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Twisted Poisson Homology of Truncated Polynomial Algebras in Four Variables

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Abstract

In this paper, we study the twisted Poisson homology of truncated polynomials algebra A in four variables, and we calculate exactly the dimension of i-th (i = 1, 2, 3, 4) twisted Poisson homology group over A by the induction on the length. The calculation methods provided in this paper can also solve truncated polynomials algebra in a few variables.

Keywords

Twisted Poisson Homology, Poisson Algebra, (Twisted) Poisson Module

1. Introduction

For a Poisson algebra, Lichnerowicsz (see [1]) first introduced the notion of Poisson cohomology in 1977. This Poisson cohomology provides important information about the structure of Poisson algebra. Launois S and Richard L (see [2]) studied the Poisson (co)homology of the algebra of the truncated polynomial in two variables and established a duality between the two. Can Zhu (see [3]) proved that this result is still true for all Frobenius Poisson algebra as follows (Theorem 1):

Theorem 1. Let S be a Frobenius Poisson algebra. Then we have the following isomorphism:

$$HP^{i}(S,S)^{*} \cong HP_{i}(S,S_{\sigma}),$$

For all $i \in N$, where S_σ is the Poisson module induced by the Frobenius isomorphism $\sigma : S \to S^{*}$ (see [3], Corollary 3.3).

In general, given a Poisson algebra, it is very difficult to calculate its Poisson cohomology. From the above Theorem, the dimension of Poisson cohomology space is determined by calculating twisted Poisson homology. So there is a natural problem: how to calculate the twisted Poisson homology of a Poisson algebra. For example, for algebra in [2], how would we calculate its twisted Poisson homology if we extended two variables to four variables or even n variables. The purpose of this paper is to provide a solution to calculate the twisted Poisson homology of truncated polynomials algebra in four variables.

In this article, we will recall some basic knowledge in the second part and show the main conclusions in the third part.

2. Preliminaries

Throughout, $\mathbb{k}$ is a field of characteristic zero.

Definition 1 [4]. A right Poisson module M over the Poisson algebra R is a k-vector space M endowed with two bilinear maps $\cdot$ and $\{-,-\}_M : M \times R \to M$ such that

1) $(M,\cdot)$ is a module over the commutative algebra R;
2) $(M,\{-,-\}_M)$ is a right Lie-module over the Lie algebra $(R,\{-,-\})$;
3) $\{xa,b\}_M = \{x,b\}_M a + x\{a,b\}$ for any $a,b \in R$ and $x \in M$;
4) $\{x,ab\}_M = \{x,a\}_M b + \{x,b\}_M a$ for any $a,b \in R$ and $x \in M$;

Left Poisson modules are defined similar. Any Poisson algebra R is naturally a right or left Poisson module over itself.

Definition 2 [5]. Let A be a Poisson algebra. In general, let $\Omega^1(A)$ be the Kähler differential module of A and $\Omega^p(A) := \wedge^p \Omega^1(A)$ be the p-th Kähler differentia forms, where $\wedge$ is the wedge product over A (also in [6]). Given a right Poisson module M over the Poisson algrbra A, there is a canonical chain complex

$$\begin{aligned}&\cdots \longrightarrow M \otimes_A \Omega^p(A) \xrightarrow{\partial_p} M \otimes_A \Omega^{p-1}(A) \xrightarrow{\partial_{p-1}} \cdots \\ &\xrightarrow{\partial_2} M \otimes_A \Omega^1(A) \xrightarrow{\partial_1} M \otimes_A \Omega^0(A) \xrightarrow{\partial_0} 0\end{aligned} \tag{1.1}$$

where for $p \geq 1$, $\partial_p : M \otimes_A \Omega^p(A) \to M \otimes_A \Omega^{p-1}(A)$ is defined as:

$$\begin{aligned}&\partial_p\left(m \otimes da_1 \wedge \cdots \wedge da_p\right) \\ &= \sum_{i=1}^{p} (-1)^{i-1} \{m,a_i\}_M \otimes da_1 \wedge \cdots \widehat{da_i} \cdots \wedge da_p \\ &\quad + \sum_{1\leq i<j\leq p} (-1)^{i+j} m \otimes \{a_i,a_j\}_A da_1 \wedge \cdots \widehat{da_i} \cdots \widehat{da_j} \cdots \wedge da_p\end{aligned}$$

The complex (1.1) is called the Poisson complex of A with values in M, and for $p \geq 0$ is p-th Poisson homology of A with values in M, denoed by $HP_p(A,M)$.

