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Fast Computation of Pareto Set for Bicriteria Linear Programs with Application to a Diet Formulation Problem

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Abstract

In case of mathematical programming problems with conflicting criteria, the Pareto set is a useful tool for a decision maker. Based on the geometric properties of the Pareto set for a bicriteria linear programming problem, we present a simple and fast method to compute this set in the criterion space using only an elementary linear program solver. We illustrate the method by solving the pig diet formulation problem which takes into account not only the cost of the diet but also nitrogen or phosphorus excretions.

Keywords

Bicriteria Linear Program, Pareto Set, Criterion Space, Weighted-Sum, Diet Formulation, Taxation System

1. Introduction

Animal diet formulation is a very important problem from an economic and environmental point of view, so it is an interesting example in operations research. Many modern animal diet formulation methods tend to take into account nitrogen and phosphorus excretions that are detrimental from an environmental point of view. Following [1], it is appropriate to apply a tax on excretions so as to change the behavior of the producers in the swine industry. These changes in behavior are studied using a formulation of the problem as a bicriteria problem and are obtained by the determination of the Pareto set of the problem. For linear models, this Pareto set is a simple polygonal line. This fact implies that changes in behavior of the producers are abrupt and correspond to particular values of the tax. In other words even in increasing the tax it can

happen that there is no change in behavior. Behavior changes happend only at very particular values of the tax. We will see that these behaviors correspond to efficient extreme points of the Pareto set, and to every extreme point corresponds a tax interval so that any value of the tax in this interval leads to the behavior given by that extreme point.

The computation and visualization of the Pareto set, also known as the efficiency set, for bicriteria linear programming problems is a useful tool for decision makers. We could try to compute this set in the decision space [2]-[10], but due to the high dimension of this space, it can be a quite large and complicated set. Methods to obtain this set are also complicated, see for example [11]. Fortunately, the geometric aspect of the Pareto set in the criterion (or outcome) space for bicriteria linear program is quite simple [12].

The outline of the paper is the following. The bicriteria problem is presented in Section 2. We will see in Section 3, that the Pareto set of a bicriteria linear problem is a simple polygonal line with $L + 1$ extreme points joined by L adjacent segments. Then in Section 4 we presents the link between the geometric structure of the Pareto set and the weighted-sums approach. Then an elementary algorithm to determine the Pareto set in the criterion space is suggested and its complexity is analyzed. Let us point out that this method uses only elementary result from a linear program solver, that is to say the optimal solution (values of the decision variables). This fact is an interesting property of the method.

Few methods exist for computing the Pareto set in the criteria space. One such method is presented in [13]. The method requires information about the dual, assume the feasible set is compact, and determine the Pareto set with at most $2L + 4$ calls to a linear program solver. Another simple method for bi-criteria problems is presented in [12] to obtain the Pareto set in the criterion space. The algorithm is based on information about the reduced costs of all nonbasic variables, which is equivalent to have information about the solution of the dual problem. For bi-criteria linear problems we could also use a parametric analysis to obtain the Pareto set [11] [14]. The last two methods require that the software used to solve a linear program send information about the dual, reduced cost or postoptimal analysis, which is not always possible for a simple linear program solver. Unfortunately, even if it seems that those two methods require around $2L$ iterations, their complexities are nowhere analyzed. Moreover they can cycle as explained in [15] (pages 281-282), and [16] (pages 162-166).

Finally, in Section 5, we compute the Pareto sets for least cost diet formulation problems for pig, or any monogastric animal, taking into account the nitrogen and/or phosphorus excretions. Tax systems related to efficient extreme points of this problem are described.

2. Bicriteria Linear Programming Problem

Let us consider the *standard form* of the *bicriteria linear programming problem* [11]

$$(P)\quad \begin{cases} & \min z_1(x) = c_1 x \\ & \min z_2(x) = c_2 x \\ \text{suject to} & \\ & Ax = b \\ & x \geq 0 \end{cases}$$

where x is a column vector in $\mathbb{R}^n$, and the c_k's ($k = 1,2$) are two row vectors $c_k = (c_{k,1}, \cdots, c_{k,n})$ in $\mathbb{R}^n$. The *feasible set* $\mathcal{S}$ in $\mathbb{R}^n$ is defined by $\mathcal{S} = \{x \in \mathbb{R}^n \mid Ax = b \text{ and } x \geq 0\}$, where A is a (m,n)-matrix, and b is a column vector in $\mathbb{R}^m$. Let C be the $(2,n)$-matrix given by

$$C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_{1,1} & \cdots & c_{1,n} \\ c_{2,1} & \cdots & c_{2,n} \end{pmatrix}.$$

The feasible set in the criterion space $\mathbb{R}^2$ is then $\mathcal{S}_c = \{z \in \mathbb{R}^2 \mid z = Cx \text{ for } x \in \mathcal{S}\} = C\mathcal{S}$. It is well-known that $\mathcal{S}$ and $\mathcal{S}_c$ are polyhedral sets in $\mathbb{R}^n$ and $\mathbb{R}^2$ respectively. Throughout this paper we will suppose that the two criteria are lower bounded on $\mathcal{S}$ which means that for $i = 1,2$ we have

$$z_i^{\min} = \min\{z_i(x) = c_i x \mid x \in \mathcal{S}\} > -\infty.$$

3. Structure of the Pareto Set

3.1. Efficiency Set

A feasible solution $x \in \mathcal{S}$ is an *efficient solution* if and only if it does not exist any other feasible solution $\overline{x} \in \mathcal{S}$ such that 1) $z_i(\overline{x}) \leq z_i(x)$ for $i = 1,2$, and 2) $z_j(\overline{x}) < z_j(x)$ for at least one $j \in \{1,2\}$. The set of all efficient solutions is called the *efficiency set* noted $\mathcal{E}$, also called *Pareto set*. The corresponding set in the criterion space is the set $\mathcal{E}_c = C\mathcal{E}$.

3.2. Geometric Structure

Under the assumption that the two cost vectors c_1 and c_2 are linearly independant, Using weighted-sums, we can replace the bicriteria linear programming problem by a single criterion linear programming problem. We consider $\lambda \in [0,1]$ and the weighted-sum function is

$$z(x;\lambda) = (1-\lambda) z_1(x) + \lambda z_2(x) = [(1-\lambda)c_1 + \lambda c_2]x,$$

and we consider the single criteria problem for $\lambda \in [0,1]$

$$(P(\lambda))\quad \begin{cases} & \min z(x;\lambda) = (1-\lambda) z_1(x) + \lambda z_2(x) = [(1-\lambda)c_1 + \lambda c_2]x \\ \text{subject to} & \\ & x \in \mathcal{S}. \end{cases}$$

The *value function* $\varphi(\lambda)$ of $(P(\lambda))$ is defined by

$$\varphi(\lambda) = \min\{z(x;\lambda) \mid x \in \mathcal{S}\}.$$

From [11] we have

$$\mathcal{E} = \bigcup_{\lambda\in(0,1)} \arg\min_{x\in\mathcal{S}} z(x;\lambda).$$

Hence the efficiency set $\mathcal{E}$ in the decision space is a connected set and is the union of faces, edges and vertices of $\mathcal{S}$. This set may be quite complex due to the high dimension of the decision space. On the other side $\mathcal{E}_c$, which is the image in $\mathbb{R}^2$ of $\mathcal{E}$ by a linear transform, is a much simpler set.

