

Numerical Treatment of Initial-Boundary Value Problems with Mixed Boundary Conditions

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Abstract

In this paper, we extend the reliable modification of the Adomian Decomposition Method coupled to the Lesnic's approach to solve boundary value problems and initial boundary value problems with mixed boundary conditions for linear and nonlinear partial differential equations. The method is applied to different forms of heat and wave equations as illustrative examples to exhibit the effectiveness of the method. The method provides the solution in a rapidly convergent series with components that can be computed iteratively. The numerical results for the illustrative examples obtained show remarkable agreement with the exact solutions. We also provide some graphical representations for clear-cut comparisons between the solutions using Maple software.

Keywords

Decomposition Method, Modified Adomian Decomposition Method, Linear and Nonlinear Partial Differential Equations, Mixed Boundary Conditions, Initial-Boundary Value Problem

1. Introduction

Mixed boundary value problems are characterized by a combination of Dirichlet and Neumann conditions along at least one boundary condition. They occur in a wide range of engineering and applied mathematics applications [1] [2] [3] [4]. These applications include the classic electrical potential and electric field conditions on a disk [5], stress and strain conditions around a punch pressing on an elastic surface [6] as well as some applications in porous media problems such as the infiltration and seepage among others [7] [8]. Historically, only very few of these problems could be solved using analytic methods. In view of this, many researchers obtained the solutions of initial and boundary value problems by

using either initial or boundary condition(s). In recent years, there have been significant developments in the use of various semi-analytical methods for partial differential equations such as the homotopy perturbation method [9] and Adomian Decomposition Method (ADM) [10]. Duan and Rach [11] developed a new resolution method of Boundary Value Problems (BVPs) for nonlinear ordinary differential equations using the ADM. It is also well-known that the ADM provides approximate analytic solutions without using the Green function concept, which greatly facilitates analytic approximations and numerical computations. Several different resolution techniques for solving BVPs for nonlinear ordinary differential equations by using the ADM were considered by Adomian and Rach [12]-[18], Adomian [19], and Wazwaz [20]-[26]. Also, for a two-point BVP for second-order nonlinear differential equations, Adomian and Rach [17] [18] proposed the double decomposition method in order to avoid solving such nonlinear algebraic equations, and Jang [27] and Ebaid [28] introduced different modified inverse linear operators. Adomian [29] suggested a modified method for the hyperbolic, parabolic and elliptic partial differential equations with initial and boundary conditions by using two equations for u , one inverting the L_t operator and the other inverting the L_x operator, then, adding them and dividing by two. Further, with regards to the mixed value problems, Lesnic and Elliot [30] proposed the inverse operator defined by $L_x^{-1} = \int_{x_0}^x \int_1^x dx' dx''$ to solve the

linear homogeneous heat equation $u_t = u_{xx}$, $x_0 < x < 1$, $t > 0$ subject to the mixed boundary conditions $u(x, t) = h_1(t)$, $u_x(1, t) = h_2(t)$, where $\beta_i = (t)$, $i = 1, 2$ are known functions. However, in this paper, we will present a modified recursion scheme based on the reliable modification of the ADM with new structure of the inverse operator applied to the (BVPs) with mixed boundary conditions using Lesnic's approach and Ebaid's method. The proposed operator allows the appearance of all the conditions in the solution thereby making the solution more realistic. The paper is arranged in the following manner: in Section 2, we analyze the ADM; Section 3 presents the modified ADM suggested by Wazwaz; in Section 4, Lesnic's approach is used to approximate solutions of some problems; the implementation of this new method to some test problems is presented in Section 5; finally, a brief conclusion is given in Section 6.

2. Analysis of the Adomian Decomposition Method with Mixed Conditions

Nonlinear partial differential equations models in mathematics and physics play an important role in theoretical sciences. The understanding of these nonlinear partial differential equations is also crucial to many applied areas such as meteorology, oceanography, and aerospace industry. Nonlinear partial differential equations are the most fundamental models in studying nonlinear phenomena.

Consider the nonlinear partial differential equation given in an operator form

$$L_x u(x, t) + L_t u(x, t) + Ru(x, t) + Nu(x, t) = g(x, t) \quad (2.1)$$

where L_x and L_t are the linear operators to be inverted, which are usually the highest order differential operators in x , and t respectively; L_x, R is the linear remainder operator; $Nu(x, t)$ is a nonlinear operator which is assumed to be analytic function, and $g(x, t)$ is the input function that is assumed to be continuous function. The solutions for $u(x, t)$ obtained from the operator equations $L_x u$ in x -direction and $L_t u$ in t -direction are called partial solutions. We further give the following illustrations:

2.1. Boundary Value Problems

Consider the general form of the single second-order nonlinear inhomogeneous temporal-spatial partial differential equation:

$$L_{xx}u(x, t) + L_t u(x, t) + Nu(x, t) = g(x, t), a \leq x \leq b, t > 0, \tag{2.2}$$

subject to the mixed boundary conditions

$$u(a, t) = h_1(t), u_x(b, t) = h_2(t) \tag{2.3}$$

where, $L_{xx}u(x, t) = \frac{\partial^2}{\partial x^2}u(x, t), L_t u(x, t) = \frac{\partial}{\partial t}u(x, t)$. We consider the x partial solution as

$$L_{xx}u(x, t) = g(x, t) - L_t u(x, t) - Nu(x, t) \tag{2.4}$$

Applying the two-fold indefinite integration inverse operator $L_{xx}^{-1} = \int dx \int dx'$ to both sides of Equation (2.4), gives

$$u(x, t) = \Phi_x + L_{xx}^{-1}g(x, t) - L_{xx}^{-1}L_t u(x, t) - L_{xx}^{-1}Nu(x, t), \tag{2.5}$$

where $\Phi_x = c_1(t) + c_2(t)x$ and the constants of integrations $c_1(t)$ and $c_2(t)$, are determined from the boundary conditions. We now decompose the following Φ_x , the linear and nonlinear terms u and Nu based on ADM as follows:

$$\begin{aligned} \Phi_x &= \sum_{n=0}^{\infty} \Phi_{x,n} = \sum_{n=0}^{\infty} (c_{1,n}(t) + c_{2,n}(t)x), \\ u(x, t) &= \sum_{n=0}^{\infty} u_n(x, t), \\ Nu(x, t) &= \sum_{n=0}^{\infty} A_n(u_0, u_1, u_2, \dots, u_n) \end{aligned} \tag{2.6}$$

where A_n 's are the Adomian polynomials determined from the definitional formula

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} N \left(\sum_{k=0}^{\infty} u_k \lambda^k \right) \Big|_{\lambda=0}, n \geq 0. \tag{2.7}$$

Substituting Equation (2.6) into Equation (2.5), yields the following recursion scheme

$$\begin{aligned} u_0 &= \Phi_{x,0} + L_{xx}^{-1}g(x, t), \\ u_{n+1} &= \Phi_{x,n+1} - L_{xx}^{-1}L_t u_n - L_{xx}^{-1}A_n, n \geq 0. \end{aligned} \tag{2.8}$$

The n -term approximation of the solution is $\varphi_n = \sum_{i=0}^{n-1} u_i(x, t)$ for $n > 0$.

Thus, $\varphi_1 = u_0, \varphi_2 = \varphi_1 + u_1, \varphi_3 = \varphi_2 + u_2, \dots$, and all φ_n 's must satisfy the boundary conditions.

The first approximate $\varphi_1 = u_0 = c_{1,0}(t) + c_{2,0}(t)x + L_{xx}^{-1}g(x,t)$, where the values $c_1(t)$ and $c_2(t)$ can be evaluated by using the boundary conditions in Equation (2.2)

$$\varphi_1|_{x=a} = h_1(t), \varphi_1'|_{x=b} = h_2(t),$$

which results in

$$\begin{aligned} c_{1,0}(t) + c_{2,0}(t)x + L_{xx}^{-1}g(x,t)|_{x=a} &= h_1(t), \\ c_{2,0}(t) + (L_{xx}^{-1}g(x,t))'|_{x=b} &= h_2(t). \end{aligned}$$

Thus φ_1 is now determined. Since u_0 and φ_1 are now completely known, we form the next term $u_1 = \Phi_{x,1} - L_{xx}^{-1}L_t u_0 - L_{xx}^{-1}A_0$, then $\varphi_2 = \varphi_1 + u_1$, and we continue in the same manner to obtain u_2, u_3, \dots, u_n for some $n \geq 0$. Substituting all these values in $u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$, we get the solution of Equation (2.1).

2.2. An Alternative Combination of the Initial and Boundary Conditions

Adomian [29] suggested a modified method for the partial differential equations with initial and boundary conditions by using two equations for u , one inverting the L_t operator and the other inverting the L_x operator, then, adding them and dividing by two. To convey the basic idea for treatment of initial and boundary conditions by ADM for solving initial boundary value problems, we consider Equations (2.2)-(2.3) with the initial conditions

$$u(x,0) = p_1(x), u_i(x,0) = p_2(x). \tag{2.9}$$

where $p_i(x), i = 1, 2$ are known functions.

