0)+v_\pi(\pi)}\right.\\
&\left.-\frac{(-1)^n\left[\frac{1}{2}v_0(\pi)Q(\pi)+v_\pi'(0)\right]-(-1)^n\cos^2(\varepsilon_n\pi)\left[\frac{1}{2}v_0(\pi)Q(\pi)-v_0'(\pi)\right]}{q_1(\pi)-\frac{1}{2}q(\pi)-v_0(0)+v_\pi(\pi)}\right\}
\end{aligned} \tag{59}
$$

So we have (58).

6. Conclusion

In this work, we study three inverse nodal problems for the Sturm-Liouville operator with different nonlocal integral conditions. We get the uniqueness theorems. And we present some constructive procedures to solve the inverse nodal problems. It makes sense for the theoretical integrity of the inverse nodal problem with nonlocal integral conditions.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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