Since we have assumed that both criteria are lower bounded on $\mathcal{S}$, it follows that $\mathcal{E}_c$ is a simple compact polygonal line. Indeed in that case $\mathcal{E}_c$ is the union of a finite number L of segments $[Q_{l-1}, Q_l]$

$$\mathcal{E}_c = \bigcup_{l=1}^{L} [Q_{l-1}, Q_l]$$

where

$$[Q_{l-1}, Q_l] = \left\{ Q \in \mathbb{R}^2 \mid Q = (1-\sigma)Q_{l-1} + \sigma Q_l \text{ for } \sigma \in [0,1] \right\},$$

and such that

$$(Q_{l-1}, Q_l) \cap (Q_{\tilde{l}-1}, Q_{\tilde{l}}) = \varnothing \quad \text{if } l \neq \tilde{l},$$

with

$$(Q_{l-1}, Q_l) = \left\{ Q \in \mathbb{R}^2 \mid Q = (1-\sigma)Q_{l-1} + \sigma Q_l \text{ for } \sigma \in (0,1) \right\}.$$

To each segment is associated a weight $\lambda_{l-1,l}$ such that the vector $(1-\lambda_{l-1,l}, \lambda_{l-1,l})^t$ is orthogonal to the segment $[Q_{l-1}, Q_l]$ in $\mathbb{R}^2$. To each point Q of $\mathcal{E}_c$ is associated an interval $\Lambda(Q)$ defined by

$$\Lambda(Q) = \begin{cases} \left[\underline{\lambda}_l, \overline{\lambda}_l\right] & \text{if } Q = Q_l \ (l = 0, \cdots, L), \\ \left[\lambda_{l-1,l}, \lambda_{l-1,l}\right] & \text{if } Q \in (Q_{l-1}, Q_l) \ (l = 1, \cdots, L), \end{cases}$$

where

$$\begin{cases} \underline{\lambda}_0 = 0, \\ \overline{\lambda}_{l-1} = \underline{\lambda}_l = \lambda_{l-1,l} \text{ for } l = 1, \cdots, L, \\ \overline{\lambda}_L = 1, \end{cases}$$

with $\overline{\lambda}_l - \underline{\lambda}_l > 0$ for $l = 0, \cdots, L$.

3.3. Weak Efficiency Set

We will call *weak efficiency set*, or *weak Pareto set*, the set defined by

$$\mathcal{E}^f = \bigcup_{\lambda\in[0,1]} \arg\min_{x\in\mathcal{S}} z(x;\lambda).$$

Obviously $\mathcal{E} \subseteq \mathcal{E}^f$. In the criteria space we will have $\mathcal{E}_c^f = C\mathcal{E}^f$. Geometrically in the criterion space $\mathbb{R}^2$, this means we add to $\mathcal{E}_c$ possibly a vertical segment or a ray from Q_0 in the positive direction of z_2, $D_0 = (0,1)$,

$$R(Q_0; D_0) = \left\{ Q_0 + \eta D_0 \mid \eta \in (0, \eta_0] \right\} \subset \mathcal{S}_c,$$

and/or a horizontal segment or a ray from Q_L in the positive direction of z_1, $D_L = (1,0)$,

$$R(Q_L;D_L)=\{Q_L+\eta D_L \mid \eta\in(0,\eta_L]\}\subset\mathcal{S}_c,$$

where η_0 and η_L are nonnegative finite or infinite values. They are the maximal values of η such that $R(Q_0;D_0)$ and $R(Q_L;D_L)$ are both subsets of $\mathcal{S}_c$. To these points on $\mathcal{E}_c$ we set

$$\Lambda(Q)=\begin{cases}[0,0] & \text{if } Q\in R(Q_0;D_0),\\ [1,1] & \text{if } Q\in R(Q_L;D_L).\end{cases}$$

3.4. Link to Parametric Analysis

The parametric analysis is based on the weighted-sum given by

$$\tilde{z}(x;\mu)=z_1(x)+\mu z_2(x)$$

for $\mu\in[0,+\infty)$, and the value function in this case is defined by

$$\tilde{\varphi}(\mu)=\min\{\tilde{z}(x;\mu)\mid x\in\mathcal{S}\}.$$

Instead of $(P(\lambda))$, we could consider the single criteria problem for $\mu\geq 0$

$$(P(\mu))\quad\begin{cases}\min \tilde{z}(x;\mu)=z_1(x)+\mu z_2(x)=(c_1+\mu c_2)x\\ \text{subject to}\\ x\in\mathcal{S}.\end{cases}$$

Since λ and μ are related by the formulae

$$\lambda=\frac{\mu}{1+\mu}\ \text{ and }\ \mu=\frac{\lambda}{1-\lambda},$$

to the efficient extreme points $\{Q_l\}_{l=0}^{L}$ on the efficiency set $\mathcal{E}_c$ correspond also the following intervals for the parameter μ

$$\tilde{\Lambda}(Q)=\begin{cases}[\underline{\mu}_l,\overline{\mu}_l] & \text{if } Q=Q_l\ \ (l=0,\cdots,L),\\ [\mu_{l-1,l},\mu_{l-1,l}] & \text{if } Q\in(Q_{l-1},Q_l)\ \ (l=1,\cdots,L),\end{cases}$$

where

$$\begin{cases}\underline{\mu}_0=0,\\ \overline{\mu}_{l-1}=\underline{\mu}_l=\mu_{l-1,l} & \text{for } l=1,\cdots,L,\\ \overline{\mu}_L=+\infty.\end{cases}$$

In many applications, the parameter μ is in fact a tax over the the second criteria (for a minimization problem). Interesting enough is to observe that the behavior change (extreme point) only for the critical values $\mu_{l-1,l}$ of the parameter μ. Indeed when μ increases and its value passes through $\mu_{l-1,l}$, the optimal point, extreme point, move from Q_{l-1} to Q_l. Thus, any level of taxes μ strictly between the values $\mu_{l-1,l}=\underline{\mu}_l$ and $\mu_{l,l+1}=\overline{\mu}_l$ causes the same behavior described by Q_l.

4. Computation of the Pareto Set

4.1. Preliminaries

Let us associate to any $Q=(z_1,z_2)\in\mathcal{S}_c$ the weighted-sum function given by

$$\varphi_Q(\lambda) = (1-\lambda)z_1 + \lambda z_2.$$

Then the value function $\varphi(\lambda)$ associated to $(P(\lambda))$ is such that

$$\begin{aligned}\varphi(\lambda) &= \min\{\varphi_Q(\lambda) \mid Q \in \mathcal{S}_c\} \\ &= \min\{\varphi_Q(\lambda) \mid Q \in \mathcal{E}_c\} \\ &= \min\{\varphi_{Q_l}(\lambda) \mid l = 0, \cdots, L\}.\end{aligned}$$

Hence we have the following results.

Theorem 4.1. [12] *Let $Q \in \mathcal{E}_c$, we have $\varphi(\lambda) = \varphi_Q(\lambda)$ if and only if $\lambda \in \Lambda(Q)$.*

Theorem 4.2. [12] *Let $Q \in \mathcal{E}_c$ and $0 \le \lambda_1 < \lambda_2 \le 1$. Then λ_1 and $\lambda_2 \in \Lambda(Q)$ if and only if $[\lambda_1, \lambda_2] \subseteq \Lambda(Q)$. It follows that Q is one of the Q_l ($l \in 0, \cdots, L$).*

Theorem 4.3. [17] *The function $\varphi(\lambda)$ is continuous, piecewise linear and concave. The abscissae of slope changes are the increasing values $\lambda_{l-1,l}$ for $l = 1, \cdots, L$.*

Let us observe that the slope associated to $\varphi_Q(\lambda)$ strictly decreases for Q going from Q_0 to Q_L on $\mathcal{E}_c$, since z_1 increases and z_2 decreases steadily. We deduce the next results.

Theorem 4.4. [12] *Let Q_i' and Q_j' be two distinct points on $\mathcal{E}_c$. For $\lambda \in [0,1]$, $(1-\lambda, \lambda)^t$ is orthogonal to the segment $[Q_i', Q_j']$ if and only if $\varphi_{Q_i'}(\lambda) = \varphi_{Q_j'}(\lambda)$.*

Theorem 4.5. *Let Q_i' and Q_j' be two distinct points on $\mathcal{E}_c$ and $\lambda \in [0,1]$, such that $(1-\lambda, \lambda)^t$ is orthogonal to the segment $[Q_i', Q_j']$. For a fixed λ, the function $\varphi_Q(\lambda)$ is constant as a function of Q on the segment $[Q_i', Q_j']$. Let us note this constant value by $\varphi_{Q_i'Q_j'}$. Moreover*

1) if $\varphi(\lambda) = \varphi_{Q_i'Q_j'}$ then $[Q_i', Q_j'] \subset \mathcal{E}_c$, $\lambda \in \Lambda(Q_i')$ and $\lambda \in \Lambda(Q_j')$;
2) if $\varphi(\lambda) > \varphi_{Q_i'Q_j'}$ then $(Q_i', Q_j') \cap \mathcal{E}_c = \varnothing$.