Firstly, we consider the t partial solution as

$$L_t u(x,t) = g(x,t) - L_{xx} u(x,t) - Nu(x,t) \tag{2.10}$$

Applying the inverse operator L_t^{-1} defined by $L_t^{-1} = \int_0^t \int_0^t dt dt$ to both sides of Equation (2.10) and using the initial conditions, gives

$$u(x,t) = u(x,0) + t u_t(x,0) + L_t^{-1} g(x,t) - L_t^{-1} L_{xx} u(x,t) - L_t^{-1} Nu(x,t) \tag{2.11}$$

Secondly, we consider the x partial solution as in Equation (2.5). Next, we averaged the partial solutions, *i.e.* add two partial solutions in Equation (2.5) and Equation (2.11) and divide by two, we obtain

$$\begin{aligned} u(x,t) = \frac{1}{2} & \left(u(x,0) + t u_t(x,0) + \Phi_x + L_t^{-1} g(x,t) + L_{xx}^{-1} g(x,t) \right. \\ & \left. - L_t^{-1} L_{xx} u(x,t) - L_{xx}^{-1} L_t u(x,t) - L_t^{-1} Nu(x,t) - L_{xx}^{-1} Nu(x,t) \right) \end{aligned} \tag{2.12}$$

Substituting Equation (2.6) into Equation (2.12), yields the following recursion scheme

$$u_0 = \frac{1}{2} \left(u(x, 0) + tu_t(x, 0) + \Phi_{x,0} + L_u^{-1}g(x, t) + L_{xx}^{-1}g(x, t) \right)$$

$$u_{n+1} = \frac{1}{2} \left(\Phi_{x,n} - L_u^{-1}L_{xx}u_n(x, t) - L_{xx}^{-1}L_uu_n(x, t) - L_u^{-1}A_n - L_{xx}^{-1}A_n \right), n \geq 0. \quad (2.13)$$

To illustrate this method for coupled linear and nonlinear partial differential equations, we take two examples in the following section.

2.3. Numerical Experiments

Example 1.

Consider the linear homogeneous heat equation

$$u_t - u_{xx} = 0, 0 \leq x \leq 1, t > 0,$$

with specified conditions $u(0, t) = 0, u_x(1, t) = \cos(1)e^{-t}$.

Rewriting the heat equation in the operator form as $L_tu - L_{xx}u = 0$.

Applying the inverse operator L_{xx}^{-1} defined by $L_{xx}^{-1} = \int dx \int dx'$, gives

$$u(x, t) = c_1(t) + c_2(t)x + L_{xx}^{-1}L_tu(x, t).$$

This introduces the recursive relations

$$u_0 = c_1(t) + c_2(t)x,$$

$$u_{n+1} = c_{1,n+1}(t) + c_{2,n+1}(t)x + L_{xx}^{-1}L_tu_n, n \geq 0.$$

The first approximant is $\varphi_1 = u_0 = c_{1,0}(t) + c_{2,0}(t)x$. Applying the x conditions to φ_1 , it is clear that $c_{1,0}(t) = 0$ and $c_{2,0}(t) = \cos(1)e^{-t}$. Thus, if the one-term approximant φ_1 were sufficient, the solution would be $\varphi_1 = u_0 = \cos(1)e^{-t}x$. The next term is

$$u_1 = c_{1,1}(t) + c_{2,1}(t)x + L_{xx}^{-1}L_tu_0$$

$$= c_{1,1}(t) + c_{2,1}(t)x + L_{xx}^{-1}L_t(\cos(1)e^{-t}x)$$

Then $\varphi_2 = u_0 + u_1$ is given by

$$\varphi_2 = \cos(1)e^{-t}x + c_{1,1}(t) + c_{2,1}(t)x - \frac{1}{6}\cos(1)e^{-t}x^3$$

Applying the condition at $x = 0$, we have $c_{1,1}(t) = 0$. From the condition on x at 1, we get

$$c_{2,1}(t) = \frac{1}{2}\cos(1)e^{-t}, \text{ thus } u_1 = \frac{1}{2}\cos(1)e^{-t}x - \frac{1}{6}\cos(1)e^{-t}x^3$$

Continuing in a similar way u_2, u_3, \dots, u_n are obtained for some n , then we get the approximate solution $u_{ap} = \sum_{n=0} u_n(x, t)$ which converged to the exact solution $u_{ex} = \cos(x)e^{-t}$.

Table 1 shows the comparison between the absolute error of the exact and approximate solutions for various values of t . **Figure 1** and **Figure 2** give the plots for the exact and approximate solutions by using ADM for $t = 0.5, t = 1.0$ and $0 \leq x \leq 1$, respectively.

Table 1. Absolute errors using ADM at $t = 0.5, t = 1.0$ and $0 \leq x \leq 1$.

x	$T = 0.5$			$T = 1.0$		
	φ_3	φ_5	φ_{10}	φ_3	φ_5	φ_{10}
0.0	0.00000000e+00	0.00000000e+00	0.00000000e+00	0.00000000e+00	0.00000000e+00	0.00000000e+00
0.1	4.65015125e-03	7.64016845e-04	8.35421477e-06	2.82045931e-03	4.63399641e-04	5.06708740e-06
0.2	9.18603225e-03	1.50922156e-03	1.65027211e-05	5.57161020e-03	9.15389147e-04	1.00094063e-05
0.3	1.34961408e-02	2.21726512e-03	2.42448756e-05	8.18582316e-03	1.34483927e-03	1.47052604e-05
0.4	1.74744502e-02	2.87071334e-03	3.13900408e-05	1.05987898e-02	1.74117565e-03	1.90390221e-05
0.5	2.10229970e-02	3.45347612e-03	3.77622790e-05	1.27510922e-02	2.09463915e-03	2.29039800e-05
0.6	2.40542893e-02	3.95120365e-03	4.32046846e-05	1.45896640e-02	2.39652615e-03	2.62049659e-05
0.7	2.64934801e-02	4.35163977e-03	4.75832475e-05	1.60691079e-02	2.63940294e-03	2.88606985e-05
0.8	2.82802497e-02	4.64492392e-03	5.07901529e-05	1.71528385e-02	2.81728877e-03	3.08057849e-05
0.9	2.93703477e-02	4.82383397e-03	5.27464362e-05	1.78140164e-02	2.92580320e-03	3.19923307e-05
1.0	2.97367443e-02	4.88396424e-03	5.34039271e-05	1.80362471e-02	2.96227405e-03	3.23911192e-05

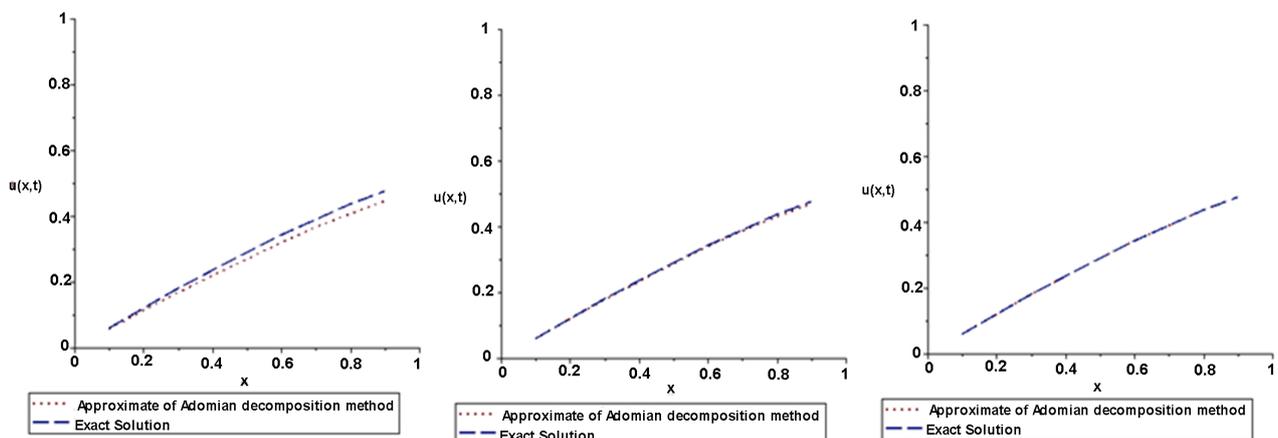


Figure 1. The exact solution and the approximate solution using ADM for $\varphi_3, \varphi_5, \varphi_{10}$ at $t = 0.5$ and $0 \leq x \leq 1$.

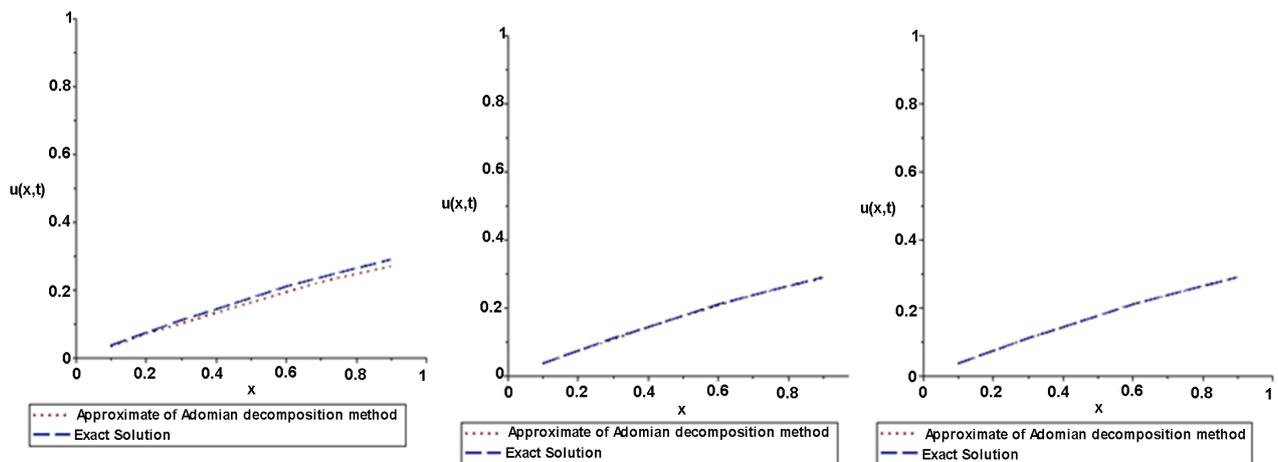


Figure 2. The exact solution and the approximate solution using ADM for $\varphi_3, \varphi_5, \varphi_{10}$ at $t = 1.0$ and $0 \leq x \leq 1$.

