Journal of Applied Mathematics and Physics, 2018, 6, 2589-2599
http://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Extremal Problems Related to Dual Gauss-John Position

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How to cite this paper: Ma, T.Y. (2018) Extremal Problems Related to Dual Gauss-John Position. *Journal of Applied Mathematics and Physics*, **6**, 2589-2599.
https://doi.org/10.4236/jamp.2018.612216

Received: December 4, 2018
Accepted: December 23, 2018
Published: December 26, 2018

Abstract

In this paper, the extremal problem, $\min\{\tilde{l}_p(\phi K): o \in \phi K \subseteq L, \phi \in \mathrm{GL}(n)\}$, of two convex bodies K and L in $\mathbb{R}^n$ is considered. For K to be in extremal position in terms of a decomposition of the identity, give necessary conditions together with the optimization theorem of John. Besides, we also consider the weaker optimization problem: $\min\{(\tilde{l}_p(\phi K))^p : \phi K \subseteq B_2^n, \phi K \cap S^{n-1} \neq \varnothing, \phi \in \mathrm{GL}(n)\}$. As an application, we give the geometric distance between the unit ball B_2^n and a centrally symmetric convex body K.

Keywords

Dual Gauss-John Position, Optimization Theorem of John, Dual $\tilde{l}_p$-Norm, Contact Pair

1. Introduction

Let γ_n be the classical Gaussian probability measure with density $\frac{1}{(\sqrt{2\pi})^n} e^{-\frac{|x|^2}{2}}$, and $\|\cdot\|_K$ is the Minkowski functional of a convex body $K \subset \mathbb{R}^n$. An important quantity on local theory of Banach space is the associated l-norm:

$$l(K) = \int_{\mathbb{R}^n} \| x \|_K \mathrm{d}\gamma_n(x).$$

The minimum of the functional

$$\int_{\mathbb{R}^n} \|x\|_{\phi K} \mathrm{d}\gamma_n(x)$$

under the constraint $\phi K \subseteq B_2^n$ is attained for $\phi = I_n$, then a convex body K is in the Gauss-John position, where $\phi \in \mathrm{GL}(n)$, B_2^n is the Euclidean unit ball and I_n is the identity mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$.

For $x \in \mathbb{R}^n \setminus \{o\}$, the map $x \otimes x : \mathbb{R}^n \to \mathbb{R}^n$ is the rank 1 linear operator $y \mapsto \langle x, y \rangle x$.

Giannopoulos et al. in [1] showed that if K is in the Gauss-John position, then there exist $m \le n(n+1)/2$ contact points $x_1, x_2, \cdots, x_m \in \partial K \cap S^{n-1}$, and constants $c_1, c_2, \cdots, c_m > 0$ such that $\sum_{i=1}^{m} c_i = 1$ and

$$\int_{\mathbb{R}^n} (x \otimes x - I_n) \| x \|_K \mathrm{d}\gamma_n(x) = \int_{\mathbb{R}^n} \| x \|_K \mathrm{d}\gamma_n(x) \left(\sum_{i=1}^{m} c_i x_i \otimes x_i \right).$$

Note that the Gauss-John position is not equivalent to the classical John position. Giannopoulos et al. [1] pointed out that, when K is in the Gauss-John position, the distance between the unit ball B_2^n and the John ellipsoid is of order $\sqrt{n / \log n}$.

Notice that the study of the classical John theorem went back to John [2]. It states that each convex body K contains a unique ellipsoid of maximal volume, and when B_2^n is the maximal ellipsoid in K, it can be characterized by points of contact between the boundary of K and that of B_2^n. John's theorem also holds for arbitrary centrally symmetric convex bodies, which was proved by Lewis [3] and Milman [4]. It was provided in [5] that a generalization of John's theorem for the maximal volume position of two arbitrary smooth convex bodies. Bastero and Romance [6] proved another version of John's representation removing smoothness condition but with assumptions of connectedness. For more information about the study of its extensions and applications, please see [7]-[13].

Recall that a convex body $\tilde{K}$ is a position of K if $\tilde{K} = \phi K + a$, for some non-degenerate linear mapping $\phi \in \mathrm{GL}(n)$ and some $a \in \mathbb{R}^n$. We say that K is in a position of maximal volume in L if $K \subseteq L$ and for any position $\tilde{K}$ of K such that $\tilde{K} \subseteq L$ we have $\mathrm{vol}_n(\tilde{K}) \le \mathrm{vol}_n(K)$, where $\mathrm{vol}_n(\cdot)$ denotes the volume of appropriate dimension.