Theorem 4.6. *Let Q_i' and Q_j' be two distinct points on $\mathcal{E}_c$. If $\lambda \in \Lambda(Q_i')$ is such that $\varphi(\lambda) = \varphi_{Q_j'}(\lambda)$ then $\lambda \in \Lambda(Q_j')$. Moreover there exists $l \in \{0, \cdots, L\}$ such that $\lambda = \lambda_{l-1,l}$ and $[Q_i', Q_j'] \subseteq [Q_{l-1}, Q_l] \subseteq \mathcal{E}_c$.*

Theorem 4.7. *Let Q_i' and Q_j' be two distinct points on $\mathcal{E}_c^f$. Let $\lambda_i' \in \Lambda(Q_i')$ and $\lambda_j' \in \Lambda(Q_j')$, and consider the following two lines*

$$\mathcal{L}_i(\lambda_i') = \{Q \in \mathbb{R}^2 \mid \varphi_Q(\lambda_i') = \varphi(\lambda_i')\}$$

and

$$\mathcal{L}_j(\lambda_j') = \{Q \in \mathbb{R}^2 \mid \varphi_Q(\lambda_j') = \varphi(\lambda_j')\}.$$

(A) If $\lambda_i' \ne \lambda_j'$, the point of intersection of $\mathcal{L}_i(\lambda_i')$ and $\mathcal{L}_j(\lambda_j')$ is $\tilde{Q}(\lambda_i', \lambda_j') = (\psi(0; \lambda_i', \lambda_j'), \psi(1; \lambda_i', \lambda_j'))$ where

$$\psi(\lambda; \lambda_i', \lambda_j') = \frac{\lambda_j' - \lambda}{\lambda_j' - \lambda_i'}\varphi(\lambda_i') + \frac{\lambda - \lambda_i'}{\lambda_j' - \lambda_i'}\varphi(\lambda_j'),$$

so

$$\psi\left(0;\lambda_i',\lambda_j'\right)=\frac{\lambda_j'\varphi\left(\lambda_i'\right)}{\lambda_j'-\lambda_i'}-\frac{\lambda_i'\varphi\left(\lambda_j'\right)}{\lambda_j'-\lambda_i'}$$

and

$$\psi\left(1;\lambda_i',\lambda_j'\right)=\frac{\lambda_j'-1}{\lambda_j-\lambda_i'}\varphi\left(\lambda_i'\right)+\frac{1-\lambda_i'}{\lambda_j'-\lambda_i'}\varphi\left(\lambda_j'\right).$$

(B) If $\lambda_i'=\lambda_j'$, then $\mathcal{L}_i\left(\lambda_i'\right)=\mathcal{L}_j\left(\lambda_j'\right)$ which contains the segment $\left[Q_i',Q_j'\right]$.

4.2. Algorithm

In this section we consider both criteria upper bounded on $\mathcal{S}$. In the forthcoming algorithm we initialize the process with the two points Q_0 and Q_L on $\mathcal{E}_c$. Then we gradually obtain a sequence of points $\left\{Q_i'\right\}_{i=0}^{I}$ on $\mathcal{E}_c$, and a sequence of intervals associated to these points $\left\{\Lambda'\left(Q_i'\right)=\left[\underline{\lambda}_i',\overline{\lambda}_i'\right]\right\}_{i=0}^{I}$ such that $\Lambda'\left(Q_i'\right)\subseteq\Lambda\left(Q_i'\right)$ and

$$\varphi\left(\lambda\right)=\varphi_{Q_i'}\left(\lambda\right)\ \text{for all}\ \lambda\in\Lambda'\left(Q_i'\right).$$

At the end of the process $I=L$ and we have $Q_l'=Q_l$ with

$$\left[\underline{\lambda}_l',\overline{\lambda}_l'\right]=\Lambda'\left(Q_l'\right)=\Lambda\left(Q_l\right)=\left[\underline{\lambda}_l,\overline{\lambda}_l\right]$$

for $l=0,\cdots,L=I$.

Algorithm (Pareto bicriteria)

STEP 0. Initialization.

(A) Enter the data of the problem.

(B) Determine $x_i^*=\arg\min_{x\in\mathcal{S}}z_i\left(x\right)$ for $i=1,2$ and set $z_i^{\min}=z_i\left(x^*\right)$. For $i,j=1,2$ and $j\neq i$ set $z_{j|i}=z_j\left(x_i^*\right)$. We get the initial point $\tilde{Q}_0=\left(z_1^{\min},z_{2|1}\right)$ which as the same first coordinate as Q_0, and $\tilde{Q}_L=\left(z_{1|2},z_2^{\min}\right)$ which as the same second coordinate as Q_L. Those two points might not be on $\mathcal{E}_c$, but are on $\mathcal{E}_c^f$.

(C) Set $Q_0'\coloneqq\tilde{Q}_0$ and $\Lambda'\left(Q_0'\right)=\left[\underline{\lambda}_0',\overline{\lambda}_0'\right]\coloneqq\left[0,0\right]$;

(D) Set $Q_1'\coloneqq\tilde{Q}_L$ and $\Lambda'\left(Q_1'\right)=\left[\underline{\lambda}_1',\overline{\lambda}_1'\right]\coloneqq\left[1,1\right]$;

(E) Set $I\coloneqq1$.

STEP 1. As long that there exists an index i such that $\underline{\lambda}_i'-\overline{\lambda}_{i-1}'>0$, select one such index i and do:

(A) Find $\lambda^*\in\left[\overline{\lambda}_{i-1}',\underline{\lambda}_i'\right]$ such that $\varphi_{Q_{i-1}'Q_i'}^*\coloneqq\varphi_{Q_{i-1}'}\left(\lambda^*\right)=\varphi_{Q_i'}\left(\lambda^*\right)$, hence $\lambda^*\in\left[0,1\right]$ such that $\left(1-\lambda^*,\lambda^*\right)^t$ is orthogonal to the segment $\left[Q_{i-1}',Q_i'\right]$ (see Theorem 4.4);

(B) Solve $\left(P\left(\lambda^*\right)\right)$, compute $Q_{\lambda^*}\coloneqq\left(z_1^{\lambda^*},z_2^{\lambda^*}\right)\in\mathcal{E}_c$ with $\varphi\left(\lambda^*\right)=\varphi_{Q_{\lambda^*}}\left(\lambda^*\right)$;

(C) Update the list of points $\left\{Q_i'\right\}_{i=0}^{I}$ and their intervals $\left\{\Lambda'\left(Q_i'\right)=\left[\underline{\lambda}_i',\overline{\lambda}_i'\right]\right\}_{i=0}^{I}$:

I) Modification of the intervals. If $\varphi\left(\lambda^*\right)=\varphi_{Q_{i-1}'Q_i'}^*$ then all the segment $\left[Q_{i-1}',Q_i'\right]$ is in $\mathcal{E}_c$ (see Theorem 4.5), and $\varphi\left(\lambda\right)$ is defined on $\left[\overline{\lambda}_{i-1}',\underline{\lambda}_i'\right]$ by

(see Theorems 4.1 and 4.5)

$$\varphi(\lambda)=\begin{cases}\varphi_{Q'_{i-1}}(\lambda) & \text{for } \lambda\in\left[\underline{\lambda}'_{i-1},\lambda^*\right]\\ \varphi_{Q'_i}(\lambda) & \text{for } \lambda\in\left[\lambda^*,\overline{\lambda}'_i\right].\end{cases}$$

We modify as follow:

a) for Q'_{i-1}: $\Lambda'(Q'_{i-1})=\left[\underline{\lambda}'_{i-1},\overline{\lambda}'_{i-1}\right]:=\left[\underline{\lambda}'_{i-1},\lambda^*\right]$;

b) for Q'_i: $\Lambda'(Q'_i)=\left[\underline{\lambda}'_i,\overline{\lambda}'_i\right]:=\left[\lambda^*,\overline{\lambda}'_i\right]$;

In the sequel no more point will be generated on $\left[Q'_{i-1},Q'_i\right]\subseteq\mathcal{E}_c$.