Example 2

Consider the nonlinear inhomogeneous wave equation

$$u_{tt} - u_{xx} - u + u^2 = xt + x^2t^2, t > 0,$$

with specified initial conditions $u(x, 0) = 1, u_t(x, 0) = x$, and the boundary conditions $u(0, t) = 1, u_x(0, t) = t$. Rewriting the wave equation in the operator form as

$$L_{tt}u - L_{xx}u - u + u^2 = xt + x^2t^2.$$

To solve initial-boundary value problem, firstly, we consider the t partial solution as $L_{tt}u = L_{xx}u + u - u^2 + xt + x^2t^2$.

Applying the inverse operator L_{tt}^{-1} defined by $L_{tt}^{-1} = \int_0^t \int_0^t dt dt$, gives

$$u(x, t) = u(x, 0) + tu_t(x, 0) + L_{tt}^{-1}(xt + x^2t^2) + L_{tt}^{-1}L_{xx}u(x, t) + L_{tt}^{-1}u - L_{tt}^{-1}u^2.$$

This introduces the recursive relations

$$u_0 = u(x, 0) + tu_t(x, 0) + L_{tt}^{-1}(xt + x^2t^2)$$

$$u_{n+1} = L_{tt}^{-1}L_{xx}u_n + L_{tt}^{-1}u_n - L_{tt}^{-1}A_n, n \geq 0.$$

where the nonlinear term u^2 can be expressed by an infinite series of the Adomian polynomials A_n given by:

$$A_0 = (u_0)^2,$$

$$A_1 = 2u_0u_1,$$

$$A_2 = (u_1)^2 + 2u_0u_2,$$

$$A_3 = 2u_1u_2 + 2u_0u_3,$$

$$A_4 = (u_2)^2 + 2u_1u_3 + 2u_0u_4,$$

⋮

So that first two terms are

$$u_0 = 1 + xt + \frac{1}{6}xt^3 + \frac{1}{12}x^2t^4,$$

$$u_1 = \frac{1}{180}t^6 - \frac{1}{12960}x^4t^{10} - \frac{1}{2529}x^3t^9 - \frac{1}{2016}x^2t^8 - \frac{1}{252}x^3t^7 - \frac{1}{72}x^2t^6 - \frac{1}{120}xt^5 - \frac{1}{12}x^2t^4 - \frac{1}{6}xt^3,$$

Secondly, we consider the x partial solution $L_{xx}u = L_{tt}u - u + u^2 - xt - x^2t^2$.

Applying the inverse operator L_{xx}^{-1} defined by $L_{xx}^{-1} = \int dx \int dx'$, gives

$$u(x, t) = \Phi_x + L_{xx}^{-1}(-xt - x^2t^2) + L_{xx}^{-1}L_{tt}u(x, t) - L_{xx}^{-1}u(x, t) + L_{xx}^{-1}u^2(x, t),$$

This gives the recursive relations

$$u_0 = \Phi_{x,0} + L_{xx}^{-1}(-xt - x^2t^2),$$

$$u_{n+1} = \Phi_{x,n} + L_{xx}^{-1}L_{tt}u_n - L_{xx}^{-1}u_n + L_{xx}^{-1}A_n, n \geq 0.$$

So that

$$\begin{aligned}
 u_0 &= c_{1,0}(t) + c_{2,0}(t)x + L_{xx}^{-1}(-xt - x^2t^2) \\
 &= c_{1,0}(t) + c_{2,0}(t)x - \frac{1}{12}x^4t^2 - \frac{1}{6}x^3t,
 \end{aligned}$$

The first approximant is $\varphi_1 = u_0$. Applying the x conditions to φ_1 it is clear that $c_{1,0}(t) = 1$ and $c_{2,0}(t) = t$. Thus, if the one-term approximant φ_1 were sufficient, the solution would be $\varphi_1 = u_0 = 1 + xt - \frac{1}{12}x^4t^2 - \frac{1}{6}x^3t$. The next term is

$$u_1 = c_{1,1}(t) + c_{2,1}(t)x + L_{xx}^{-1}L_t u_0 - L_{xx}^{-1}u_0 + L_{xx}^{-1}A_0.$$

Then $\varphi_2 = u_0 + u_1$ is given by:

$$\begin{aligned}
 \varphi_2 &= 1 + xt + c_{1,1}(t) + c_{2,1}(t)x - \frac{1}{180}x^6 + \frac{1}{12960}t^4x^{10} + \frac{1}{2592}t^3x^9 \\
 &\quad + \frac{1}{2016}t^2x^8 - \frac{1}{252}t^3x^7 - \frac{1}{72}t^2x^6 - \frac{1}{120}tx^5,
 \end{aligned}$$

Applying the condition at $x = 0$, we have $c_{1,1}(t) = 0$. From the condition on x at 0, we get $c_{2,1}(t) = 0$, thus

$$\begin{aligned}
 u_1 &= -\frac{1}{180}x^6 + \frac{1}{12960}t^4x^{10} + \frac{1}{2592}t^3x^9 + \frac{1}{2016}t^2x^8 \\
 &\quad - \frac{1}{252}t^3x^7 - \frac{1}{72}t^2x^6 - \frac{1}{120}tx^5 + \frac{1}{12}t^2x^4 + \frac{1}{6}tx^3,
 \end{aligned}$$

Next, we average the partial solutions, i.e. add two partial solutions and divide by two, so we obtain

$$\begin{aligned}
 u_0 &= \frac{1}{2} \left(2 + 2xt + \frac{1}{12}x^2t^4 + \frac{1}{6}xt^3 - \frac{1}{12}x^4t^2 - \frac{1}{6}x^3t \right) \\
 u_1 &= \frac{1}{2} \left(\frac{1}{180}t^6 - \frac{1}{12960}x^4t^{10} - \frac{1}{2592}x^3t^9 - \frac{1}{2016}x^2t^8 - \frac{1}{252}x^3t^7 - \frac{1}{72}x^2t^6 \right. \\
 &\quad - \frac{1}{120}xt^5 - \frac{1}{12}x^2t^4 - \frac{1}{6}xt^3 - \frac{1}{180}x^6 + \frac{1}{12960}t^4x^{10} + \frac{1}{2592}t^3x^9 \\
 &\quad \left. + \frac{1}{2016}t^2x^8 - \frac{1}{252}t^3x^7 - \frac{1}{72}t^2x^6 - \frac{1}{120}tx^5 + \frac{1}{12}t^2x^4 + \frac{1}{6}tx^3 \right),
 \end{aligned}$$

Continuing in a similar way u_2, u_3, \dots, u_n are obtained for some n , then we get the approximate solution $u_{ap} = \sum_{n=0} u_n(x, t)$ which is converge to the exact solution $u_{ex} = 1 + xt$.

Table 2 shows the comparison between the absolute error of exact and approximate solutions for various values of t . **Figure 3** and **Figure 4** give the plots for the exact solution and the approximate solution by application of (ADM) for $t = 0.5, t = 1.0$

3. Modified Adomian Decomposition Method with Mixed Conditions

Wazwaz in [31], suggested a modification to the recurrence relations of the Standard Adomian Decomposition Method (ADM). He assumed that the first

Table 2. Absolute errors using ADM at $t = 0.5, t = 1.0$ and $0 \leq x \leq 1$.

x	$t = 0.5$			$t = 1.0$		
	φ_3	φ_5	φ_{10}	φ_3	φ_5	φ_{10}
0.0	1.16795669e-06	9.91381644e-11	7.38008938e-21	3.03130511e-04	1.94227703e-07	1.98867397e-13
0.1	1.11741220e-06	1.34624036e-10	1.32310675e-20	3.13576647e-04	5.24225154e-07	1.80979784e-14
0.2	1.02259378e-06	1.68442683e-10	1.87267309e-20	3.10560067e-04	8.92768322e-07	3.35618411e-13
0.3	9.15093801e-07	1.98929868e-10	2.27716130e-20	2.91108055e-04	1.27778455e-06	7.19015636e-13
0.4	1.07394957e-06	2.47291825e-10	2.45567197e-20	2.52542863e-04	1.64754249e-06	1.09372711e-12
0.5	2.78526691e-06	6.80058898e-10	1.54926759e-19	1.94233658e-04	1.95990772e-06	1.34001536e-12
0.6	1.02972732e-05	4.84484741e-09	1.31137692e-17	1.22320508e-04	2.16910519e-06	1.29712536e-12
0.7	3.49347643e-05	3.38038332e-08	6.42509911e-16	6.00678450e-05	2.27490316e-06	7.88380042e-13
0.8	1.02662323e-04	1.89000465e-07	1.89050311e-14	6.75505137e-05	2.54957010e-06	2.03620129e-13
0.9	2.66686450e-04	8.69375166e-07	3.75110306e-13	2.75330767e-04	4.36439340e-06	1.01270536e-12
1.0	6.26917478e-04	3.41245963e-06	5.43693717e-12	9.37415208e-04	1.27237836e-05	4.38984895e-11

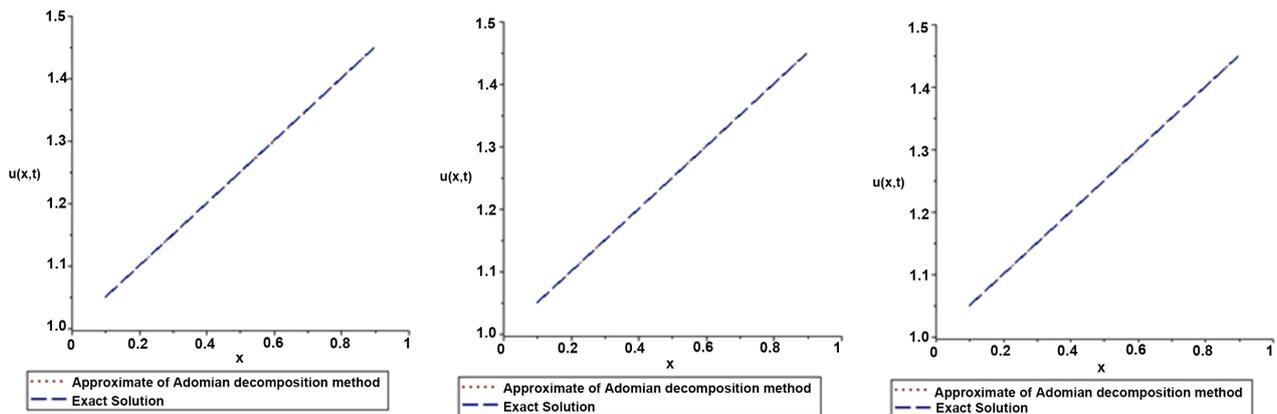


Figure 3. The exact solution and the approximate solution using ADM for $\varphi_3, \varphi_5, \varphi_{10}$ at $t = 0.5$.