Recently, Li and Leng in [14] generalized the Gauss-John position to a general situation. For $p \ge 1$, denote l_p-norm by

$$l_p(K) = \left(\int_{\mathbb{R}^n} \| x \|_K^p \, \mathrm{d}\gamma_n(x) \right)^{\frac{1}{p}}. \tag{1.1}$$

They consider the following extremal problem:

$$\min \left\{ l_p(\phi K) : o \in \phi K \subseteq L, \phi \in \mathrm{GL}(n) \right\}, \tag{1.2}$$

where L is a given convex body in $\mathbb{R}^n$ and K is a convex body containing the origin o such that $o \in K \subseteq L$.

Li and Leng [14] showed that let L be a given convex body in $\mathbb{R}^n$ and K be a convex body such that $o \in K \subseteq L$. If K is in extremal position of (1.2), then there exist $m \le n^2$ contact pairs $(x_i, y_i)_{1 \le i \le m}$ of (K, L), and constants $c_1, c_2, \cdots, \cdots, c_m > 0$ such that

$$I_n = \int_{\mathbb{R}^n} (x \otimes x) \mathrm{d}\mu(x) - p \sum_{i=1}^{m} c_i x_i \otimes y_i, \quad \sum_{i=1}^{m} c_i = 1,$$

where $\mathrm{d}\mu(x)$ is the probability measure on $\mathbb{R}^n$ with normalized density

$$\mathrm{d}\mu(x)=\|x\|_K^p \,\mathrm{d}\gamma_n(x)/(l_p(K))^p.$$

In this paper, we first present a dual concept of l_p-norm $l_p(K)$. The generalizations of John's theorem and Li and Leng [14] play a critical role. It would be impossible to overstate our reliance on their work.

For $p\ge 1$, we define the dual $\tilde{l}_p$-norm of convex body K by

$$\tilde{l}_p(K)=\left(\int_{\mathbb{R}^n}\rho_K(x)^p\,\mathrm{d}\gamma_n(x)\right)^{\frac{1}{p}}, \tag{1.3}$$

where ρ_K is the radial function of the star body K about the origin.

Now, we consider the extremal problem:

$$\min\left\{\tilde{l}_p(\phi K): o\in\phi K\subseteq L, \phi\in \mathrm{GL}(n)\right\}, \tag{1.4}$$

where L is a given convex body in $\mathbb{R}^n$ and K is a convex body containing the origin o such that $o\in K\subseteq L$.

Then we prove that the necessary conditions for K to be in extremal position in terms of a decomposition of the identity.

Theorem 1.1. Let L be a given convex body in $\mathbb{R}^n$ and K be a convex body such that $o\in K\subseteq L$. If K is in extremal position of (1.4), then there exist $m\le n^2$ contact pairs $(x_i,y_i)_{1\le i\le m}$ of (K,L), and $c_1,c_2,\cdots,c_m>0$ such that

$$I_n=\int_{\mathbb{R}^n}(x\otimes x)\mathrm{d}\tilde{\mu}(x)-p\sum_{i=1}^m c_i x_i\otimes y_i,\quad \sum_{i=1}^m c_i=1,$$

where $\mathrm{d}\tilde{\mu}(x)$ is the probability measure on $\mathbb{R}^n$ with normalized density

$$\mathrm{d}\tilde{\mu}(x)=\|x\|_K^{-p}\,\mathrm{d}\gamma_n(x)/(\tilde{l}_p(K))^p.$$

Next the following result is obtained, which is an restriction that is weaker than the extremal problem (1.4):

$$\min\left\{\left(\tilde{l}_p(\phi K)\right)^p:\phi K\subseteq B_2^n,\phi K\cap S^{n-1}\ne\varnothing,\phi\in\mathrm{GL}(n)\right\}. \tag{1.5}$$

Theorem 1.2. Let K be a given convex body in $\mathbb{R}^n$. If I_n is the solution of the extremal problem (1.5), then there exist contact points u,u' of K and B_2^n such that

$$\langle u',\theta\rangle^2\le\left(\tilde{l}_p(K)\right)^p\int_{\mathbb{R}^n}\|x\|_K^{-p-1}\left\langle\nabla h_{K^o}(x),\theta\right\rangle\left\langle x,\theta\right\rangle d\gamma_n(x)\le\left\langle u,\theta\right\rangle^2, \tag{1.6}$$

for every $\theta\in S^{n-1}$.