II) Point insertion and interval modification. If $\varphi(\lambda^*)<\varphi^*_{Q'_{i-1}Q'_i}$ then $Q_{\lambda^*}\notin\left[Q'_{i-1},Q'_i\right]$, insert the point and modify intervals as follows (see Theorem 4.6):

a) Insert Q_{λ^*} between Q'_{i-1} and Q'_i in the list with $\Lambda'(Q_{\lambda^*})=\left[\underline{\lambda}^*,\overline{\lambda}^*\right]:=\left[\lambda^*,\lambda^*\right]$;

b) Set $I:=I+1$;

c) If $\varphi_{Q_{\lambda^*}}(\overline{\lambda}'_{i-1})=\varphi_{Q'_{i-1}}(\overline{\lambda}'_{i-1})=\varphi(\overline{\lambda}'_{i-1})$, then $\left[Q'_{i-1},Q_{\lambda^*}\right]\subseteq\mathcal{E}_c$ and any $\lambda\in\left[\overline{\lambda}'_{i-1},\lambda^*\right]$ is in $\Lambda'(Q_{\lambda^*})$, hence we modify $\Lambda'(Q_{\lambda^*})$ by setting $\overline{\lambda}^*:=\overline{\lambda}'_{i-1}$;

d) If $\varphi_{Q_{\lambda^*}}(\underline{\lambda}'_i)=\varphi_{Q'_i}(\underline{\lambda}'_i)=\varphi(\underline{\lambda}'_i)$, then $\left[Q_{\lambda^*},Q'_i\right]\subseteq\mathcal{E}_c$ and any $\lambda\in\left[\lambda^*,\underline{\lambda}'_i\right]$ is in $\Lambda'(Q_{\lambda^*})$, hence we modify $\Lambda'(Q_{\lambda^*})$ by setting $\overline{\lambda}^*:=\underline{\lambda}'_i$.

STEP 2. For any i such that $\overline{\lambda}'_i-\underline{\lambda}'_i=0$, remove Q'_i from the list and set $I:=I-1$.

STEP 3. End of the process (and $I=L$). The output is the list $\left\{Q_l;\left[\underline{\lambda}_l,\overline{\lambda}_l\right]\right\}_{l=0}^{L}$.

Let us observe that this process use only optimal solutions of $(P(\lambda))$, optimal values of the decision variables, which is easily obtained from any elementary linear program solver.

Remark 4.8. *This algorithm produces at each iteration an inner and an outer approximation. The inner approximation is the polygonal line joining the* Q'_i *for* $i=0,\cdots,I$. *The outer approximation is the polygonal line joining the points* Q'_0, $\tilde{Q}\left(\overline{\lambda}'_0,\underline{\lambda}'_1\right)$, Q'_1, $\tilde{Q}\left(\overline{\lambda}'_1,\underline{\lambda}'_2\right)$, Q'_2, $\cdots$, Q'_{I-2}, $\tilde{Q}\left(\overline{\lambda}'_{I-2},\underline{\lambda}'_{I-1}\right)$, Q'_{I-1}, $\tilde{Q}\left(\overline{\lambda}'_{I-1},\underline{\lambda}'_I\right)$, Q'_I, *as long as the* $\tilde{Q}\left(\overline{\lambda}'_{i-1},\underline{\lambda}'_i\right)$*'s are well determined* (*see Theorem* 4.7). *At the end of the algorithm the two approximations agree.*

4.3. Complexity

In this section we are going to determine the maximum number of calls to a linear program solver to completely determine the Pareto set, or equivalently its $L+1$ efficient extreme points $\{Q_l\}_{l=0}^{L}$. The result is given in the last theorem of this section and says that it takes at most $2L+3$ calls to a linear program solver to generates the $L+1$ extreme points $\{Q_l\}_{l=0}^{L}$.

We will use the following ordering on $\mathcal{E}_c^f$. For any two distinct points Q'_i and Q'_j on $\mathcal{E}_c^f$, we will say that Q'_i precedes Q'_j on $\mathcal{E}_c^f$, or equivalently that Q'_j follows Q'_i on $\mathcal{E}_c^f$, if moving from on $\mathcal{E}_c^f$ in the direction from

$\tilde{Q}_0$ to $\tilde{Q}_L$ we move from Q'_i to Q'_j. We will note $Q'_i <_\varepsilon Q'_j$ or equivalently $Q'_j >_\varepsilon Q'_i$.

Theorem 4.9. *The algorithm generates at most* 3 *points on* $[Q_{l-1}, Q_l]$ *on* $\mathcal{E}_c$ *and two of these points are* Q_{l-1} *and* Q_l.

Proof. Let us remark that the algorithm will eventually find a point in $[Q_{l-1}, Q_l]$ for any $l = 1, \cdots, L$. Let Q^1 be the first point generated by the algorithm in $[Q_{l-1}, Q_l]$. This first point can be generated at STEP 0, an initial point, if $\tilde{Q}_0 = Q_0 \in [Q_0, Q_1]$ for $l = 1$ or $\tilde{Q}_L = Q_L \in [Q_{L-1}, Q_L]$ for $l = L$. Otherwise, it is generated through STEP 1-C-II, with $Q'_{i-1} <_\varepsilon Q_{l-1}$ and $Q_l <_\varepsilon Q'_i$. Then this point is included in the list, and there are three cases to study:

1) $Q^1 = Q_{l-1} = Q_{\lambda^*}$ for a $\lambda^* \in \Lambda(Q_{l-1}) = \left[\underline{\lambda}_{l-1} = \lambda_{l-2,l-1}, \overline{\lambda}_{l-1} = \lambda_{l-1,l}\right]$ and we have $\Lambda'(Q^1) = \left[\lambda^{**}, \lambda^*\right]$ with $\lambda^{**} \le \lambda^*$. We will have $\lambda^{**} = \lambda^*$, or $\lambda^{**} = \lambda_{l-2,l-1}$ if the lower bound is modified through STEP 1-C-II-c (if $Q'_{i-1} \in [Q_{l-2}, Q_{l-1})$ and $\lambda_{l-2,l-1} \in \Lambda'(Q'_{i-1})$).

2) $Q^1 = Q_l = Q_{\lambda^*}$ for a $\lambda^* \in \Lambda(Q_l) = \left[\underline{\lambda}_l = \lambda_{l-1,l}, \overline{\lambda}_l = \lambda_{l,l+1}\right]$ and we have $\Lambda'(Q^1) = \left[\lambda^*, \lambda^{**}\right]$ with $\lambda^* \le \lambda^{**}$. We will have $\lambda^{**} = \lambda^*$, or $\lambda^{**} = \lambda_{l,l+1}$ if the upper bound is modified through STEP 1-C-II-d (if $Q'_i \in (Q_l, Q_{l+1}]$ and $\lambda_{l,l+1} \in \Lambda'(Q'_i)$).

3) $Q^1 \in (Q_{l-1}, Q_l)$ for $\lambda^* = \lambda_{l-1,l}$ and we have $\Lambda'(Q^1) = \left[\lambda^*, \lambda^*\right]$.

Let Q^2 be the second point generated by the algorithm in $[Q_{l-1}, Q_l]$. Q^1 must be one of the two points used to generate Q^2, and hence $\lambda^* \ne \lambda_{l-1,l}$. This point Q^2 is generated through STEP 1-C-II, and it is included in the list. There are two cases to study:

1) $Q'_{i-1} <_\varepsilon Q_{l-1}$ and $Q'_i = Q^1 \in (Q_{l-1}, Q_l]$, we will have $\lambda^* < \lambda_{l-1,l}$ and $\lambda^* \in \left[\underline{\lambda}_{l-1}, \overline{\lambda}_{l-1} = \lambda_{l-1,l}\right) \subset \Lambda(Q_{l-1})$. Consequently $Q^2 = Q_{\lambda^*} = Q_{l-1}$ and $\Lambda'(Q^2) = \left[\lambda^{**}, \lambda^*\right]$ with λ^{**} modified as in the preceding case. Moreover if $\lambda_{l-1,l} \in \Lambda'(Q^1)$ we will modify the upper bound to get $\Lambda'(Q^2) = \left[\lambda^{**}, \lambda_{l-1,l}\right]$.

2) $Q'_{i-1} = Q^1 \in [Q_{l-1}, Q_l)$ and $Q'_i >_\varepsilon Q_l$, we will have $\lambda^* > \lambda_{l-1,l}$ and $\lambda^* \in \left(\underline{\lambda}_l = \lambda_{l-1,l}, \overline{\lambda}_l\right] \subset \Lambda(Q_l)$. Consequently $Q^2 = Q_{\lambda^*} = Q_l$ and $\Lambda'(Q^2) = \left[\lambda^*, \lambda^{**}\right]$ with λ^{**} modified as in the preceding case. Moreover if $\lambda_{l-1,l} \in \Lambda'(Q^1)$ we will modify the lower bound to get $\Lambda'(Q^2) = \left[\lambda_{l-1,l}, \lambda^{**}\right]$.