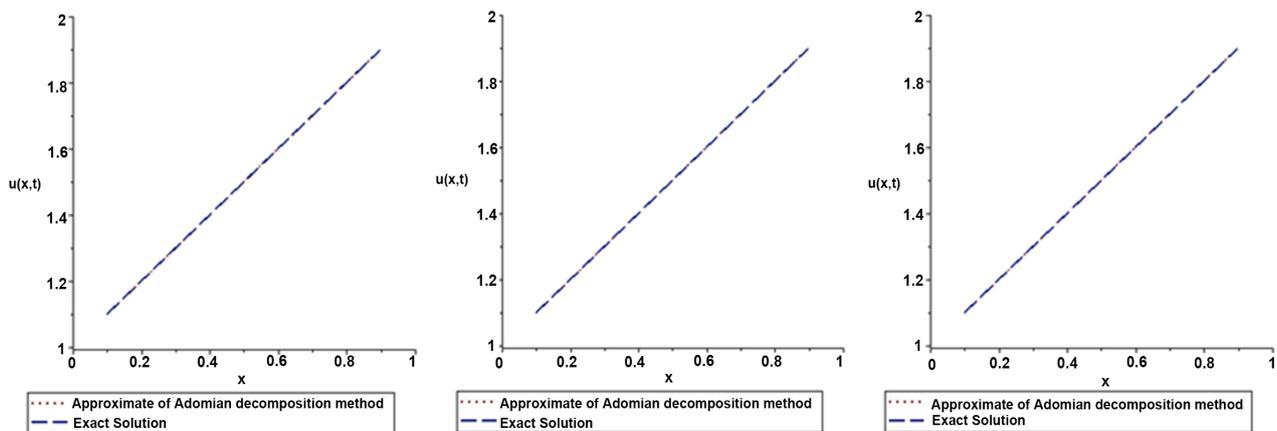


Figure 4. The exact solution and the approximate solution using ADM for $\varphi_3, \varphi_5, \varphi_{10}$ at $t = 1.0$.

term u_0 can be set as function f and divided it into two parts, namely f_1 and f_2 . Under this assumption, we set $f = f_1 + f_2$. Also, based on this, we propose a slight variation only on the components u_0 and u_1 , the variation is that only the part f_1 is assigned to the zeroth component u_0 , whereas the remaining part f_2 is combined with the other terms to define u_1 . This reduction in the number of terms of u_0 will result in reduction of the computational work and will accelerate the convergence. Further, this slight variation in the definition of the components u_0 and u_1 may provide the solution by using two iterations only. Furthermore, the calculations below will show that sometimes there is no need to evaluate the so-called Adomian polynomials required for the nonlinear differential equations. An important observation that can be made here is that the success of this method depends mainly on the proper choice of the parts f_1 and f_2 . We have been unable to establish any criterion to judge what forms of f_1 and f_2 can be assumed to yield the acceleration demanded. It appears that trials are the only criteria that can be applied so far.

3.1. Boundary Value Problems

Based on the recurrence relation in Equation (2.8)

$$u_0 = \Phi_{x,0} + L_{xx}^{-1}g(x,t),$$

$$u_{n+1} = \Phi_{x,n+1} - L_{xx}^{-1}L_u u_n - L_{xx}^{-1}A_n, \quad n \geq 0.$$

We can set $u_0 = \Phi_{x,0} + L_{xx}^{-1}g(x,t) = f$, then we divide it into two parts, so that we formulate the modified recursive algorithm as follows:

$$u_0 = f_1,$$

$$u_1 = f_2 + \Phi_{x,1} - L_{xx}^{-1}L_u u_0 - L_{xx}^{-1}A_0,$$

$$u_{n+1} = \Phi_{x,n+1} - L_{xx}^{-1}L_u u_n - L_{xx}^{-1}A_n, \quad n \geq 1. \tag{3.1}$$

Comparing the recursive scheme in Equation (2.8) of the ADM with the scheme in Equation (3.1) of the modified technique leads to the conclusion that in Equation (2.8) the zeroth component was defined by the function f whereas in Equation (3.1), the zeroth component u_0 was defined by only a part f_1 of f , the remaining part f_2 of f is added to the definition of the component u_1 in (3.1).

3.2. An alternative Combination of the Initial and Boundary Conditions

Based on the recurrence relation in Equation (2.13), we can set

$$u_0 = \frac{1}{2}(u(x,0) + tu_t(x,0) + \Phi_{x,0} + L_u^{-1}g(x,t) + L_{xx}^{-1}g(x,t)) = f$$

Then we divide it into two parts, so that we formulate the modified recursive algorithm as follows:

$$u_0 = f_1,$$

$$u_1 = f_2 + \frac{1}{2}(\Phi_{x,1} - L_u^{-1}L_{xx}u_0 - L_{xx}^{-1}L_u u_0 - L_u^{-1}A_0 - L_{xx}^{-1}A_0),$$

$$u_{n+1} = \frac{1}{2} \left(\Phi_{x,n+1} - L_t^{-1} L_{xx} u_n - L_{xx}^{-1} L_t u_n - L_t^{-1} A_n - L_{xx}^{-1} A_n \right), \quad n \geq 1. \quad (3.2)$$

Comparing the recursive scheme in Equation. (2.13) of the ADM with the recursive scheme in Equation (3.2) of the modified technique leads to the conclusion that in (2.13) the zeroth component was defined by the function f whereas in (3.2), the zeroth component u_0 is defined only by a part f_1 of f the remaining part f_2 of f is added to the definition of the component u_1 in (2.13).

In order to demonstrate the efficiency and applicability of the proposed method, we study example of nonlinear partial differential equations here.

3.3. Numerical Experiments

Example 3

Consider the nonlinear inhomogeneous heat equation

$$u_t = u_{xx} + u^2 - x^2 t^2 + x, \quad t > 0,$$

with specified conditions $u(x, 0) = 0, u(0, t) = 0, u_x(0, t) = t$.

Rewriting the heat equation in the operator form as

$$L_t u = L_{xx} u + u^2 - x^2 t^2 + x.$$

Applying the inverse operator L_{xx}^{-1} defined by $L_{xx}^{-1} = \int dx \int dx'$, we get

$$u(x, t) = \Phi_x + L_{xx}^{-1} (x^2 t^2 - x) + L_{xx}^{-1} L_t u(x, t) - L_{xx}^{-1} u^2(x, t),$$

This gives the recursive relations

$$u_0 = \Phi_{x,0} + L_{xx}^{-1} (x^2 t^2 - x),$$

$$u_{n+1} = \Phi_{x,n+1} + L_{xx}^{-1} L_t u_n - L_{xx}^{-1} A_n, \quad n \geq 0.$$

So that

$$\begin{aligned} u_0 &= c_{1,0}(t) + c_{2,0}(t)x + L_{xx}^{-1} (x^2 t^2 - x) \\ &= c_{1,0}(t) + c_{2,0}(t)x + \frac{1}{12} x^4 t^2 - \frac{1}{6} x^3, \end{aligned}$$

The first approximant is $\varphi_1 = u_0$. Applying the x conditions to φ_1 it is clear that $c_{1,0}(t) = 0$ and $c_{2,0}(t) = t$. Thus, if the one-term approximant φ_1 were sufficient, the “solution” will be $\varphi_1 = u_0 = xt + \frac{1}{12} x^4 t^2 - \frac{1}{6} x^3$. We set $u_0 = f$, then we divide it into two parts, so that

$$u_0 = f_1 = xt,$$

$$\begin{aligned} u_1 &= f_2 + c_{1,1}(t) + c_{2,1}(t)x + L_{xx}^{-1} L_t u_0 - L_{xx}^{-1} A_0 \\ &= \left(\frac{1}{12} x^4 t^2 - \frac{1}{6} x^3 \right) + c_{1,1}(t) + c_{2,1}(t)x + L_{xx}^{-1} L_t u_0 - L_{xx}^{-1} A_0. \end{aligned}$$

Then $\varphi_2 = u_0 + u_1$ is given by:

$$\varphi_2 = xt + \frac{1}{12} x^4 t^2 - \frac{1}{6} x^3 + c_{1,1}(t) + c_{2,1}(t)x + L_{xx}^{-1} L_t u_0 - L_{xx}^{-1} A_0$$

Applying the condition at $x = 0$, we have $c_{1,1}(t) = 0$. From the condition on

x at 0, we get $c_{2,1}(t) = 0$, thus $u_1 = 0$, and $u_{n+1} = 0, n \geq 1$. Thus, the approximate solution is $u = u_0 = xt$, which is the exact solution.