The rest of this paper is organized as follows: In Section 2, some basic notation and preliminaries are provided. We prove Theorem 1.1 and Theorem 1.2 in Section 3. In particular, as an application of the extremal problem of

$$\min\left\{\left(\tilde{l}_p(\phi K)\right)^p: o\in\phi K\subseteq B_2^n,\phi\in\mathrm{GL}(n)\right\}, \tag{1.7}$$

Section 3 shows the geometric distance between the unit ball B_2^n and a centrally symmetric convex body K.

2. Notation and Preliminaries

In this section, we present some basic concepts and various facts that are needed in our investigations. We shall work in $\mathbb{R}^n$ equipped with the canonical Euclidean scalar product $\langle \cdot,\cdot \rangle$ and write $|\cdot|$ for the corresponding Euclidean norm. We denote the unit sphere by S^{n-1}.

Let K be a convex body (compact, convex sets with non-empty interiors) in $\mathbb{R}^n$. The support function of K is defined by

$$h_K(x) = \max\{\langle x, y\rangle : y \in K\}, \quad x \in \mathbb{R}^n.$$

Obviously, $h_{\phi K}(x) = h_K(\phi^t x)$ for $\phi \in \mathrm{GL}(n)$, where ϕ^t denotes the transpose of ϕ.

A set $K \subset \mathbb{R}^n$ is said to be a star body about the origin, if the line segment from the origin to any point $x \in K$ is contained in K and K has continuous and positive radial function $\rho_K(\cdot)$. Here, the radial function of K, $\rho_K : S^{n-1} \to [0,\infty)$, is defined by

$$\rho_K(u) = \max\{\lambda : \lambda u \in K\}.$$

Note that if K be a star body (about the origin) in $\mathbb{R}^n$, then K can be uniquely determined by its radial function $\rho_K(\cdot)$ and vice verse. If $\alpha > 0$, we have

$$\rho_K(\alpha x) = \alpha^{-1}\rho_K(x) \quad \text{and} \quad \rho_{\alpha K}(x) = \alpha\rho_K(x).$$

More generally, from the definition of the radial function it follows immediately that for $\phi \in \mathrm{GL}(n)$ the radial function of the image $\phi K = \{\phi y : y \in K\}$ of star body K is given by $\rho_{\phi K}(x) = \rho_K(\phi^{-1}x)$, for all $x \in \mathbb{R}^n$.

If $K, L \in \mathcal{S}_o^n$ and $\lambda, \mu \ge 0$ (not both zero), then for $p > 0$, the L_p-radial combination, $\lambda K \tilde{+}_p \mu L \in \mathcal{S}_o^n$, is defined by (see [15])

$$\rho(\lambda K \tilde{+}_p \mu L, \cdot)^p = \lambda\rho(K,\cdot)^p + \mu\rho(L,\cdot)^p. \tag{2.1}$$

If a star body K contains the origin o as its interior point, then the Minkowski functional $\|\cdot\|_K$ of K is defined by

$$\| x \|_K = \min\{\lambda > 0 : x \in \lambda K\}.$$

In this case,

$$\| x \|_K = \rho_K^{-1}(x) = h_{K^\circ}(x),$$

where K° denotes the polar set of K, which is defined by

$$K^\circ = \{x \in \mathbb{R}^n : \langle x, y\rangle \le 1 \text{ for all } y \in K\}.$$

It is easy to verify that for $\phi \in \mathrm{GL}(n)$,

$$(\phi K)^\circ = \phi^{-t} K^\circ,$$

where ϕ^{-t} denotes the reverse of the transpose of ϕ. Obviously, $(K^\circ)^\circ = K$ (see [13] for details).

Let K and L be two convex bodies in $\mathbb{R}^n$. According to [4], if $o \in K \subseteq L \subseteq \mathbb{R}^n$, we call a pair $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ a contact pair for (K, L) if it satisfies:

1) $x \in K \cap \partial L$,
2) $y \in L^{\circ} \cap \partial K^{\circ}$,
3) $\langle x, y \rangle = 1$.