Definition 3 [5]. Let $D \in \chi^1(A)$ be a Poisson derivation, and M be a right Poisson A-module. Define a new bilinear map $\{-,-\}_{M^D} : M \times R \to M$ as

$$\{m,a\}_{M^D} := \{m,a\}_M + m \cdot D(a)$$

Then the A-module with $\{-,-\}_{M^D}$ is a right Poisson A-module, which is called the twisted Poisson module of M twisted by the Poisson derivation D, denote by M^D.

3. Twisted Poisson Homology of Truncated Polynomial

We consider the truncated polynomials algebra

$$A:=\Bbbk[x_1,x_2,x_3,x_4]/\langle x_ix_j-x_jx_i,x_i^2\rangle$$

with the Poisson bracket $\{x_i,x_j\}_A=\lambda_{ij}x_ix_j$, $\forall 1\le i<j\le 4,\lambda_{ij}\in\Bbbk$. The fact that $\lambda_{ij}=-\lambda_{ji}$ is clear from the definition of Poisson bracket. We can get the modular derivation $D(x_i)=\left(\left(\lambda_{i\,i+1}+\lambda_{i\,i+2}+\cdots+\lambda_{i\,n}\right)-\left(\lambda_{1\,i}+\lambda_{2\,i}+\cdots+\lambda_{i-1\,i}\right)\right)x_i$ in [7]. Then we define a new bilinear map $\{-,-\}_{A^D}$ as $\{x_i,x_j\}_{A^D}:=\{x_i,x_j\}_A+x_iD(x_j)$. By definition 3, A^D becomes a twisted Poisson right A-module with $\{-,-\}_{A^D}$. Motivated by this result and definition 2, we obtain a new canonical chain complex over A:

$$0\longrightarrow A^D\otimes_A\Omega^4(A)\xrightarrow{\delta_4^\pi}A^D\otimes_A\Omega^3(A)\xrightarrow{\delta_3^\pi}A^D\otimes_A\Omega^2(A)$$
$$\xrightarrow{\delta_2^\pi}A^D\otimes_A\Omega^1\xrightarrow{\delta_1^\pi}(A)\,A^D\otimes_A\Omega^0(A)\xrightarrow{\delta_0^\pi}0$$

where for $1\le p\le 4$, $\delta_p^\pi:A^D\otimes_A\Omega^p(A)\to A^D\otimes_A\Omega^{p-1}(A)$ is defined as:

$$\begin{aligned}&\delta_p^\pi\left(a_0\otimes da_1\wedge\cdots\wedge da_p\right)\\&=\sum_{i=1}^{p}(-1)^{i-1}\{a_0,a_i\}_{A^D}\otimes da_1\wedge\cdots\widehat{da_i}\cdots\wedge da_p\\&\quad+\sum_{1\le i<j\le p}(-1)^{i+j}a_0\otimes\{a_i,a_j\}_A\,da_1\wedge\cdots\widehat{da_i}\cdots\widehat{da_j}\cdots\wedge da_p\end{aligned}$$

Now we can get some conclusions as follows.

Proposition 3.1. $HP_0(A,A^D)=\Bbbk(x_1x_2x_3x_4)$, $\dim\left(HP_0(A,A^D)\right)=1$.

Proof. $A^D\otimes\Omega^1(A)\xrightarrow{\delta_1^\pi}A^D\otimes\Omega^0(A)\xrightarrow{\delta_0^\pi}0$

$$\delta_1^\pi:m\otimes dx_i\mapsto\{m,x_i\}_{A^D}$$

First of all, we have that $\ker\delta_0^\pi=A^D\otimes\Omega^0(A)$, which leads to

$$HP_0(A,A^D)=\ker\delta_0^\pi/\operatorname{Im}\delta_1^\pi=A^D\otimes\Omega^0(A)/\operatorname{Im}\delta_1^\pi,$$

So we just need to consider which elements in $A^D\otimes\Omega^0(A)$ have the inverse image.