4. Improvement of the Inverse Operator with Mixed Boundary Conditions

In 1999, Lesnic and Elliott [30] employed ADM for solving some inverse boundary value problems in heat conduction, also they applied the modification method to deal with noisy input data and obtain a stable approximate solution. In [32] Lesnic investigated the application of the decomposition method involving computational algebra for solving more complicated problems with Dirichlet, Neumann or mixed boundary conditions. In [30] the authors defined the operator L_t^{-1} and as definite integrals given by

$$L_t^{-1} = \int_0^t dt', \quad L_x^{-1} = \int_{x_0}^x dx' \int_1^{x'} dx'' \tag{4.1}$$

to solve the linear homogeneous heat equation $u_t = u_{xx}, x_0 < x < 1, t > 0$, subject to the mixed boundary conditions $u(x_0, t) = h_1(t), u_x(1, t) = h_2(t)$.

We will make an extension to the inverse operator (4.1) given in [30] to all cases of problems, so that we consider in this section, two types of problems: boundary value problems and initial-boundary value problems.

4.1. Boundary Value Problems

Consider Equations. (2.2)-(2.3), and applying the inverse operator L_{xx}^{-1} defined by

$$L_{xx}^{-1} = \int_a^x dx' \int_b^{x'} dx'' \text{ to both sides of Equation (2.4), gives}$$

$$u(x, t) = u(a, t) + (x - a) \frac{\partial u(b, t)}{\partial x} + L_{xx}^{-1}g(x, t) - L_{xx}^{-1}L_{tt}u(x, t) - L_{xx}^{-1}Nu(x, t) \tag{4.2}$$

i.e. the boundary conditions can be used directly to solve the boundary value problem in x -direction.

Substituting $u(x, t) = \sum_{n=0}^{\infty} u_n(x, t)$, and $Nu(x, t) = \sum_{n=0}^{\infty} A_n(u_0, u_1, u_2, \dots, u_n)$ into Equation (4.2), gives

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x, t) &= u(a, t) + (x - a) \frac{\partial u(b, t)}{\partial x} + L_{xx}^{-1}g(x, t) \\ &\quad - L_{xx}^{-1}L_{tt} \sum_{n=0}^{\infty} u_n(x, t) - L_{xx}^{-1} \sum_{n=0}^{\infty} A_n(x, t) \end{aligned}$$

This yields the recursive relations

$$\begin{aligned} u_0 &= u(a, t) + (x - a) \frac{\partial u(b, t)}{\partial x} + L_{xx}^{-1}g(x, t), \\ u_{n+1} &= -L_{xx}^{-1}L_{tt}u_n - L_{xx}^{-1}A_n, n \geq 0. \end{aligned} \tag{4.3}$$

4.2. An Alternative Combination of the Initial and Boundary Conditions

Here, we take the average of the t partial solution as in Equation (2.11), and the x

partial solution as in Equation (4.2) to get

$$\begin{aligned}
 u(x, t) = \frac{1}{2} & \left(u(x, 0) + u_t(x, 0)t + u(a, t) + (x - a) \frac{\partial u(b, t)}{\partial x} \right. \\
 & + L_t^{-1}g(x, t) + L_{xx}^{-1}g(x, t) - L_t^{-1}L_{xx}u(x, t) - L_{xx}^{-1}L_tu(x, t) \\
 & \left. - L_t^{-1}Nu(x, t) - L_{xx}^{-1}Nu(x, t) \right) \tag{4.4}
 \end{aligned}$$

Substituting $u(x, t) = \sum_{n=0}^{\infty} u_n(x, t)$, and $Nu(x, t) = \sum_{n=0}^{\infty} A_n(u_0, u_1, u_2, \dots, u_n)$ into Equation (4.3), gives

$$\begin{aligned}
 \sum_{n=0}^{\infty} u_n(x, t) = \frac{1}{2} & \left(u(x, 0) + u_t(x, 0)t + u(a, t) + (x - a) \frac{\partial u(b, t)}{\partial x} \right. \\
 & + L_t^{-1}g(x, t) + L_{xx}^{-1}g(x, t) - L_t^{-1}L_{xx} \sum_{n=0}^{\infty} u_n(x, t) \\
 & \left. - L_{xx}^{-1}L_t \sum_{n=0}^{\infty} u_n(x, t) - L_t^{-1} \sum_{n=0}^{\infty} A_n(x, t) - L_{xx}^{-1} \sum_{n=0}^{\infty} A_n(x, t) \right)
 \end{aligned}$$

so that the recurrence relations are

$$\begin{aligned}
 u_0 &= \frac{1}{2} \left(u(x, 0) + u_t(x, 0)t + u(a, t) + (x - a) \frac{\partial u(b, t)}{\partial x} + L_t^{-1}g(x, t) + L_{xx}^{-1}g(x, t) \right) \\
 u_{n+1} &= \frac{1}{2} \left(-L_t^{-1}L_{xx}u_n - L_{xx}^{-1}L_tu_n - L_t^{-1}A_n - L_{xx}^{-1}A_n \right), \quad n \geq 0. \tag{4.5}
 \end{aligned}$$

To give a clear overview of these methods, we have chosen several differential equations. The examples will be approached by the Lesnic’s Approach and the modified technique for comparison reasons. We also compare the approximate solution with the exact solution.

4.3. Numerical Experiments

Example 4

Consider the linear homogeneous heat equation

$$u_t - u_{xx} = 0, \quad 0 \leq x \leq 1, \quad t > 0,$$

with specified conditions $u(x, 0) = e^x, u(0, t) = e^t, u_x(1, t) = e^{1+t}$.

Rewriting the heat equation in the operator form as $L_t u = L_{xx} u$. Applying the inverse operator L_{xx}^{-1} defined by $L_{xx}^{-1} = \int_0^x dx' \int_1^{x'} dx''$, we get the recursive relations

$$\begin{aligned}
 u_0 &= u(0, t) + x \frac{\partial u(1, t)}{\partial x}, \\
 u_{n+1} &= L_{xx}^{-1}L_t u_n, \quad n \geq 0.
 \end{aligned}$$

So that

$$\begin{aligned}
 u_0 &= e^t + xe^{1+t}, \\
 u_1 &= \frac{1}{6}e^{1+t}x^3 + \frac{1}{2}e^t x^2 - e^t x - \frac{1}{2}e^{1+t}x,
 \end{aligned}$$

$$u_2 = \frac{1}{120}e^{1+t}x^5 + \frac{1}{24}e^t x^4 + \frac{1}{3}\left(\frac{-1}{2}e^t - \frac{1}{4}e^{1+t}\right)x^3 + \frac{5}{24}xe^{1+t} + \frac{1}{3}e^t x,$$

Continuing in a similar way u_2, u_3, \dots, u_n are obtained for some n , then we get the approximate solution $u_{ap} = \sum_{n=0}^{\infty} u_n(x, t)$ which converged to the exact solution $u_{ex} = e^{x+t}$.

Table 3 shows the comparison between the absolute error of exact and approximate solutions for various values of t . **Figure 5** and **Figure 6** show the results for the exact solution and the approximate solution by application of Lesnic's approach for $t = 0.5, t = 1.0$.

Example 5

Consider the linear homogeneous heat equation

$$u_t = u_{xx} + u, \quad t > 0,$$

with specified conditions $u(0, t) = e^{(1-\pi^2)t}, u_x(0, t) = 0$.

Rewriting the heat equation in the operator form a $L_t u = L_{xx} u + u$.

Applying the inverse operator L_{xx}^{-1} defined by $L_{xx}^{-1} = \int_0^x dx' \int_1^{x'} dx''$, we get the recursive relations

$$u_0 = u(0, t) + x \frac{\partial u(0, t)}{\partial x},$$

$$u_{n+1} = L_{xx}^{-1} L_t u_n - L_{xx}^{-1} u_n, \quad n \geq 0.$$

So that

$$u_0 = e^{(1-\pi^2)t},$$

$$u_1 = \frac{1}{2} \left((1-\pi^2) e^{(1-\pi^2)t} - e^{(1-\pi^2)t} \right) x^2,$$

$$u_2 = \frac{1}{4} \left(\frac{1}{6} (1-\pi^2)^2 e^{(1-\pi^2)t} - \frac{1}{3} (1-\pi^2) e^{(1-\pi^2)t} + \frac{1}{6} e^{(1-\pi^2)t} \right) x^4,$$

Continuing in a similar way u_3, u_4, \dots, u_n are obtained for some n , then we get the approximate solution $u_{ap} = \sum_{n=0}^{\infty} u_n(x, t)$ which converged to the exact solution $u_{ex} = e^{(1-\pi^2)t} \cos(\pi x)$.

Table 4 shows the comparison between the absolute error of exact and approximate solutions for various values of t . **Figure 7** and **Figure 8** show the results for the exact solution and the approximate solution by application of Lesni's approach for $t = 0.5, t = 1.0$.

Example 6

Consider the nonlinear inhomogeneous wave equation

$$u_{tt} - u_{xx} + u^2 = 2x^2 - 2t^2 + x^4 t^4, \quad t > 0,$$

with specified initial condition $u(x, 0) = 0, u_t(x, 0) = 0$ and the boundary conditions $u(0, t) = 0, u_x(0, t) = 0$.