If $x, y \in \mathbb{R}^n$, we denote by $x \otimes y$ the rank one projection defined by $x \otimes y(u) = \langle x, u \rangle y$ for all $u \in \mathbb{R}^n$.

The geometric distance $\delta_G(K, L)$ of the convex bodies K and L is defined by

$$\delta_G(K, L) = \inf\{\alpha\beta : \alpha > 0, \beta > 0, (1/\beta)L \subset K \subset \alpha L\}.$$

3. Proof of Main Results

First, we prove that $\tilde{l}_p(\cdot)$ is a norm with respect to L_p-radial combination in $\mathcal{S}_o^n$. Apparently, $\tilde{l}_p(K) \ge 0$ and $\tilde{l}_p(K) = 0$ if and only if $K = \{o\}$. At the same time, $\tilde{l}_p(cK) = c\tilde{l}_p(K)$ if real constant $c > 0$. In addition, it is follows that

$$\tilde{l}_p(K \tilde{+}_p L) \le \tilde{l}_p(K) + \tilde{l}_p(L).$$

Indeed, we have

$$\begin{aligned}
\tilde{l}_p(K \tilde{+}_p L) &= \left(\int_{\mathbb{R}^n} \rho^p_{K \tilde{+}_p L}(x) \mathrm{d}\gamma_n(x)\right)^{\frac{1}{p}} \\
&= \left(\int_{\mathbb{R}^n} \rho^p_K(x) \mathrm{d}\gamma_n(x) + \int_{\mathbb{R}^n} \rho^p_L(x) \mathrm{d}\gamma_n(x)\right)^{\frac{1}{p}} \\
&\le \left(\int_{\mathbb{R}^n} \rho^p_K(x) \mathrm{d}\gamma_n(x)\right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}^n} \rho^p_L(x) \mathrm{d}\gamma_n(x)\right)^{\frac{1}{p}} \\
&= \tilde{l}_p(K) + \tilde{l}_p(L).
\end{aligned}$$

Therefore, $\tilde{l}_p(\cdot)$ is a norm with respect to L_p-radial combination and $\mathcal{S}_o^n$ is normed space for $\tilde{l}_p(\cdot)$.

Now, we prove the optimization theorem of John [2] (see [10] also).

Lemma 3.1. Let $\mathcal{F} : \mathbb{R}^N \to \mathbb{R}$ be a C^1-function. Let S be a compact metric space and $\mathcal{G} : \mathbb{R}^N \times S \to \mathbb{R}$ be continuous. Suppose that for every $s \in S$, $\nabla_z \mathcal{G}(z, s)$ exists and is continuous on $\mathbb{R}^N \times S$.

Let $\mathcal{A} = \{z \in \mathbb{R}^N : \mathcal{G}(z, s) \ge 0, \text{ for all } s \in S\}$ and $z_0 \in \mathcal{A}$ satisfy

$$\mathcal{F}(z_0) = \min_{z \in \mathcal{A}} \mathcal{F}(z).$$

Then, either $\nabla_z \mathcal{F}(z_0) = 0$, or, for some $1 \le m \le N$, there exist $s_1, s_2, \cdots, s_m \in S$ and $\lambda_1, \lambda_2, \cdots, \lambda_m \in \mathbb{R}$ such that $\mathcal{G}(z_0, s_i) = 0, \lambda_i \ge 0$ for $1 \le i \le m$, and

$$\nabla_z \mathcal{F}(z_0) = \sum_{i=1}^m \lambda_i \nabla_z \mathcal{G}(z_0, s_i).$$

Using a similar argument as that in [1], we give the proof of Theorem 1.1.

Proof of Theorem 1.1. For $N = n^2$, we define $\mathcal{F} : \mathbb{R}^N \to \mathbb{R}$ by

$$\mathcal{F}(\phi) = \tilde{l}_p(\phi K) = \left(\int_{\mathbb{R}^n} \| \phi^{-1} x \|_K^{-p} \, \mathrm{d}\gamma_n(x)\right)^{\frac{1}{p}}, \tag{3.1}$$

where $\phi \in \mathbb{R}^N$ is the linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$. Clearly $\mathcal{F}$ is C^1. For $S = K \times L^{\circ}$, define $\mathcal{G} : \mathbb{R}^N \times S \to \mathbb{R}$ by