We proceed by the induction on the length of the elements in $A^D\otimes\Omega^0(A)$.

Remark: We make an agreement on the length: the length of $1\otimes da_1\wedge\cdots\wedge da_n$ is 0; the length of $a_1a_2\cdots a_i\otimes da_j\wedge\cdots\wedge da_n$ is *i*.

1) The image of element of length 0

$$1\otimes dx_1\mapsto\{1,x_1\}_{A^D}=D(x_1)=(\lambda_{12}+\lambda_{13}+\lambda_{14})x_1$$
$$1\otimes dx_2\mapsto\{1,x_2\}_{A^D}=D(x_2)=(\lambda_{23}+\lambda_{24}-\lambda_{12})x_2$$
$$1\otimes dx_3\mapsto\{1,x_3\}_{A^D}=D(x_3)=(\lambda_{34}-\lambda_{13}-\lambda_{23})x_3$$
$$1\otimes dx_4\mapsto\{1,x_4\}_{A^D}=D(x_4)=-(\lambda_{14}+\lambda_{24}+\lambda_{34})x_4$$

Hence, we have: $x_1,x_2,x_3,x_4\in\operatorname{Im}\delta_1^\pi$.

2) The image of element of length 1

$$x_1\otimes dx_2\mapsto\{x_1,x_2\}_{A^D}=\{x_1,x_2\}_A+x_1D(x_2)=(\lambda_{23}+\lambda_{24})x_1x_2$$

$$x_3 \otimes dx_1 \mapsto \{x_3, x_1\}_{A^D} = \{x_3, x_1\}_A + x_3 D(x_1) = (\lambda_{12} + \lambda_{14}) x_1 x_3$$

$$x_1 \otimes dx_4 \mapsto \{x_1, x_4\}_{A^D} = \{x_1, x_4\}_A + x_1 D(x_4) = -(\lambda_{24} + \lambda_{34}) x_1 x_4$$

$$x_2 \otimes dx_3 \mapsto \{x_2, x_3\}_{A^D} = \{x_2, x_3\}_A + x_2 D(x_3) = (\lambda_{34} - \lambda_{13}) x_2 x_3$$

$$x_2 \otimes dx_4 \mapsto \{x_2, x_4\}_{A^D} = \{x_2, x_4\}_A + x_2 D(x_4) = -(\lambda_{14} + \lambda_{34}) x_2 x_4$$

$$x_3 \otimes dx_4 \mapsto \{x_3, x_4\}_{A^D} = \{x_3, x_4\}_A + x_3 D(x_4) = -(\lambda_{14} + \lambda_{24}) x_3 x_4$$

Obviously, we get: $x_i x_j \in \operatorname{Im} \delta_1^\pi, \forall 1 \le i < j \le 4$.

3) The image of element of length 2

$$x_1 x_2 \otimes dx_3 \mapsto \{x_1 x_2, x_3\}_{A^D} = \{x_1 x_2, x_3\}_A + x_1 x_2 D(x_3) = \lambda_{34} x_1 x_2 x_3$$

$$x_1 x_2 \otimes dx_4 \mapsto \{x_1 x_2, x_4\}_{A^D} = \{x_1 x_2, x_4\}_A + x_1 x_2 D(x_4) = -\lambda_{34} x_1 x_2 x_4$$

$$x_1 x_3 \otimes dx_4 \mapsto \{x_1 x_3, x_4\}_{A^D} = \{x_1 x_3, x_4\}_A + x_1 x_3 D(x_4) = -\lambda_{24} x_1 x_3 x_4$$

$$x_2 x_3 \otimes dx_4 \mapsto \{x_2 x_3, x_4\}_{A^D} = \{x_2 x_3, x_4\}_A + x_2 x_3 D(x_4) = -\lambda_{14} x_2 x_3 x_4$$

Hence, we can see: $x_i x_j x_k \in \operatorname{Im} \delta_1^\pi, \forall 1 \le i < j < k \le 4$.