Rewriting the wave equation in the operator form as

$$L_{tt} u - L_{xx} u + u^2 = 2x^2 - 2t^2 + x^4 t^4.$$

Table 3. Absolute errors of Lesnic’s approach at $t = 0.5, t = 1.0$.

x	$t = 0.5$			$t = 1.0$		
	φ_3	φ_5	φ_{10}	φ_3	φ_5	φ_{10}
0.0	0.00000000e+00	0.00000000e+00	0.00000000e+00	0.00000000e+00	0.00000000e+00	0.00000000e+00
0.1	4.24902315e-02	6.97705740e-03	7.62907729e-05	7.00545484e-02	1.15032229e-02	1.25782220e-04
0.2	8.39308011e-02	1.37823116e-02	1.50703014e-04	1.38378497e-01	2.27231903e-02	2.48467264e-04
0.3	1.23299225e-01	2.02481910e-02	2.21404446e-04	2.03286055e-01	3.33836232e-02	3.65034220e-04
0.4	1.59625814e-01	2.62154825e-02	2.86654167e-04	2.63178475e-01	4.32202036e-02	4.72612822e-04
0.5	1.92017276e-01	3.15372522e-02	3.44845510e-04	3.16582967e-01	5.19961385e-02	5.68554127e-04
0.6	2.19678089e-01	3.60824631e-02	3.94545612e-04	3.62187938e-01	5.94899244e-02	6.50495743e-04
0.7	2.41929405e-01	3.97392012e-02	4.34530692e-04	3.98874156e-01	6.55188663e-02	7.16419995e-04
0.8	2.58225232e-01	4.24174302e-02	4.63816184e-04	4.25741433e-01	6.99345194e-02	7.64703609e-04
0.9	2.68165604e-01	4.40512072e-02	4.81680983e-04	4.42130336e-01	7.26281622e-02	7.94157682e-04
1.0	2.71506419e-01	4.46003060e-02	4.87685197e-04	4.47638409e-01	7.35334732e-02	8.04056958e-04

Table 4. Absolute errors of Lesnic’s approach at $t = 0.5, t = 1.0$.

x	$t = 0.5$			$t = 1.0$		
	φ_3	φ_5	φ_{10}	φ_3	φ_5	φ_{10}
0.0	0.00000000e+00	0.00000000e+00	0.00000000e+00	0.00000000e+00	0.00000000e+00	0.00000000e+00
0.1	1.58048860e-08	3.05774246e-14	4.00000000e-31	1.87405030e-10	3.62569093e-16	4.00000000e-33
0.2	1.00618440e-06	3.12411864e-11	4.47807730e-25	1.19307420e-08	3.70439590e-13	5.30984100e-27
0.3	1.13607976e-05	1.79481652e-09	1.48748759e-21	1.34709647e-07	2.12818773e-11	1.76377518e-23
0.4	6.30539648e-05	3.17061819e-08	4.68360013e-19	7.47656780e-07	3.75953232e-10	5.55353720e-21
0.5	2.36780141e-04	2.93320054e-07	4.05459549e-17	2.80759946e-06	3.47801646e-09	4.80770055e-19
0.6	6.93626458e-04	1.80143646e-06	1.55080207e-15	8.22461404e-06	2.13603726e-08	1.83884976e-17
0.7	1.71018201e-03	8.33538786e-06	3.37515258e-14	2.02783311e-05	9.88361203e-08	4.00205716e-16
0.8	3.71364715e-03	3.13376896e-05	4.86129475e-13	4.40342408e-05	3.71583868e-07	5.76423702e-15
0.9	7.31350601e-03	1.00509330e-04	5.10789012e-12	8.67192471e-05	1.19178044e-06	6.05663528e-14
1.0	1.33266620e-02	2.84314481e-04	4.18457517e-11	1.58019709e-04	3.37123368e-06	4.96182279e-13

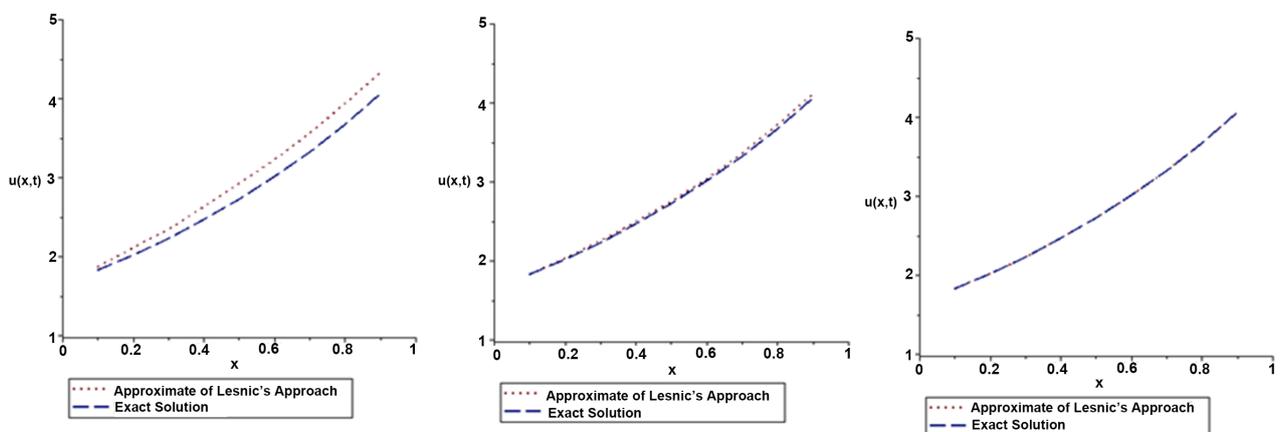


Figure 5. The exact solution and the approximate solution of Lesnic’s approach for $\varphi_3, \varphi_5, \varphi_{10}$ at $t = 0.5$.

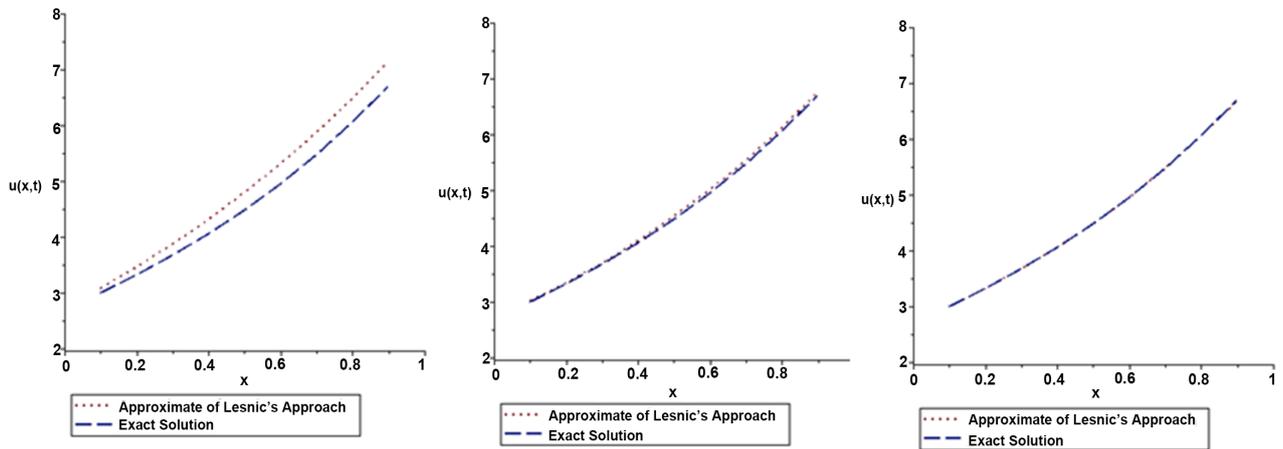


Figure 6. The exact solution and the approximate solution of Lesnic's approach for $\varphi_3, \varphi_5, \varphi_{10}$ at $t = 1.0$.

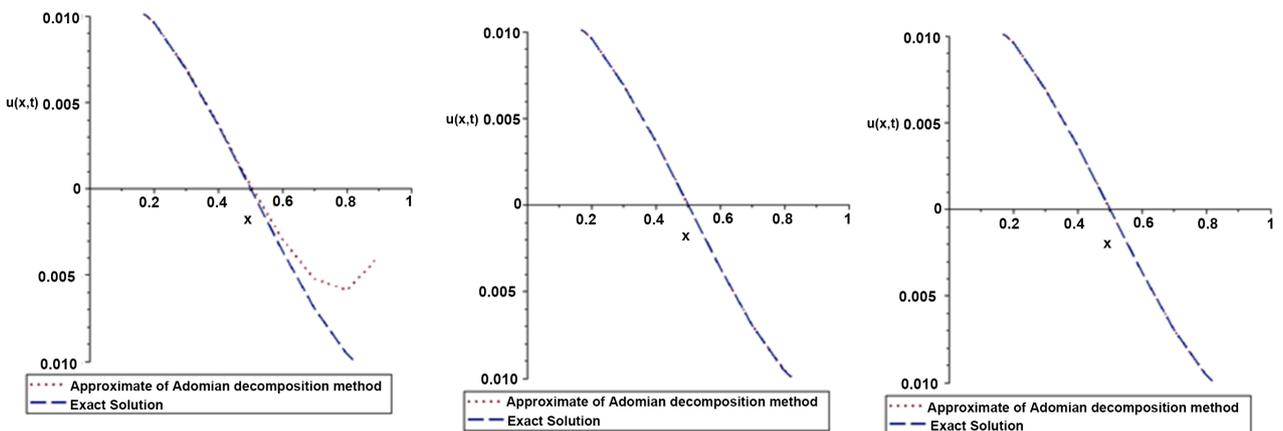


Figure 7. The exact solution and the approximate solution of Lesnic's approach for $\varphi_3, \varphi_5, \varphi_{10}$ at $t = 0.5$.