$$\mathcal{G}(\phi,(x,y)) = 1 - \langle \phi x, y \rangle.$$

The set

$$\mathcal{A} = \{z \in \mathbb{R}^N : \mathcal{G}(z,s) \geq 0, s \in S\}$$

is just the set of elements $\phi \in \mathbb{R}^N$ such that $\phi K \subseteq L$. If K is in extremal position of $\min\{\tilde{l}_p(\phi K) : o \in \phi K \subseteq L, \phi \in \mathrm{GL}(n)\}$, then $\mathcal{F}$ attains its minimum on $\mathcal{A}$ at I_n, namely,

$$\mathcal{F}(I_n) = \tilde{l}_p(K) = \min\{\tilde{l}_p(\phi K) : o \in \phi K \subseteq L, \phi \in \mathrm{GL}(n)\}.$$

Now we prove $\nabla_\phi \mathcal{F}(I_n)$. It follows from (3.1) that

$$\begin{aligned}
\mathcal{F}(\phi) &= \left(\int_{\mathbb{R}^n} \| \phi^{-1} x \|_K^{-p} \, \mathrm{d}\gamma_n(x)\right)^{\frac{1}{p}} \\
&= \left((2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \| \phi^{-1} x \|_K^{-p} \, e^{-\frac{|x|^2}{2}} \mathrm{d}x\right)^{\frac{1}{p}} \\
&= \left((2\pi)^{-\frac{n}{2}} (\det \phi) \int_{\mathbb{R}^n} \| x \|_K^{-p} \, e^{-\frac{|\phi x|^2}{2}} \mathrm{d}x\right)^{\frac{1}{p}}.
\end{aligned}$$

It is easy to obtain that for non-degenerate ϕ, we have

$$\nabla_\phi \mathcal{G}(\phi,(x,y)) = -\nabla_\phi \langle \phi x, y \rangle = \nabla_\phi \langle x \otimes y, \phi \rangle = -x \otimes y$$

and

$$\begin{aligned}
\nabla_\phi \mathcal{F}(\phi) &= \frac{1}{p}\left((2\pi)^{-\frac{n}{2}} (\det \phi) \int_{\mathbb{R}^n} \| x \|_K^{-p} \, e^{-\frac{|\phi x|^2}{x}} \mathrm{d}x\right)^{-\frac{1}{q}} \\
&\quad \times \Bigg[(2\pi)^{-\frac{n}{2}} (\det \phi)(\phi^{-1})^* \int_{\mathbb{R}^n} \| x \|_K^{-p} \, e^{-\frac{|\phi x|^2}{x}} \mathrm{d}x \\
&\quad - (2\pi)^{-\frac{n}{2}} (\det \phi) \int_{\mathbb{R}^n} \| x \|_K^{-p} \, e^{-\frac{|\phi x|^2}{x}} x \otimes x \mathrm{d}x \Bigg],
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $(\phi^{-1})^*$ denotes conjugate of transposed transformation of ϕ^{-1}, and ϕ^{-1} is inverse transform of $\phi \in \mathrm{GL}(n)$.

Since $\mathcal{F}$ attains its minimum on $\mathcal{A}$ at $z_0 = I_n$, combining with Lemma 3.1, it follows that for some $m \leq N$, there exist $\lambda_i \geq 0$, $s_i \in S$, $s_i = (x_i, y_i)$, $1 \leq i \leq m$, such that

$$\langle x_i, y_i \rangle = 1 - \mathcal{G}(I_n, (x_i, y_i)) = 1, \ 1 \leq i \leq m,$$

and

$$\begin{aligned}
\nabla_\phi \mathcal{F}(I_n) &= \frac{1}{p}\left(\tilde{l}_p(K)\right)^{-\frac{p}{q}} \int_{\mathbb{R}^n} (I_n - x \otimes x) \| x \|_K^{-p} \, \mathrm{d}\gamma_n(x) \\
&= \sum_{i=1}^m \lambda_i \nabla_\phi \mathcal{G}(I_n, (x_i, y_i)) \qquad (3.2) \\
&= -\sum_{i=1}^m \lambda_i x_i \otimes y_i.
\end{aligned}$$