4) The image of element of length 3

$$\delta_1^\pi (x_1 x_2 x_3 \otimes dx_4) = 0$$

$$\delta_1^\pi (x_1 x_2 x_4 \otimes dx_3) = 0$$

$$\delta_1^\pi (x_1 x_3 x_4 \otimes dx_2) = 0$$

$$\delta_1^\pi (x_2 x_3 x_4 \otimes dx_1) = 0$$

Hence, $x_1 x_2 x_3 x_4$ have no inverse image under the map δ_1^π. Thus $x_1 x_2 x_3 x_4 \notin \operatorname{Im} \delta_1^\pi$, and $x_1 x_2 x_3 x_4 \in HP_0(A, A^D)$.

Combined with (1)-(4), it is obvious that

$$HP_0(A, A^D) = \mathbb{k}(x_1 x_2 x_3 x_4), \quad \dim\left(HP_0(A, A^D)\right) = 1 .\#$$

Proposition 3.2. $HP_1(A, A^D) = \mathbb{k}(x_i x_j x_k \otimes dx_t), 1 \le i < j < k \le 4$,

$$\dim\left(HP_1(A, A^D)\right) = 4 .$$

Proof. $A^D \otimes \Omega^2(A) \xrightarrow{\delta_2^\pi} A^D \otimes \Omega^1(A) \xrightarrow{\delta_1^\pi} A^D \otimes \Omega^0(A)$

$$\delta_1^\pi : m \otimes dx_i \mapsto \{m, x_i\}_{A^D}$$

$$\delta_2^\pi : m \otimes dx_i \wedge dx_j \mapsto \{m, x_i\}_{A^D} \otimes dx_j - \{m, x_j\}_{A^D} \otimes dx_i - m \otimes d\{x_i, x_j\}$$

In this part, we need to consider two questions: 1) what is the form of the element in $\ker \delta_1^\pi$; 2) whether the element in $\ker \delta_1^\pi$ has the inverse image.

We distinguish four cases below.

1) The element with the length of 0

Since $\delta_1^\pi (1 \otimes dx_1) = \{1, x_1\}_{A^D} = (\lambda_{12} + \lambda_{13} + \lambda_{14}) x_1 \ne 0$, we have $1 \otimes dx_1 \notin \ker \delta_0^\pi$, and all elements with the form as $1 \otimes dx_i$ have the same situation.

2) The element with the length of 1

$$\delta_1^\pi (C_{12} x_1 \otimes dx_2 + C_{21} x_2 \otimes dx_1) = \left[C_{12}(\lambda_{23} + \lambda_{24}) + C_{21}(\lambda_{13} + \lambda_{14})\right] x_1 x_2$$

Let $C_{12}x_1 \otimes dx_2 + C_{21}x_2 \otimes dx_1 \in \ker \delta_1^\pi$, then $C_{12}(\lambda_{23}+\lambda_{24}) + C_{21}(\lambda_{13}+\lambda_{14}) = 0$, we can easy get:

$$-(\lambda_{13}+\lambda_{14})x_1 \otimes dx_2 + (\lambda_{23}+\lambda_{24})x_2 \otimes dx_1 \in \ker \delta_1^\pi$$

Similarly, $-(\lambda_{12}+\lambda_{14})x_1 \otimes dx_3 + (\lambda_{32}+\lambda_{34})x_3 \otimes dx_1 \in \ker \delta_1^\pi$

$$\vdots$$

$$-(\lambda_{13}+\lambda_{23})x_3 \otimes dx_4 + (\lambda_{14}+\lambda_{24})x_4 \otimes dx_3 \in \ker \delta_1^\pi$$

Now we prove that these elements with the length of 1 in $\ker \delta_1^\pi$ have inverse image under the map δ_2^π.

$$\begin{aligned}\delta_2^\pi(1 \otimes dx_1 \wedge dx_2) &= \{1, x_1\}_{A^D} \otimes dx_2 - \{1, x_2\}_{A^D} \otimes dx_1 - 1 \otimes d\{x_1, x_2\} \\ &= D(x_1) \otimes dx_2 - D(x_2) \otimes dx_1 - 1 \otimes d\{x_1, x_2\} \\ &= (\lambda_{13}+\lambda_{14})x_1 \otimes dx_2 - (\lambda_{23}+\lambda_{24})x_2 \otimes dx_1\end{aligned}$$