Two points of $[Q_{l-1}, Q_l]$ are now in the list. We can have a point $Q' \in (Q_{l-1}, Q_l)$ with $\Lambda'(Q') = \left[\lambda_{l-1,l}, \lambda_{l-1,l}\right]$ or an extreme point Q_{l-1}, with $\Lambda'(Q_{l-1}) = \left[\underline{\lambda}'_{l-1}, \lambda_{l-1,l}\right]$, or Q_l, with $\Lambda'(Q_l) = \left[\lambda_{l-1,l}, \overline{\lambda}'_l\right]$. Otherwise the two points are the extreme points Q_{l-1} and Q_l. In that case, if $Q'_{i-1} = Q_{l-1}$ and $Q'_i = Q_l$, it can happen that $\underline{\lambda}'_i - \overline{\lambda}'_{i-1} = \lambda_{l-1,l} - \lambda_{l-1,l} = 0$ and we will have terminated with the interval $[Q_{l-1}, Q_l]$. Otherwise let us note Q^3 the third point generated in $[Q_{l-1}, Q_l]$. There are two cases to study:

1) We have only one extreme point Q_{l-1}, or Q_l, of the segment in the list and $Q' \in (Q_{l-1}, Q_l)$. As in the preceding paragraph, we will introduce it in the list, and depending of the case, by passing through STEP 1-C-II, $Q^3 = Q_{l-1}$ if $Q'_{i-1} <_\varepsilon Q_{l-1}$ and $Q'_i = Q'$, or $Q^3 = Q_l$ if $Q'_{i-1} = Q'$ and $Q'_i >_\varepsilon Q_l$. Moreover,

we will have respectively $\Lambda'(Q_{l-1}) = \left[\underline{\lambda}'_{l-1}, \lambda_{l-1,l}\right] \subseteq \Lambda(Q_{l-1})$ or $\Lambda'(Q_l) = \left[\lambda_{l-1,l}, \overline{\lambda}'_l\right] \subseteq \Lambda(Q_l)$.

2) We already have two extreme points Q_{l-1} and Q_l in the list. In that case $Q'_{i-1} = Q_{l-1}$ and $Q'_i = Q_l$ and we will have $\lambda^* = \lambda_{l-1,l}$ and $Q_{\lambda^*} \in [Q_{l-1}, Q_l]$. We pass through STEP 1-C-I and we modify the intervals to get $\Lambda'(Q_{l-1}) = \left[\underline{\lambda}'_{l-1}, \lambda_{l-1,l}\right] \subseteq \Lambda(Q_{l-1})$ et $\Lambda'(Q_l) = \left[\lambda_{l-1,l}, \overline{\lambda}'_l\right] \subseteq \Lambda(Q_l)$. Since $\overline{\lambda}'_{l-1} = \lambda_{l-1,l} = \underline{\lambda}'_l$, $Q^3 = Q_{\lambda^*}$ is not added to the list.

In the sequel, the algorithm generate no more point on $[Q_{l-1}, Q_l]$ because if we have two points $Q'_{i-1} = Q_{l-1}$ and $Q'_i = Q_l$ we have $\underline{\lambda}'_i - \overline{\lambda}'_{i-1} = \lambda_{l-1,l} - \lambda_{l-1,l} = 0$, or else, if we have three points, $Q'_{i-1} = Q_{l-1}$ and $Q'_i \in (Q_{l-1}, Q_l)$ and $Q'_{i+1} = Q_l$ we have $\underline{\lambda}'_i - \overline{\lambda}'_{i-1} = \lambda_{l-1,l} - \lambda_{l-1,l} = 0$ and $\underline{\lambda}'_{i+1} - \overline{\lambda}'_i = \lambda_{l-1,l} - \lambda_{l-1,l} = 0$.

Theorem 4.10. *If* $\tilde{Q}_0 <_\varepsilon Q_0$, *respectively* $\tilde{Q}_L >_\varepsilon Q_L$, *then* $\tilde{Q}_0$, *respectively* $\tilde{Q}_L$, *is eventually removed of the list without any supplementary call to the linear program solver.*

Proof. When Q_0 is introduced in the list, there is no supplementary call for $\left[\tilde{Q}_0, Q_0\right]$. Similarly for the interval $\left[Q_L, \tilde{Q}_L\right]$ when Q_L is introduced in the list. The points $\tilde{Q}_0$ and $\tilde{Q}_L$ are removed from the list at STEP 2 since $\Lambda'\left(\tilde{Q}_0\right) = [0,0]$ and $\Lambda'\left(\tilde{Q}_L\right) = [1,1]$.

Theorem 4.11. *The algorithm generates the extreme points* $\{Q_l\}_{l=0}^{L}$ *of the Pareto set in at most* $2L+3$ *calls to a linear program solver.*

Proof. The initialization STEP 0 requires 2 calls. For STEP 1, as we generate the Q_l for $l = 0, \cdots, L$ and possibly one supplementary call for each segment $[Q_{l-1}, Q_l]$ for $l = 1, \cdots, L$, there is at most $2L+1$ calls. Hence the algorithm requires at most $2L+3$ calls.

5. A Real World Application: Pig Diet Formulation

To illustrate our method of computation of the Pareto set we consider the pig diet formulation problem taking into account not only the cost of the diet but also environmental considerations, such as the reduction of nitrogen or phosphorus excretions. One way to analyze this problem is to rewrite the problem as bicriteria problem. Hence the Pareto set indicates the effect of the reduction of excretions, nitrogen or phosphorus, on the cost of the diet. This information is certainly useful for a decision maker which have to choose a diet which decrease the excretions without being too expensive [1]. Even if in thispaper we describe the problem for the swine industry, the method could be applied to any monogastric animal: pig, rabbit, chicken, etc.

5.1. Classical Model

The least cost diet problem, introduced in [18], is a classical linear programming problem [19] [20] [21]. A decision variable x_j is assigned to each ingredient and represents the amount (in kg) of the j^{th} ingredient per unit weight (1 kg) of the feed. Together, they form the decision vector $x = (x_j)_{j=1}^{n}$ in our model. The model's objective function is the diet cost. A vector of unit costs $c = (c_j)_{j=1}^{n}$ is

used, where each c_j represents the unit cost of the j^{th} ingredient (euro/kg or \$/kg). Thus the total cost of a unit of weight (1 kg) of diet $x=\left(x_j\right)_{j=1}^n$ is $z=cx=\sum_{j=1}^n c_j x_j$ which must be minimized over the set of feasible diets denoted by $\mathcal{S}$. The classic least cost animal diet formulation model is:

$$\left(P_{\text{diet}}\right)\begin{cases}\min z=cx\\ \text{subject to}\\ x\in\mathcal{S}=\left\{x\in\mathbb{R}^n \mid Ax\begin{matrix}\leqq\\ \geqq\end{matrix}b \text{ et } x\geq 0\right\}.\end{cases}$$

The constraints impose some bounds on the quantity of the different ingredients in the diet. For example a unit of feed is produced (a 1 kg mix), expressed by the constraint $\sum_{j=1}^n x_j=1$. Some ingredients, or combinations of ingredients, can be imposed on the diet. These restrictions give rise to equality constraints (=) or inequality constraints (≥ or ≤). More specifically, to satisfy protein requirements, the following constraints are introduced for the *L* groups of amino acids contained in the ingredients. We set

$$\sum_{j=1}^n aa_{lj}^{dig}x_j\geq b_l^* \quad (l=1,\cdots,L)$$

where aa_{lj}^{dig} represents the amount of digestible amino acid *l* contained in a unit of ingredient *j* and b_l^* is the minimum amount of digestible amino acid *l* required. Finally, the diet must satisfy the digestible phosphorus requirements b_{ph}^* given by

$$\sum_{j=1}^n ph_j^{dig}x_j\geq b_{ph}^*$$

where ph_j^{dig} is the amount of digestible phosphorus contained in a unit of ingredient *j*.

5.2. Modelling of Nitrogen and Phosphorus Excretions

Nitrogen and phosphorus excretions are directly related to the excess of amounts of protein (amino acids) and phosphorus in the diet. Hence, we have to establish the protein and the phosphorus contents of the diet and take into account the parts that are actually assimilated.