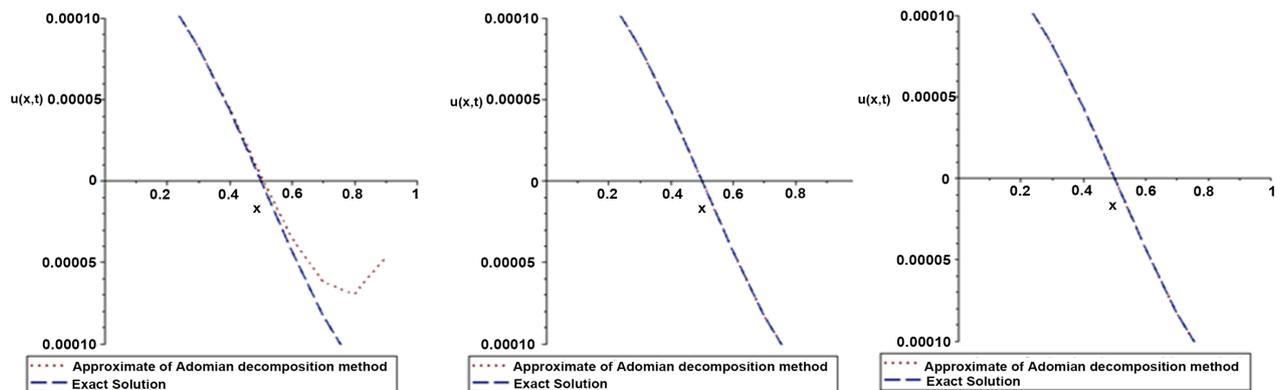


Figure 8. The exact solution and the approximate solution of Lesnic's approach for $\varphi_3, \varphi_5, \varphi_{10}$ at $t = 1.0$.

Firstly, we consider the t partial solution as

$$L_{tt}u = L_{xx}u - u^2 + 2x^2 - 2t^2 + x^4t^4.$$

Applying the inverse operator L_u^{-1} defined by $L_u^{-1} = \int_0^t \int_0^t dt dt$, gives the recursive relations

$$u_0 = u(x, 0) + tu_t(x, 0) + L_u^{-1}(2x^2 - 2t^2 + x^4t^4),$$

$$u_{n+1} = L_u^{-1}L_{xx}u_n - L_u^{-1}A_n, n \geq 0.$$

So that

$$u_0 = \frac{1}{30}x^4t^6 - \frac{1}{6}t^4 + x^2t^2,$$

$$u_1 = \frac{11}{840}x^2t^8 + \frac{1}{6}t^4 - \frac{1}{163800}x^8t^{14} + \frac{1}{11880}x^4t^{12}$$

$$- \frac{1}{10}\left(\frac{1}{135}x^6 + \frac{1}{324}\right)t^{10} - \frac{1}{30}x^4t^6,$$

Secondly, we consider the x partial solution

$$L_{xx}u = L_u u + u^2 - 2x^2 + 2t^2 - x^4t^4.$$

Applying the inverse operator L_{xx}^{-1} defined by $L_{xx}^{-1} = \int_0^x dx' \int_0^{x'} dx''$, gives the recursive relations

$$u_0 = u(0, t) + x \frac{\partial u(0, t)}{\partial x} + L_{xx}^{-1}(-2x^2 + 2t^2 - x^4t^4),$$

$$u_{n+1} = L_{xx}^{-1}L_t u_n - L_{xx}^{-1}A_n, n \geq 0.$$

So that

$$u_0 = -\frac{1}{30}t^4x^6 - \frac{1}{6}x^4 + x^2t^2,$$

$$u_1 = -\frac{11}{840}t^2x^8 + \frac{1}{6}x^4 + \frac{1}{163800}t^8x^{14} + \frac{1}{11880}t^4x^{12}$$

$$+ \frac{1}{10}\left(\frac{-1}{135}t^6 + \frac{1}{324}\right)x^{10} + \frac{1}{30}t^4x^6,$$

Next, we average the partial solutions, i.e. add two partial solutions and divide by two, so we obtain

$$u_0 = \frac{1}{2}\left(\frac{1}{30}x^4t^6 - \frac{1}{6}t^4 + 2x^2t^2 - \frac{1}{30}t^4x^6 - \frac{1}{6}x^4\right),$$

$$u_1 = \frac{1}{2}\left(\frac{11}{840}x^2t^8 + \frac{1}{6}t^4 - \frac{1}{163800}x^8t^{14} + \frac{1}{11880}x^4t^{12}\right.$$

$$- \frac{1}{10}\left(\frac{1}{135}x^6 + \frac{1}{324}\right)t^{10} - \frac{1}{30}x^4t^6 - \frac{11}{840}t^2x^8$$

$$+ \frac{1}{6}x^4 + \frac{1}{163800}t^8x^{14} + \frac{1}{11880}t^4x^{12}$$

$$\left. + \frac{1}{10}\left(\frac{-1}{135}t^6 + \frac{1}{324}\right)x^{10} + \frac{1}{30}t^4x^6\right),$$

Continuing in a similar way u_2, u_3, \dots, u_n are obtained for some n , then we get the approximate solution $u_{ap} = \sum_{n=0}^{\infty} u_n(x, t)$ which converged to the exact solution $u_{ex} = x^2t^2$.

Table 5 shows the comparison between the absolute error of exact and approximate solutions for various values of t . **Figure 9** and **Figure 10** show the

Table 5. Absolute errors of Lesnic’s approach at $t = 0.5, t = 1.0$.

x	$t = 0.5$			$t = 1.0$		
	φ_3	φ_5	φ_{10}	φ_3	φ_5	φ_{10}
0.0	2.92793246e-07	1.01879027e-11	6.85095911e-25	2.99609298e-04	6.66358850e-07	1.17411886e-14
0.1	2.92797760e-07	1.01884440e-11	5.49993680e-25	2.99751384e-04	6.67335460e-07	6.46368619e-15
0.2	2.92714604e-07	1.01838277e-11	4.25441720e-24	2.99906957e-04	6.68628865e-07	9.47122029e-15
0.3	2.90660076e-07	1.01582155e-11	1.04069553e-23	2.99261398e-04	6.65318536e-07	3.60495597e-14
0.4	2.59880611e-07	1.03640472e-11	1.88856065e-23	2.96428943e-04	6.49193205e-07	7.21721640e-14
0.5	6.38490216e-09	2.03770775e-11	1.39263400e-24	2.89265750e-04	6.08842328e-07	1.14059558e-13
0.6	1.55441030e-06	2.05585573e-10	1.04956960e-20	2.73991554e-04	5.30208113e-07	1.53190969e-13
0.7	8.35705522e-06	2.34523011e-09	1.45619577e-18	2.42057261e-04	4.00247226e-07	1.74249490e-13
0.8	3.27155910e-05	2.00278856e-08	1.03741414e-16	1.70952966e-04	2.32477815e-07	1.53300820e-13
0.9	1.07359772e-04	1.33275317e-07	4.44838968e-15	8.65289576e-07	2.20560700e-07	2.08420940e-14
1.0	3.09754605e-04	7.26354463e-07	1.27588322e-13	4.18259832e-04	1.50572369e-06	1.53639602e-12

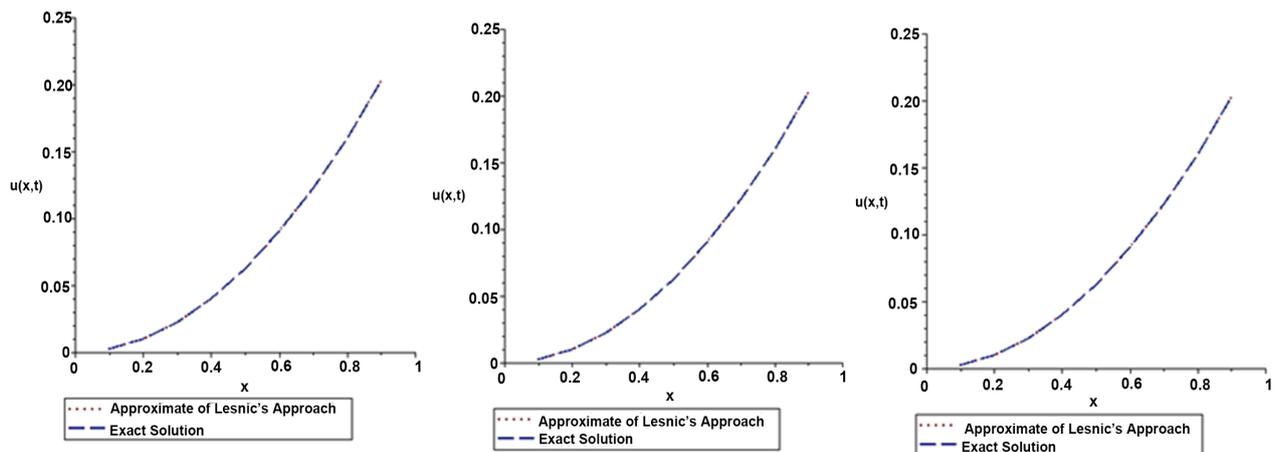


Figure 9. The exact solution and the approximate solution of Lesnic’s approach for $\varphi_3, \varphi_5, \varphi_{10}$ at $t = 0.5$.

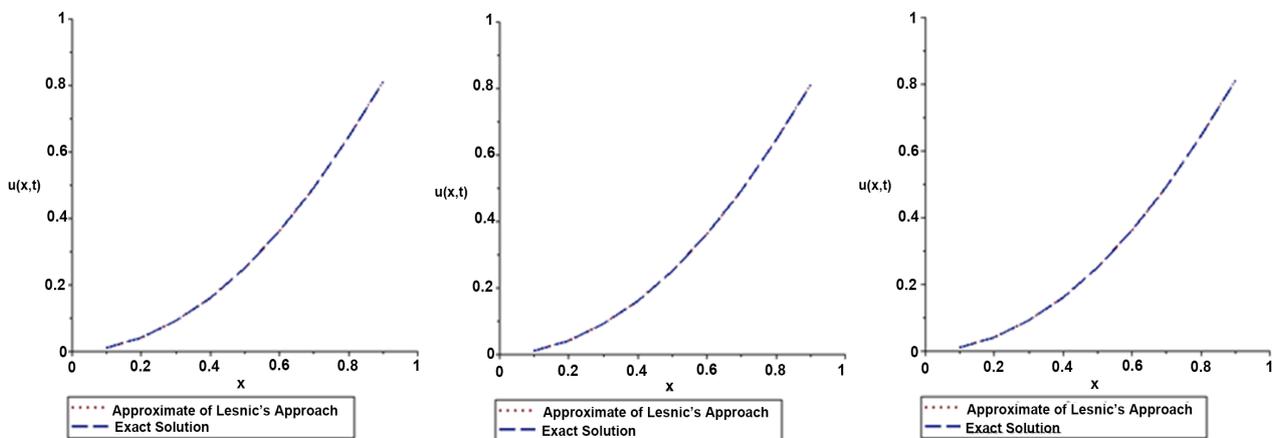


Figure 10. The exact solution and the approximate solution of Lesnic’s approach for $\varphi_3, \varphi_5, \varphi_{10}$ at $t = 1.0$.

results for the exact solution and the approximate solution by application of Lesnic's approach for $t = 0.5, t = 1.0$.