$$\begin{aligned}\delta_2^\pi(1 \otimes dx_1 \wedge dx_3) &= \{1, x_1\}_{A^D} \otimes dx_3 - \{1, x_3\}_{A^D} \otimes dx_1 - 1 \otimes d\{x_1, x_3\} \\ &= D(x_1) \otimes dx_3 - D(x_3) \otimes dx_1 - 1 \otimes d\{x_1, x_3\} \\ &= (\lambda_{12}+\lambda_{14})x_1 \otimes dx_2 - (\lambda_{32}+\lambda_{34})x_3 \otimes dx_1\end{aligned}$$

$$\vdots$$

$$\begin{aligned}\delta_2^\pi(1 \otimes dx_3 \wedge dx_4) &= \{1, x_3\}_{A^D} \otimes dx_4 - \{1, x_4\}_{A^D} \otimes dx_3 - 1 \otimes d\{x_3, x_4\} \\ &= D(x_3) \otimes dx_4 - D(x_4) \otimes dx_3 - 1 \otimes d\{x_3, x_4\} \\ &= -(\lambda_{13}+\lambda_{23})x_3 \otimes dx_4 + (\lambda_{14}+\lambda_{24})x_4 \otimes dx_3\end{aligned}$$

3) The element with the length of 2

$$\begin{aligned}&\delta_1^\pi(C_{123}x_1x_2 \otimes dx_3 + C_{132}x_1x_3 \otimes dx_2 + C_{231}x_2x_3 \otimes dx_1) \\ &= (C_{123}\lambda_{34} + C_{132}\lambda_{24} + C_{231}\lambda_{14})x_1x_2x_3\end{aligned}$$

If $C_{123}\lambda_{34} + C_{132}\lambda_{24} + C_{231}\lambda_{14} = 0$ (1.2), so that

$$C_{123}x_1x_2 \otimes dx_3 + C_{132}x_1x_3 \otimes dx_2 + C_{231}x_2x_3 \otimes dx_1 \in \ker \delta_1^\pi$$

For (1.2), let C_{123} be a free variable, we can infer that

$$\begin{cases} C_{123}\lambda_{34} + C_{132}\lambda_{24} = 0 \\ C_{123}\lambda_{34} + C_{231}\lambda_{14} = 0 \end{cases} \Rightarrow \begin{cases} C_{132} = \dfrac{-C_{123}\lambda_{34}}{\lambda_{24}} \\ C_{231} = \dfrac{-C_{123}\lambda_{34}}{\lambda_{14}} \end{cases} \Rightarrow \begin{cases} x_1x_2 \otimes dx_3 - \dfrac{\lambda_{34}}{\lambda_{24}}x_1x_3 \otimes dx_2 \in \ker \delta_1^\pi \\ x_1x_2 \otimes dx_3 - \dfrac{\lambda_{34}}{\lambda_{14}}x_2x_3 \otimes dx_1 \in \ker \delta_1^\pi \end{cases}$$

similarly,

$$x_1x_3 \otimes dx_4 - \frac{\lambda_{24}}{\lambda_{23}}x_1x_4 \otimes dx_3 \in \ker \delta_1^\pi$$

$$x_1x_3 \otimes dx_4 + \frac{\lambda_{24}}{\lambda_{12}}x_3x_4 \otimes dx_1 \in \ker \delta_1^\pi$$

$$\vdots$$

We can find the inverse image of all elements as the above by following

$$\delta_2^\pi\left(\frac{1}{\lambda_{24}}x_1\otimes dx_2\wedge dx_3\right)=x_1x_2\otimes dx_3-\frac{\lambda_{34}}{\lambda_{24}}x_1x_3\otimes dx_2$$

$$\delta_2^\pi\left(\frac{1}{\lambda_{14}}x_2\otimes dx_1\wedge dx_3\right)=x_1x_2\otimes dx_3-\frac{\lambda_{34}}{\lambda_{14}}x_2x_3\otimes dx_1$$

$$\delta_2^\pi\left(-\frac{1}{\lambda_{23}}x_1\otimes dx_3\wedge dx_4\right)=x_1x_3\otimes dx_4-\frac{\lambda_{24}}{\lambda_{23}}x_1x_4\otimes dx_3$$