The protein content of a diet $x=\left(x_j\right)_{j=1}^n$ is $q_{pr}x=\sum_{j=1}^n pr_j x_j$, where pr_j is the amount of protein per unit of ingredient *j*. The total excretion of protein $r_{pr}\left(x\right)$ is then given by the amount in protein of the diet from which we remove the amount of protein effectively digested given by $\sum_{l=1}^L b_l^*=b_{pr}^*$, then

$$r_{pr}\left(x\right)=q_{pr}x-b_{pr}^*.$$

Hence decreasing the total excretion $r_{pr}\left(x\right)$ is equivalent to decrease the protein content $q_{pr}x$ of the diet while maintained fixed the needs b_{pr}^* in protein.

As for the nitrogen, the amount of phosphorus of a unit weight diet $x=\left(x_j\right)_{j=1}^n$ is $q_{ph}x=\sum_{j=1}^n ph_j x_j$, where ph_j is the amount of phosphorus per

unit of ingredient *j*. The amount b_{ph}^{*} is the the amount of phosphorus which is actually digested. In this way the phosphorus excretion $r_{ph}(x)$ is given by the phosphorus content of the diet from which we remove the amount of phosphorus which is actually digested

$$r_{ph}(x) = q_{ph}x - b_{ph}^{*}.$$

Hence, decreasing the phosphorus excretion $r_{ph}(x)$ is equivalent to decreasing the phosphorus content $q_{pr}x$ of the diet while maintained fixed the needs b_{ph}^{*} in phosphorus.

5.3. Data

The ingredients and their corresponding variables are described in Table 1. Table 2 contains the entire model together with the values of the technical coefficients of the model.

5.4. Software

The algorithm was programmed in MATLAB, which includes in its standard library the linear program solver called Linprog. This software can use the simplex method or an interior point method.

5.5. Two Criteria Models and Results

At first we analyse the relation between the cost of the diet and the two different excretions (nitrogen and phosphorus). As a curiosity, we also consider the interactions between the two kind of excretions: nitrogen and phosphorus.

Table 1. List of available ingredients.

Type	Ingredient	Variable
Cereals	Oats	x_1
	Hard wheat	x_2
	Corn	x_3
	Barley	x_4
Oleaginous	Soybean meal	x_5
	Colza meal	x_6
Animal byproducts	Meat and bones meal	x_7
	Animal fat	x_8
Minerals	Dicalcique phosphate	x_9
	Calcium carbonate	x_{10}
	Sodium chloride	x_{11}
Synthetic amino acids	L-lysine	x_{12}
	DL-methione	x_{13}
	L-threonine	x_{14}
	L-tryptophane	x_{15}
Premix	Fixed quantity 5 g/kg	x_{16}

Table 2. Technical coefficients of the model: $b_{pr}^{*} = 134.04$ g/kg and $b_{ph}^{*} = 2.84$ g/kg.

	INGREDIENTS																	
	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}	x_{16}	type	b
	MATRIX A																	
Total weight	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	=	1
Energy	2820	3310	3390	3070	3520	2760	2695	8281.25	0	0	0	4780	5640	4120	6570	0	≥	3400 kcal/kg
Sodium	0.2	0.1	0.04	0.1	0.3	0.4	8	0	1.8	0.8	395	0	0	0	0	0	≥	1.29 g/kg
	0.2	0.1	0.04	0.1	0.3	0.4	8	0	1.8	0.8	395	0	0	0	0	0	≤	2.5 g/kg
Calcium	0.9	0.8	0.4	0.7	3.4	8.3	76	0	220	385	3	0	0	0	0	0	≥	6.66 g/kg
Fixed ingredient	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	=	0.005 kg/kg
Restricted	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	≤	0.400 kg/kg
Ingredients	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	≤	0.030 kg/kg
	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	≤	0.050 kg/kg
	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	≤	0.600 kg/kg
	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	≤	0.600 kg/kg
	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	≤	0.250 kg/kg
	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	≤	0.050 kg/kg
Digestible amino acids																		
Lysine	7	2.5	1.9	2.8	24.9	13.6	25.5	0	0	0	0	798	0	0	0	0	≥	9.73 g/kg
Thréonine	5.7	2.7	2.5	2.6	15.3	10.8	16.2	0	0	0	0	0	0	990	0	0	≥	6.07 g/kg
Méthionine	2.5	1.5	1.5	1.4	5.9	6	7	0	0	0	0	0	990	0	0	0	≥	2.63 g/kg
Méthionine + cystine	5.3	3.7	3.3	3.4	11.6	12.6	10.4	0	0	0	0	0	990	0	0	0	≥	5.53 g/kg
Tryptophane	2.2	1.1	0.4	1	5.2	3.3	2.8	0	0	0	0	0	0	0	985	0	≥	1.77 g/kg
Isoleucine	5.8	3.4	2.7	2.9	18.7	10.6	13.4	0	0	0	0	0	0	0	0	0	≥	5.31 g/kg
Valine	8.7	4	3.6	4.1	19.3	13.1	21.3	0	0	0	0	0	0	0	0	0	≥	6.61 g/kg
Leucine	10.2	6.4	9.5	5.7	29.8	18.6	31.9	0	0	0	0	0	0	0	0	0	≥	9.84 g/kg
Phenylalanine	7	4.5	3.7	4.1	20.6	10.9	18.4	0	0	0	0	0	0	0	0	0	≥	5.81 g/kg
Phenylalanine + tyrosine	12.1	7.1	6.8	6.5	34.5	18.8	29.6	0	0	0	0	0	0	0	0	0	≥	9.12 g/kg
Histidine	4.3	2.2	2.1	1.8	10.9	7.4	9.5	0	0	0	0	0	0	0	0	0	≥	3.11 g/kg
Arginine	10.7	4.7	3.5	4	31.6	17.6	31	0	0	0	0	0	0	0	0	0	≥	4.55 g/kg
Non essential amino acids	69	54.7	36.3	46.6	206.1	144.9	254.4	0	0	0	0	0	0	0	0	0	≥	72.4 g/kg
Digestible phosphorus	3.44	1.7	0.364	1.02	1.922	2.394	31.4	0	180.4	0.2	0	0	0	0	0	0	≥	2.84 g/kg
	TECHNICAL COEFFICIENTS FOR INGREDIENTS FOR EACH CRITERION																	
Prices ($/kg)	0.242	0.321	0.32	0.288	0.455	0.317	0.532	1.236	0.906	0.08	0.198	2.808	5.75	3.75	57	5.269		
Nitrogen excretions (g/kg)	170	105	81	101	453	337	540	0	0	0	0	798	990	990	985	0		
Phosphorus excretions (g/kg)	8.4	3.4	2.6	3.4	6.2	11.4	38.8	0	185	0.2	0	0	0	0	0	0		

5.5.1. Cost and Excretions

We have considered two separate bicriteria models. We look for least cost diets while taking into account the nitrogen excretion for the first model and the phosphorus excretion for the second model. For each of these two bicriteria problems, the Pareto curve indicates the diet cost increase caused by an excretion decrease.

While considering the nitrogen excretion, the problem is :

$$\left(P_{c,pr}\right)\begin{cases}\min z_1 = cx \\ \min z_2 = q_{pr}x \\ \text{subject to} \\ x \in \mathcal{S}.\end{cases}$$

Table 3 presents the set of efficient extreme points of the Pareto set in the criterion space, and the Pareto curve is sketched in Figure 1. For this problem, the algorithm detects $L=10$ segments and 11 efficient extreme points

$$Q_l = \left(z_{l,1}, z_{l,2}\right) = \left(z_{l,\text{cost}}, z_{l,\text{nitrogen excretion}}\right)$$

for $l=0,\cdots,L=10$. A total of 22 calls to the linear program solver was required (the predicted maximum is $2L+3=23$).

From its associated weighted-sum model given by

$$\left(P_{c,pr}(\lambda)\right)\begin{cases}\min z(x;\lambda) = (1-\lambda)z_1(x) + \lambda z_2(x) = \left[(1-\lambda)c + \lambda q_{pr}\right]x \\ \text{subject to} \\ x \in \mathcal{S},\end{cases}$$

Table 3. Efficient extreme points in the criterion space $\mathbb{R}^2$ and the corresponding taxes for $\left(P(c,pr)\right)$. and the corresponding taxes.