From $\langle x_i, y_i \rangle = 1, x_i \in K \subseteq L, y_i \in L^\circ \subseteq K^\circ$, we yield $x_i \in \partial L$ and $y_i \in \partial K^\circ$. Taking the trace in (3.2), we have

$$\begin{aligned}
&\mathrm{Tr}\left(\nabla_\phi \mathcal{F}(I_n)\right) \\
&= \mathrm{Tr}\left(\frac{1}{p}\left(\tilde{l}_p(K)\right)^{-\frac{p}{q}} \int_{\mathbb{R}^n} (I_n - x\otimes x)\|x\|_K^{-p}\,\mathrm{d}\gamma_n(x)\right) \\
&= \frac{1}{p}\left(\tilde{l}_p(K)\right)^{-\frac{p}{q}}\left[n\int_{\mathbb{R}^n}\|x\|_K^{-p}\,\mathrm{d}\gamma_n(x) - \int_{\mathbb{R}^n}|x|^2\|x\|_K^{-p}\,\mathrm{d}\gamma_n(x)\right] \\
&= \frac{1}{p}\left(\tilde{l}_p(K)\right)^{-\frac{p}{q}}\left[n\int_0^\infty r^{n-p-1}e^{-\frac{r^2}{2}}\mathrm{d}r - \int_0^\infty r^{n-p+1}e^{-\frac{r^2}{2}}\mathrm{d}r\right]\int_{S^{n-1}}\|\theta\|_K^{-p}\mathrm{d}S(\theta) \\
&= \frac{1}{p}\left(\tilde{l}_p(K)\right)^{-\frac{p}{q}}\left(p\int_{\mathbb{R}^n}\|x\|_K^{-p}\,\mathrm{d}\gamma_n(x)\right) = \tilde{l}_p(K).
\end{aligned}$$

Suppose $\lambda_i = c_i\tilde{l}_p(K)$. Together with (3.2), we obtain

$$\int_{\mathbb{R}^n}(x\otimes x - I_n)\|x\|_K^{-p}\mathrm{d}\gamma_n(x) = p(\tilde{l}_p(K))^p(\sum_{i=1}^m c_i x_i\otimes y_i),$$

where $\sum_{i=1}^m c_i = 1$. This completes the proof. □

If $L = B_2^n$ and $\mathcal{G}(\phi, x) = 1 - |\phi x|^2$, then using the same method in the proof of Theorem 1.1, we obtain

Corollary 3.2. Let K be a convex body such that $o \in K \subseteq B_2^n$. If K is in extremal position of (1.7), then there exist contact points $u_1, u_2, \cdots, u_m \in \partial K \cap S^{n-1}$ with $m \le n^2$ and $c_1, c_2, \cdots, c_m > 0$, such that,

$$I_n = \int_{\mathbb{R}^n}(x\otimes x)\mathrm{d}\tilde{\mu}(x) - p\sum_{i=1}^m c_i u_i\otimes u_i, \quad \sum_{i=1}^m c_i = 1,$$

where $\mathrm{d}\tilde{\mu}(x)$ is the probability measure on $\mathbb{R}^n$ with normalized density

$$\mathrm{d}\tilde{\mu}(x) = \|x\|_K^{-p}\,\mathrm{d}\gamma_n(x)/(\tilde{l}_p(K))^p.$$

Proof of Theorem 1.2. Suppose that $\phi \in L(\mathbb{R}^n, \mathbb{R}^n)$ and $\varepsilon > 0$ is small enough. Then

$$\phi_1 := (\min_{u\in S^{n-1}}\|u - \varepsilon\phi u\|_K)(I_n - \varepsilon\phi)^{-1}$$

satisfies $\phi_1 K \subseteq B_2^n, \phi_1 K \cap S^{n-1} \ne \varnothing$. Therefore

$$\int_{\mathbb{R}^n}\|x - \varepsilon\phi x\|_K^{-p}\mathrm{d}\gamma_n(x) \le (\tilde{l}_p(K))^p(\min_{u\in S^{n-1}}\|u - \varepsilon\phi u\|_K)^{-p}.$$

Let u_ε be a point on S^{n-1} at which the minimum is attained. Observe that

$$\|x - \varepsilon\phi x\|_K^{-p} = \|x\|_K^{-p} + \varepsilon p\|x\|_K^{-p-1}\langle\nabla h_{K^\circ}(x), \phi x\rangle + O(\varepsilon^2)$$

and

$$|u_\varepsilon - \varepsilon\phi u_\varepsilon|^{-p} = 1 + \varepsilon p\langle u_\varepsilon, \phi u_\varepsilon\rangle + O(\varepsilon^2).$$