$$\delta_2^\pi\left(\frac{1}{\lambda_{12}}x_3\otimes dx_1\wedge dx_4\right)=x_1x_3\otimes dx_4+\frac{\lambda_{24}}{\lambda_{12}}x_3x_4\otimes dx_1$$

$$\vdots$$

4) The element with the length of 3

For $\forall 1\le i<j<k\le 4,\ \delta_1^\pi\left(x_ix_jx_k\otimes dx_t\right)=0$, indeed, for $\forall 1\le i<j\le 4, 1\le k<t\le 4$, $\delta_2^\pi\left(x_ix_j\otimes dx_k\wedge dx_t\right)=0$. Since the map δ_p^π keeps the variable unchanged, so we can't find the inverse image of $x_ix_jx_k\otimes dx_t$ under the map δ_2^π.

In conclusion, only $x_ix_jx_k\otimes dx_t\in HP_1\left(A,A^D\right)$, and $\dim\left(HP_0\left(A,A^D\right)\right)=C_4^1=4$.

Similar to the proof of proposition 3.2, we can prove the following

Proposition 3.3. $HP_2\left(A,A^D\right)=\mathbb{k}\left(x_ix_j\otimes dx_k\wedge dx_t\right)$, ($1\le i<j\le 4, 1\le k<t\le 4$) $\dim\left(HP_2\left(A,A^D\right)\right)=6$.

Proof. $A^D\otimes\Omega^3(A)\xrightarrow{\delta_3^\pi}A^D\otimes\Omega^2(A)\xrightarrow{\delta_2^\pi}A^D\otimes\Omega^1(A)$

$$\delta_2^\pi: m\otimes dx_i\wedge dx_j\mapsto\left\{m,x_i\right\}_{A^D}\otimes dx_j-\left\{m,x_j\right\}_{A^D}\otimes dx_i-m\otimes d\left\{x_i,x_j\right\}$$

$$\begin{aligned}\delta_3^\pi: m\otimes dx_i\wedge dx_j\wedge dx_k\mapsto&\left\{m,x_i\right\}_{A^D}\otimes dx_j\wedge dx_k-\left\{m,x_j\right\}_{A^D}\otimes dx_i\wedge dx_k\\&+\left\{m,x_k\right\}_{A^D}\otimes dx_i\wedge dx_j-m\otimes d\left\{x_i,x_j\right\}\wedge dx_k\\&+m\otimes d\left\{x_i,x_k\right\}\wedge dx_j-m\otimes d\left\{x_j,x_k\right\}\wedge dx_i\end{aligned}$$

1) The element with the length of 0

When we calculate the 1-th twisted Poisson homology group, we have found that each element of length 0 in $A^D\otimes\Omega^2(A)$ has an image under the map δ_2^π, and never belongs to $\ker\delta_2^\pi$.

2) The element with the length of 1

$$\begin{aligned}&\delta_2^\pi\left(k_1x_1\otimes dx_3\wedge dx_4+k_3x_3\otimes dx_1\wedge dx_4+k_4x_4\otimes dx_1\wedge dx_3\right)\\&=\left(k_3\lambda_{12}-k_1\lambda_{23}\right)x_1x_3\otimes dx_4+\left(k_1\lambda_{24}+k_4\lambda_{12}\right)x_1x_4\otimes dx_3+\left(k_3\lambda_{24}+k_4\lambda_{23}\right)x_3x_4\otimes dx_1\end{aligned}$$

$x_1x_3\otimes dx_4$, $x_1x_4\otimes dx_3$ and $x_3x_4\otimes dx_1$ are linear independence. If and only if such that $k_3\lambda_{12}-k_1\lambda_{23}=0$, $k_1\lambda_{24}+k_4\lambda_{12}=0$ and $k_3\lambda_{24}+k_4\lambda_{23}=0$ at same time, *i.e.*,

$$k_3=\frac{k_1\lambda_{23}}{\lambda_{12}},\quad k_4=-\frac{k_1\lambda_{24}}{\lambda_{12}},$$

we get that $x_1\otimes dx_3\wedge dx_4+\frac{k_1\lambda_{23}}{\lambda_{12}}x_3\otimes dx_1\wedge dx_4-\frac{k_1\lambda_{24}}{\lambda_{12}}x_4\otimes dx_1\wedge dx_3\in\ker\delta_2^\pi$.