	Pareto set					Taxation system	
l	$\underline{\lambda}_l$	$\overline{\lambda}_l$	$z_{l,\text{cost}}$ \$/kg	$z_{l,\text{nitrogen excretion}}$ g/kg	$z_{l,\text{phosphorus excretion}}$ g/kg	$\underline{\mu}_l$	$\overline{\mu}_l$
0	0	0.02617	0.40062	0.19021	6.21226	0	0.02687
1	0.02617	0.13661	0.40072	0.18661	6.18977	0.02687	0.15823
2	0.13661	0.15375	0.40147	0.18184	6.15847	0.15823	0.18168
3	0.15375	0.30996	0.40292	0.17385	6.03610	0.18168	0.44920
4	0.30996	0.49911	0.40759	0.16347	5.67457	0.44920	0.99643
5	0.49911	0.76922	0.40816	0.16289	5.67073	0.99643	3.33314
6	0.76922	0.81451	0.40820	0.16288	5.66990	3.33314	4.39120
7	0.81451	0.81847	0.41580	0.16115	5.44005	4.39120	4.50866
8	0.81847	0.85167	0.41608	0.16108	5.43126	4.50866	5.74169
9	0.85167	0.99010	0.41798	0.16075	5.36141	5.74169	100.013
10	0.99010	1.00000	0.42713	0.16066	5.29463	100.013	$+\infty$

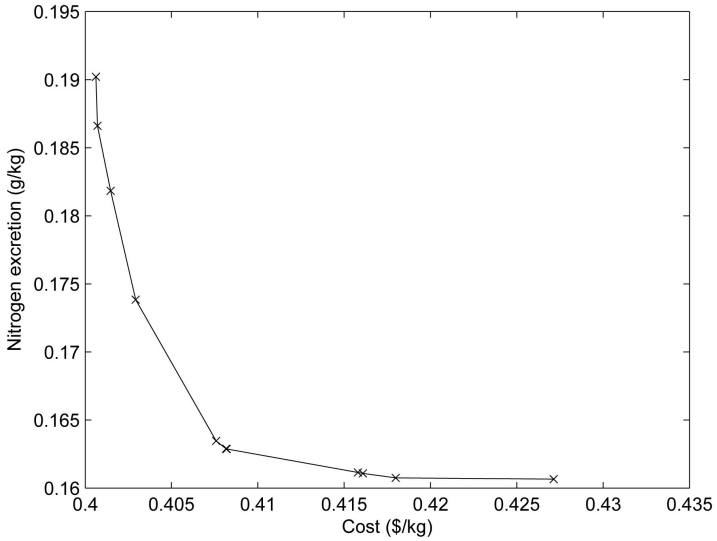

Figure 1. Pareto curve: nitrogen excretion vs diet cost.

we get the following expression for its value function $\varphi(\lambda)$, defined for $\lambda \in [0,1]$, by

$$\varphi(\lambda) = \min\{z(x;\lambda) \mid x \in \mathcal{S}\} = (1-\lambda) z_{l,\text{cost}} + \lambda z_{l,\text{nitrogen excretion}}$$

defined for $\lambda \in \left[\underline{\lambda}_l, \overline{\lambda}_l\right]$, and $l = 0, \cdots, L = 10$. So this expression depends on the interval $\left[\underline{\lambda}_l, \overline{\lambda}_l\right]$ in which λ is.

For the parametric model given by

$$\left(P_{c,pr}(\mu)\right) \begin{cases} & \min \tilde{z}(x;\mu) = z_1(x) + \mu z_2(x) = \left(c + \mu q_{ph}\right) x \\ \text{subject to} & \\ & x \in \mathcal{S}, \end{cases}$$

we get the following expression for its value function $\tilde{\varphi}(\mu)$ defined for $\mu \in [0, +\infty)$ by

$$\tilde{\varphi}(\mu) = \min\{\tilde{z}(x;\mu) \mid x \in \mathcal{S}\} = z_{l,\text{cost}} + \mu z_{l,\text{nitrogen excretion}}$$

defined for $\mu \in \left[\underline{\mu}_l, \overline{\mu}_l\right]$, and $l = 0, \cdots, L = 10$. So this expression depends on the interval $\left[\underline{\mu}_l, \overline{\mu}_l\right]$ in which μ is.

So we see that for any tax value in $\left[\underline{\mu}_l, \overline{\mu}_l\right]$ we will always have the same expression for the value function $\tilde{\varphi}(\mu)$, or the same behavior given by the efficient extreme point $Q_l = \left(z_{l,\text{cost}}, z_{l,\text{nitrogen excretion}}\right)$, and the change in the behavior will happend only when the taxation level μ passes through the extremities $\underline{\mu}_l$ or $\overline{\mu}_l$ of this interval

A similar analysis holds for the second bicriteria problem with phosphorus excretion. Indeed, for the phosphorus excretion problem, the model is:

$$\left(P_{c,ph}\right) \begin{cases} & \min z_1 = cx \\ & \min z_3 = q_{ph} x \\ \text{subject to} & \\ & x \in \mathcal{S}. \end{cases}$$

Table 4 presents the efficient extreme points in the criterion space while the Pareto curve is sketched in Figure 2. For this problem, the algorithm detects $L=22$ segments and 23 extreme points

$$Q_l=\left(z_{l,1},z_{l,3}\right)=\left(z_{l,\text{cost}},z_{l,\text{phosphorus excretion}}\right)$$

for $l=0,\cdots,L=22$. A total of 45 calls to the linear program solver was required (the predicted maximum is $2L+3=47$).

From its associated weighted-sum model given by

$$\left(P_{c,ph}(\lambda)\right)\begin{cases}\min z(x;\lambda)=(1-\lambda)z_1(x)+\lambda z_3(x)=\left[(1-\lambda)c+\lambda q_{ph}\right]x\\ \text{subject to}\\ x\in\mathcal{S},\end{cases}$$

Table 4. Efficient extreme points in the criterion space $\mathbb{R}^2$ for $\left(P(c,ph)\right)$, and the corresponding taxes.

	Pareto Set					Taxation system	
l	$\underline{\lambda}_l$	$\overline{\lambda}_l$	$z_{l,\text{cost}}$ ($/kg)	$z_{l,\text{phosphorus excretion}}$ (g/kg)	$z_{l,\text{nitrogenex cretion}}$ (g/kg)	$\underline{\mu}_l$	$\overline{\mu}_l$
0	0	0.00428	0.40062	6.21226	0.19021	0	0.00430
1	0.00428	0.00452	0.40072	6.18977	0.18661	0.00430	0.00454
2	0.00452	0.00456	0.40164	5.98711	0.18707	0.00454	0.00458
3	0.00456	0.00500	0.40196	5.91713	0.18927	0.00458	0.00502
4	0.00500	0.00528	0.40219	5.87162	0.18854	0.00502	0.00531
5	0.00528	0.00628	0.40310	5.69979	0.18742	0.00531	0.00632
6	0.00628	0.00708	0.40365	5.61223	0.18802	0.00632	0.00713
7	0.00708	0.00783	0.40379	5.59297	0.18676	0.00713	0.00789
8	0.00783	0.00919	0.40400	5.56609	0.18566	0.00789	0.00927
9	0.00919	0.01003	0.40541	5.41416	0.18737	0.00927	0.01013
10	0.01003	0.01458	0.40601	5.35505	0.18975	0.01013	0.01479
11	0.01458	0.02357	0.40633	5.33336	0.18640	0.01479	0.02414
12	0.02357	0.09694	0.40798	5.26498	0.17596	0.02414	0.10734
13	0.09694	0.11478	0.41768	5.17458	0.16744	0.10734	0.12967
14	0.11478	0.12931	0.42351	5.12967	0.16700	0.12967	0.14852
15	0.12931	0.14182	0.42429	5.12440	0.16609	0.14852	0.16526
16	0.14182	0.48610	0.43631	5.05165	0.16364	0.16526	0.94589
17	0.48610	0.49168	0.74777	4.72237	0.26885	0.94589	0.96727
18	0.49168	0.62773	0.79624	4.67226	0.28486	0.96727	1.68624
19	0.62773	0.69486	1.12394	4.47793	0.38921	1.68624	2.27723
20	0.69486	0.99962	1.30843	4.39691	0.45147	2.27723	2662.91
21	0.99962	0.99998	2.06125	4.39663	0.46634	2662.91	59645.9
22	0.99998	1.00000	15.32799	4.39641	0.42199	59645.9	$+\infty$