Since $u_\varepsilon \in S^{n-1}$ and $\|\cdot\|_K \ge |\cdot|$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} p\|x\|_K^{-p-1}\left\langle \nabla h_{K^\circ}(x), \phi x\right\rangle \mathrm{d}\gamma_n(x) + O(\varepsilon) \\
&\le \left(\tilde{l}_p(K)\right)^p \frac{\left(\min_{u\in S^{n-1}}\|u-\varepsilon\phi u\|_K\right)^{-p}-1}{\varepsilon} \\
&\le \left(\tilde{l}_p(K)\right)^p \frac{|u_\varepsilon - \varepsilon\phi u_\varepsilon|^{-p}-1}{\varepsilon} \\
&= \left(\tilde{l}_p(K)\right)^p \left(p\left\langle u_\varepsilon, \phi u_\varepsilon\right\rangle + O(\varepsilon)\right).
\end{aligned} \tag{3.3}$$

If u is a contact point of K and B_2^n, then

$$1+\varepsilon\|\phi\| \ge \|u-\varepsilon\phi u\|_K \ge \|u_\varepsilon - \varepsilon\phi u_\varepsilon\|_K \ge \|u_\varepsilon\|_K - \varepsilon\|\phi\|.$$

It follows that

$$1 \le \|u_\varepsilon\|_K \le 1+2\varepsilon\|\phi\|. \tag{3.4}$$

In order to obtain a sequence $\varepsilon_k \to 0$ and a point $u \in S^{n-1}$ such that $u_{\varepsilon_k} \to u$. If $k\to\infty$, it follows from (3.4) that $\|u\|_K = \lim_{k\to\infty}\|u_{\varepsilon_k}\| = 1$. Namely, u is a contain point of K and B_2^n. By (3.3), we obtain

$$\int_{\mathbb{R}^n} \|x\|_K^{-p-1}\langle \nabla h_{K^\circ}(x), \phi x\rangle \mathrm{d}\gamma_n(x) \le (\tilde{l}_p(K))^p\langle u, \phi u\rangle.$$

Taking ϕ for $-\phi$, we can find another contact point u' of K and B_2^n such that

$$\int_{\mathbb{R}^n} \|x\|_K^{-p-1}\langle \nabla h_{K^\circ}(x), \phi x\rangle \mathrm{d}\gamma_n(x) \ge (\tilde{l}_p(K))^p\langle u', \phi u'\rangle.$$

Choosing $\phi_\theta(x) = \langle x, \theta\rangle\theta$ with $\theta \in S^{n-1}$, we get (1.6). □

4. Estimate of the Distance

Lemma 4.1. (see [16]) Let $x=(x_1,x_2,\cdots,x_n)\in\mathbb{R}^n$ and $y=(y_1,y_2,\cdots,y_n)\in\mathbb{R}^n$. If

$$0<m_1\le x_k\le M_1,\ 0<m_2\le y_k\le M_2,\ k=1,\cdots,n,$$

then

$$\left(\sum_{k=1}^n x_k^2\right)\left(\sum_{k=1}^n y_k^2\right) \le \left(\frac{\sqrt{\frac{M_1M_2}{m_1m_2}}+\sqrt{\frac{m_1m_2}{M_1M_2}}}{2}\right)^2\left(\sum_{k=1}^n x_k y_k\right)^2.$$

Lemma 4.1 implies that if $x, y\in\mathbb{R}^n$, then there exist a constant $c\in(0,1)$ such that

$$|\left\langle x, y\right\rangle| \ge c|x||y|. \tag{4.1}$$

Suppose that K is a centrally symmetric convex body in $\mathbb{R}^n$ such that K is in the extremal position of (1.7). Now we estimate the geometric distance between K and B_2^n.

Theorem 4.1. Let $K\subseteq B_2^n$ be a centrally symmetric convex body in $\mathbb{R}^n$. If K is in the extremal position of (1.7) and $1\le p<3$, then

$$\tilde{c}_{n,p}B_2^n \subseteq K \subseteq B_2^n,$$

where

$$\tilde{c}_{n,p}=\frac{\tilde{l}_p(B_2^n)}{\sqrt{n}}\left(\frac{\sqrt{\pi}(cp+1)}{2^{1-\frac{p}{2}}\Gamma(\frac{3-p}{2})}\right)^{\frac{1}{p}},\ c\in(0,1).$$