Similarly,

$$\delta_3^\pi\left(\frac{1}{\lambda_{12}}\otimes dx_1\wedge dx_3\wedge dx_4\right)=x_1\otimes dx_3\wedge dx_4+\frac{k_1\lambda_{23}}{\lambda_{12}}x_3\otimes dx_1\wedge dx_4-\frac{k_1\lambda_{24}}{\lambda_{12}}x_4\otimes dx_1\wedge dx_3.$$

Obviously, the preimage of $x_1\otimes dx_3\wedge dx_4+\frac{k_1\lambda_{23}}{\lambda_{12}}x_3\otimes dx_1\wedge dx_4-\frac{k_1\lambda_{24}}{\lambda_{12}}x_4\otimes dx_1\wedge dx_3$ is $\frac{1}{\lambda_{12}}\otimes dx_1\wedge dx_3\wedge dx_4$. Thus, this element in $\ker\delta_2^\pi$ does not belong to $HP_2\left(A,A^D\right)$. It is clear that the element with the same form have the same situation.

3) The element with the length of 2

For $1\le i<j\le 4, 1\le k<t\le 4$, $\delta_2^\pi\left(x_ix_j\otimes dx_k\wedge dx_t\right)=0$. Also $\delta_3^\pi\left(x_i\otimes dx_j\wedge dx_k\wedge dx_t\right)=0$.

This means that, the element $x_ix_j\otimes dx_k\wedge dx_t$ has no inverse image under the map δ_3^π.

Thus, only $x_ix_j\otimes dx_k\wedge dx_t\in HP_2\left(A,A^D\right)$, $\left(1\le i<j\le 4, 1\le k<t\le 4\right)$.

In conclusion, it suffices to show that $HP_2\left(A,A^D\right)=\Bbbk\left(x_ix_j\otimes dx_k\wedge dx_t\right)$, moreover,

$$\dim\left(HP_2\left(A,A^D\right)\right)=C_4^2=6$$

Proposition 3.4. $\dim\left(HP_3\left(A,A^D\right)\right)=3$, $\dim\left(HP_4\left(A,A^D\right)\right)=0$.

Proof. $0\longrightarrow A^D\otimes\Omega^4(A)\xrightarrow{\delta_4^\pi}A^D\otimes\Omega^3(A)\xrightarrow{\delta_3^\pi}A^D\otimes\Omega^2(A)$

$$\begin{aligned}\delta_4^\pi\left(1\otimes dx_1\wedge dx_2\wedge dx_3\wedge dx_4\right)&=\left(-\lambda_{12}-\lambda_{14}-\lambda_{13}\right)x_1\otimes dx_2\wedge dx_3\wedge dx_4\\&+\left(\lambda_{21}+\lambda_{23}+\lambda_{24}\right)x_2\otimes dx_1\wedge dx_3\wedge dx_4\\&+\left(\lambda_{13}+\lambda_{23}+\lambda_{43}\right)x_3\otimes dx_1\wedge dx_2\wedge dx_4\\&+\left(-\lambda_{14}-\lambda_{34}-\lambda_{24}\right)x_4\otimes dx_1\wedge dx_2\wedge dx_3\\&:=t_1X_1+t_2X_2+t_3X_3+t_4X_4\end{aligned}$$

When we calculate the 2-th twisted Poisson homology group, we have noticed that each element like $1\otimes dx_i\wedge dx_j\wedge dx_k$ always has a image under the map δ_3^π, that means $1\otimes dx_i\wedge dx_j\wedge dx_k\notin\ker\delta_3^\pi$. On the other hand, $\delta_3^\pi\left(x_i\otimes dx_j\wedge dx_k\wedge dx_t\right)=0$, implies that, $\dim\left(\ker\delta_3^\pi\right)=4$, indeed, $\dim\left(Im\delta_4^\pi\right)=1$, so that $\dim\left(HP_3\left(A,A^D\right)\right)=3$.

Obviously, since $HP_4\left(A,A^D\right)=0$, we have $\dim\left(HP_4\left(A,A^D\right)\right)=0$.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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