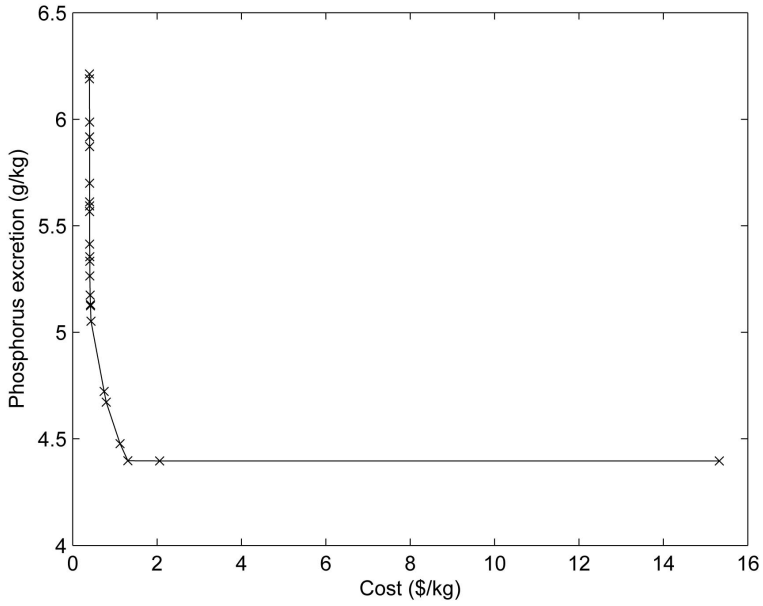

Figure 2. Pareto curve: phosphorus excretion vs diet cost.

we get the following expression for its value function $\varphi(\lambda)$ defined for $\lambda \in [0,1]$ by

$$\varphi(\lambda) = \min\{z(x;\lambda) \mid x \in \mathcal{S}\} = (1-\lambda) z_{l,\text{cost}} + \lambda z_{l,\text{phosphorus excretion}}$$

defined for $\lambda \in \left[\underline{\lambda}_l, \overline{\lambda}_l\right]$, and $l = 0, \cdots, L = 22$. So this expression depends on the interval $\left[\underline{\lambda}_l, \overline{\lambda}_l\right]$ in which λ is.

For the parametric model given by

$$\left(P_{c,ph}(\mu)\right) \begin{cases} & \min \tilde{z}(x;\mu) = z_1(x) + \mu z_3(x) = \left(c + \mu q_{ph}\right) x \\ \text{subject to} & \\ & x \in \mathcal{S}, \end{cases}$$

we get the following expression for its value function $\tilde{\varphi}(\mu)$ defined for $\mu \in [0, +\infty)$ by

$$\tilde{\varphi}(\mu) = \min\{\tilde{z}(x;\mu) \mid x \in \mathcal{S}\} = z_{l,\text{cost}} + \mu z_{l,\text{phosphorus excretion}}$$

defined for for $\mu \in \left[\underline{\mu}_l, \overline{\mu}_l\right]$, and $l = 0, \cdots, L = 22$. So this expression depends on the interval $\left[\underline{\mu}_l, \overline{\mu}_l\right]$ in which μ is.

So we see that for any tax value in $\left[\underline{\mu}_l, \overline{\mu}_l\right]$ we will always have the same expression for the value function $\tilde{\varphi}(\mu)$, or the same behavior given by the efficient extreme point $Q_l = \left(z_{l,\text{cost}}, z_{l,\text{phosphorus excretion}}\right)$, and the change in the behavior will happend only when the taxation level μ passes through the extremities $\underline{\mu}_l$ or $\overline{\mu}_l$ of this interval.

These problems of taxation are nice examples of abrupt (discrete) changes in behavior depending on the level of taxation of one criterion.

5.5.2. The Two Kinds of Excretion as Criteria

As a curiosity, we have computed the Pareto set for the bicriteria problem where

the two kinds of excretions are considered. This bicriteria problem is given by

$$\left(P_{pr,ph}\right)\begin{cases}\min z_2 = q_{pr}x \\ \min z_3 = q_{ph}x \\ \text{subject to} \\ x \in \mathcal{S}.\end{cases}$$

Table 5 presents the set of efficient extreme points of the Pareto set in the criteria space. Its corresponding Pareto curve is sketched in Figure 3. This table shows the opposite effect of trying to reduce simultaneously both excretions. Minimizing one excretion leads to an increse in the other excretion. For this problem, the algorithm detects $L=5$ segments and 6 extreme points. A total of 12 calls to the linear program solver was required (the predicted maximum is $2L+3=13$).

For each $l=0,\cdots,5$, the value function is

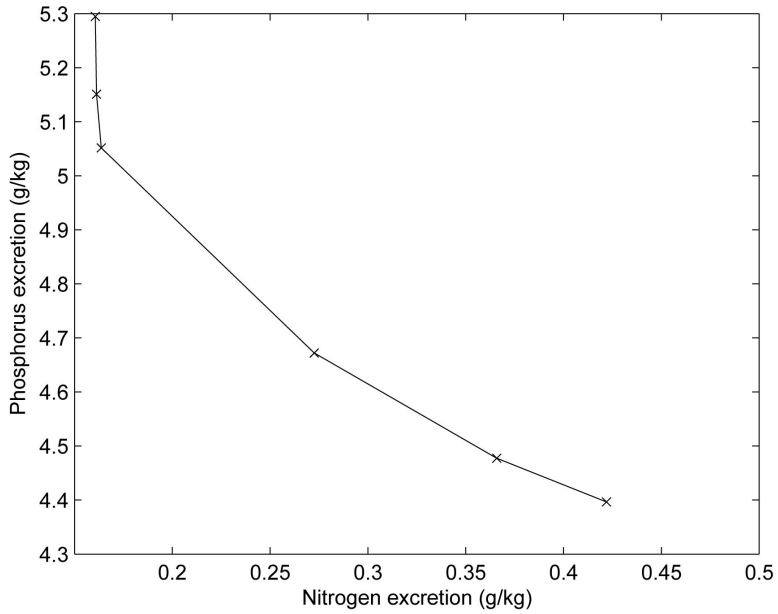

Figure 3. Pareto curve: phosphorus excretion vs nitrogen excretion.

Table 5. Efficient extreme points in the criterion space $\mathbb{R}^2$ for $\mathbb{R}^2$ pour $\left(P(pr,ph)\right)$.

Pareto set					
l	$\underline{\lambda}_l$	$\overline{\lambda}_l$	$z_{l,\text{nitrogen excretion}}$ g/kg	$z_{l,\text{phosphorus excretion}}$ g/kg	$z_{l,\text{cost}}$ \$/kg
0	0	0.00410	0.16066	5.29463	0.42713
1	0.00410	0.02350	0.16125	5.15082	0.44249
2	0.02350	0.22291	0.16364	5.05164	0.43631
3	0.22291	0.32435	0.27253	4.67205	6.65922
4	0.32435	0.40870	0.36591	4.47753	12.20160
5	0.40870	1.00000	0.42199	4.39641	15.32799

$$\varphi(\lambda) = (1-\lambda) z_{l,\text{nitrogen excretion}} + \lambda z_{l,\text{phosphorus excretion}}$$

for $\lambda \in \left[\underline{\lambda}_l, \overline{\lambda}_l\right]$. For all value of the parameter λ in the interval $\left[\underline{\lambda}_l, \overline{\lambda}_l\right]$ we will have the same expression for the value functionn $\varphi(\lambda)$ or the same behavior $\left(z_{l,\text{nitrogen excretion}}, z_{l,\text{phosphorus excretion}}\right)$ and a change in the behavior will happend for values of the parameter λ corresponding to the extremities $\underline{\lambda}_l$ ou $\overline{\lambda}_l$ of this interval.

Let us observe that the last line of **Table 3** ($l = 10$) corresponds to the first line of **Table 5** ($l = 0$) and the last line of **Table 4** ($l = 22$) corresponds to the last line of **Table 5** ($l = 5$).

6. Conclusion

In this paper we have considered bicriteria linear programming problems and have presented an elementary and efficient algorithm to compute the Pareto set in the criterion space. We have illustrated the method on a real important application. This application also suggests that it could be interresting to extend the method to three-criteria problems. Moreover it could be interesting to compare our method to other methods to find the Pareto set in the criterion space, but it is out of the scope of this paper and could be a nice subject for a future research.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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