Proof. It follows from Corollary 3.2 that K satisfies

$$I_n=\int_{\mathbb{R}^n}(x\otimes x)\mathrm{d}\tilde{\mu}(x)-p\sum_{i=1}^m c_i u_i\otimes u_i,\quad \sum_{i=1}^m c_i=1,$$

where $\mathrm{d}\tilde{\mu}(x)$ is the probability measure on $\mathbb{R}^n$ with normalized density

$$\mathrm{d}\tilde{\mu}(x)=\|x\|_K^{-p}\ \mathrm{d}\gamma_n(x)/(\tilde{l}_p(K))^p.$$

For $y\in K^\circ$ and $u_i\in S^{n-1}$. By (4.1), there exists a constant $c\in(0,1)$ such that $|\langle y,u_i\rangle|\ge c|y|$. So we obtain

$$\int_{\mathbb{R}^n}(|\langle x,y\rangle|^2-|y|^2)\mathrm{d}\tilde{\mu}(x)\ge cp|y|^2\sum_{i=1}^m c_i=cp|y|^2.$$

That is,

$$(cp+1)|y|^2\le\int_{\mathbb{R}^n}|\langle x,y\rangle|^2\ \mathrm{d}\tilde{\mu}(x).$$

Since $\|x\|_K\ge|\langle x,y\rangle|$, we have

$$\begin{aligned}\int_{\mathbb{R}^n}|\langle x,y\rangle|^2\|x\|_K^{-p}\ \mathrm{d}\gamma_n(x)&\le\int_{\mathbb{R}^n}|\langle x,y\rangle|^{2-p}\ \mathrm{d}\gamma_n(x)\\&=(2\pi)^{-\frac{n}{2}}\int_{S^{n-1}}|\langle\theta,y\rangle|^{2-p}\ \mathrm{d}S(\theta)\int_0^\infty r^{n-p+1}e^{-\frac{r^2}{2}}\mathrm{d}r\\&=2^{1-\frac{p}{2}}\pi^{-\frac{1}{2}}\Gamma(\frac{3-p}{2})|y|^{2-p}.\end{aligned}$$

From John's theorem, for every centrally symmetric convex body K in $\mathbb{R}^n$, there is a corresponding to the ball λB_2^n such that $\lambda B_2^n\subseteq K\subseteq\sqrt{n}\lambda B_2^n\ (\lambda>0)$. Take $\lambda=1/\sqrt{n}$. We obtain $\frac{1}{\sqrt{n}}B_2^n\subseteq K\subseteq B_2^n$. Thus,

$$\frac{1}{\sqrt{n}}\tilde{l}_p(B_2^n)\le\tilde{l}_p(K)\le\tilde{l}_p(B_2^n).$$

Therefore, we get

$$|y|\le\frac{\sqrt{n}}{\tilde{l}_p(B_2^n)}\left(\frac{2^{1-\frac{p}{2}}\Gamma(\frac{3-p}{2})}{\sqrt{\pi}(cp+1)}\right)^{\frac{1}{p}},$$

and the result yields. □

Giannopoulos et al. in [5] proved that if K is in a position of maximal volume in L, then $K\subseteq L\subseteq nK$, which is equivalent to $\frac{1}{n}\|x\|_K\le\|x\|_L\le\|x\|_K$ for all $x\in\mathbb{R}^n$. Hence it follows that

$$1 \le \frac{\tilde{l}_p(L)}{\tilde{l}_p(K)} \le n.$$

Furthermore, let $\phi \in \mathrm{GL}(n)$. Since $\phi K \subseteq B_2^n$ is in the maximal volume position of K contained in B_2^n, we have $\frac{1}{\sqrt{n}} B_2^n \subseteq \phi K \subseteq B_2^n$. Thus

$$\frac{1}{\sqrt{n}} \le \frac{\tilde{l}_p(\phi K)}{\tilde{l}_p(B_2^n)} \le 1.$$

Finally, we propose the following concept of l_0-norm: Let K be a convex body in $\mathbb{R}^n$, we define l_0-norm by

$$l_0(K) = \exp\left(\int_{\mathbb{R}^n} log \| x \|_K \ \gamma_n(x)\right).$$

We propose an open question as follows: How should we solve the extreme problem

$$\min\{l_0(\phi K) : o \in \phi K \subseteq L, \phi \in \mathrm{GL}(n)\}?$$

Funding

This work is supported by the National Natural Science Foundation of China (Grant No.11561020).

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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