Example 7

Consider the nonlinear inhomogeneous heat equation

$$u_t = u_{xx} + u^2 - u - t^2 e^{2x} + e^x, \quad t > 0,$$

with specified conditions $u(x, 0) = 0, u(0, t) = t, u_x(0, t) = t$. Rewriting the heat equation in the operator form as

$$L_t u = L_{xx} u + u^2 - u - t^2 e^{2x} + e^x.$$

Applying the inverse operator L_{xx}^{-1} defined by $L_{xx}^{-1} = \int_0^x dx' \int_0^{x'} dx''$, we get the recursive relations

$$u_0 = u(0, t) + x \frac{\partial u(0, t)}{\partial x} + L_{xx}^{-1} (t^2 e^{2x} - e^x),$$

$$u_{n+1} = L_{xx}^{-1} L_t u_n + L_{xx}^{-1} u_n - L_{xx}^{-1} A_n, \quad n \geq 0.$$

So that

$$u_0 = t + xt + 1 - \frac{1}{4}t^2 - \frac{1}{2}t^2x + x + \frac{1}{4}t^2e^{2x} - e^x,$$

We set $u_0 = f$ then we divide it into two parts, so that

$$u_0 = f_1 = t + xt,$$

$$\begin{aligned} u_1 &= f_2 + L_{xx}^{-1} L_t u_0 + L_{xx}^{-1} u_0 - L_{xx}^{-1} A_0 \\ &= \left(1 - \frac{1}{4}t^2 - \frac{1}{2}t^2x + x + \frac{1}{4}t^2e^{2x} - e^x \right) + \frac{1}{6}x^3 + \frac{1}{2}x^2 \\ &\quad - \frac{1}{12}t^2x^4 - \frac{1}{3}t^2x^3 - \frac{1}{2}t^2x^2 + \frac{1}{6}tx^3 + \frac{1}{2}tx^2, \end{aligned}$$

Continuing in a similar way u_2, u_3, \dots, u_n are obtained for some n , then we get the approximate solution $u_{ap} = \sum_{n=0}^{\infty} u_n(x, t)$ which converged to the exact solution $u_{ex} = te^x$.

Table 6 shows the comparison between the absolute error of exact and approximate solutions Lesnic's approach and modified of Lesnic's approach for various values of t . **Figure 11** and **Figure 12** show the results for the exact solution and the approximate solution by application of Lesnic's approach and modified of Lesnic's Approach for $t = 0.5$.

5. Conclusion

The main goal of this paper is the development of an efficient and accurate method to solve linear and nonlinear initial-boundary value problems with mixed boundary conditions. The method developed provides the solution in the form of rapidly convergent series relative to the Adomian decomposition method. It also introduces a slight change in the formulation of Adomian recursive relation. The method presented is applied to several examples of boundary-value problems and initial boundary-value problems with mixed type boundary conditions.

Table 6. Absolute errors of Lesnic’s approach and modified of Lesnic’s Approach at $t = 0.5$.

x	Lesnic’s Approach			modified of Lesnic’s Approach		
	φ_3	φ_5	φ_{10}	φ_3	φ_5	φ_{10}
0.0	0.00000000e+00	2.00000000e-24	0.00000000e+00	0.00000000e+00	1.00000000e-27	1.00000000e-16
0.1	2.00447357e-08	3.57237376e-14	1.37823812e-10	3.81571653e-09	1.06888605e-16	1.24058539e-17
0.2	1.33381463e-06	3.85833086e-11	8.00849170e-11	2.67012987e-07	3.14421657e-13	1.83039464e-16
0.3	1.57920700e-05	2.34517027e-09	1.19984480e-11	3.31116702e-06	4.40980463e-11	5.19918722e-17
0.4	9.22165384e-05	4.38853494e-08	7.93648411e-11	2.01789433e-05	1.27255431e-09	2.58912426e-16
0.5	3.65597018e-04	4.30762657e-07	1.49935927e-10	8.32226565e-05	1.66889704e-08	7.73424325e-16
0.6	1.13467005e-03	2.81244230e-06	1.95254487e-10	2.67910691e-04	1.35122395e-07	1.65874377e-14
0.7	2.97453012e-03	1.38657115e-05	2.35238261e-10	7.26544297e-04	7.88954237e-07	4.24060812e-13
0.8	6.89226862e-03	5.56852385e-05	2.49626415e-10	1.73726451e-03	3.63635665e-06	7.09170229e-12
0.9	1.45351895e-02	1.91314569e-04	1.32152517e-09	3.77230615e-03	1.40126738e-05	8.56616462e-11
1.0	2.84628000e-02	5.81448247e-04	1.18704774e-08	7.59013963e-03	4.69348011e-05	8.00241318e-10

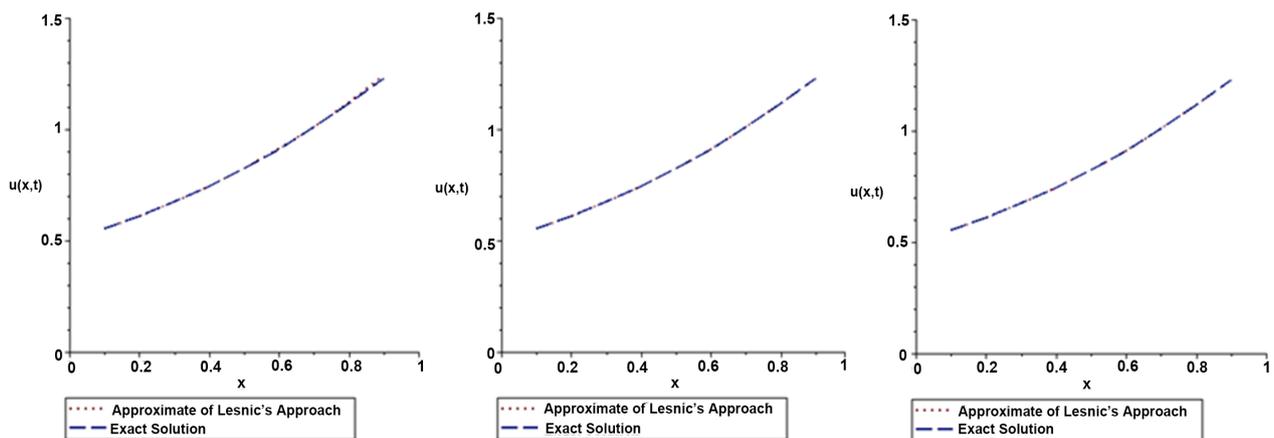


Figure 11. The exact solution and the approximate solution of Lesnic’s approach for $\varphi_3, \varphi_5, \varphi_{10}$ at $t = 0.5$.

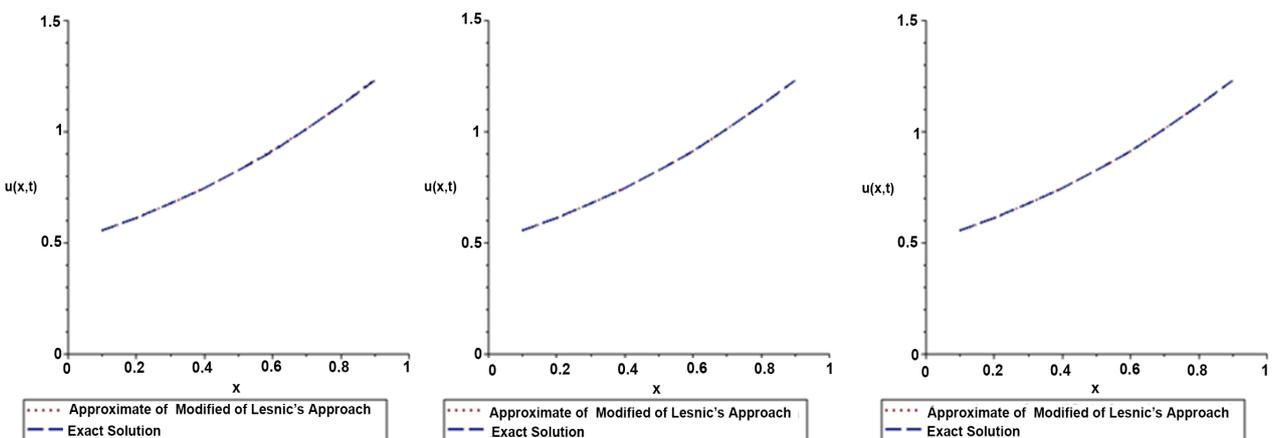


Figure 12. The exact solution and the approximate solution of Modified of Lesnic’s approach for $\varphi_3, \varphi_5, \varphi_{10}$ at $t = 0.5$.

The accuracy and computational efficiency of the proposed method are verified by numerical examples. Thus, the method is highly recommended for varieties of mixed boundary value problems.

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