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Risk-Neutral Pricing of European Call Options: A Specious Concept

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Abstract

Risk-neutral pricing of European call options is investigated from a mathematical point-of-view and is found to be a specious concept[1]. Risk-neutral pricing of European call options is an approximation in which all terms of order $(\alpha - r)/\sigma$ are ignored, where $\alpha - r$ is the risk premium and σ is the volatility.

Keywords

Risk-Neutral Pricing, European Call Options, Girsanov's Theorem, Specious

1. Introduction

The concept of risk-neutral pricing of European call options is investigated from a mathematical approach. It is found that risk-neutral pricing used in the pricing of European call options is a specious concept [1] [2] [3] that is only approximately correct and that ignores terms of $(\alpha - r)/\sigma$, where $\alpha - r$ is the risk premium and σ^2 is the variance of the one-day return of the asset that underlies the call option. The risk premium equals $(\alpha - r)$ with α the true drift rate of the underlying option and r the risk-free rate.

2. Background

First some notation and background information. Let the probability density function (pdf) for a random variable $\boldsymbol{x}$ be $f_{\boldsymbol{x}}(x)$. The probability that a measurement of the random variable $\boldsymbol{x}$ takes a value between x and $x + \mathrm{d}x$, $P\{x < \boldsymbol{x} \le x + \mathrm{d}x\}$, is given by $f_{\boldsymbol{x}}(x)\mathrm{d}x$. The cumulative density function (CDF)

[1]Specious "refers to something that appears at first encounter to be genuine or to be soundly argued or reasoned" [1]. A specious argument is "an argument that seems correct only if you do not think about it carefully" [2]. Something specious is "seemingly well-reasoned, plausible or true, but actually fallacious" [3].

$F_{\boldsymbol{x}}(a) = P\{\boldsymbol{x} \le a\} = \int_{-\infty}^{a} f_{\boldsymbol{x}}(\xi)\,\mathrm{d}\xi$.

Let S_t be the value of an asset at time *t*. Let the 1-day return be $R_t = \ln(S_t/S_{t-1})$ and the *n*-day return be $R_{t,n} = \ln(S_t/S_{t-n})$. Note that

$$\begin{aligned} R_{t,n} &= \ln(S_t/S_{t-n}) = \ln(S_t/S_{t-1} \times S_{t-1}/S_{t-2} \times \cdots \times S_{t-n+1}/S_{t-n}) \\ &= R_t + R_{t-1} + R_{t-2} + \cdots + R_{t-n+1} \end{aligned} \tag{1}$$

and, for simplicity in notation, that $R_t = R_{t,1}$.

When returns over non-lapping time periods are independent, the pdf for the *n*-day return is the *n*-fold convolution of the pdf of the 1-day return since the *n*-day return is the sum of *n* independent 1-day returns. Variances add under convolution, and hence the variance of *n*-day returns is *n* times the variance that describes the distribution of the 1-day returns. Since the normal distribution is stable under self-convolution, if the 1-day return distribution is a normal distribution with a mean μ and a variance σ^2, then the pdf for an *n*-day return is a normal distribution with a mean $= n\mu$ and with a variance $= n\sigma^2$ [4].

The returns of assets are better described by Student's *t*-distributions than by normal distributions. However, Student's *t*-distributions are not stable under self-convolution and have fat tails. The fat tails lead to integrals that diverge and these integrals are needed to price options. Both of these characteristics make pricing with Student's *t*-distributions difficult [4] [5] [6] [7] [8]. For simplicity and to be specific, the pricing of European call options for returns with normal distributions is discussed in this note. In particular, attention is paid to the mathematical basis for risk-neutral pricing of European call options. The results apply to fat tailed distributions as well [9].

Following [10] [11], the price of a European call option at time T is $C_T = E\left\{(\boldsymbol{S}_T - K_T)^+\right\}$. In the expression for C_T, $E\left\{(\boldsymbol{S}_T - K_T)^+\right\}$ is the expectation of the maximum value of 0 or the difference $(\boldsymbol{S}_T - K_T)$ of the value of the asset $\boldsymbol{S}_T$ and the strike price K_T at time *T*. At time $t = 0$, when the option is purchased, the value of the option is $C_0 = \exp(-rT)C_T$ as the sale of the option is a cash transaction and therefore the value of the option at time *T*, C_T, is discounted by the time value of money, with *r* the risk-free rate.

The expression for C_T follows from the arbitrage theorem [10] [11] [12] [13] [14].

2.1. Black-Scholes Option Pricing Formula

The Black-Scholes option pricing formula gives prices for European call options and is obtained under the constraints of: 1) no arbitrage; 2) the price of an asset is described by a geometric Brownian motion process with support $[-\infty, +\infty]$; 3) risk-neutral pricing; and, 4) the future price is a martingale. Constraint 1) requires, from the arbitrage theorem [12] [13] [14], that $C_T = E\left\{(\boldsymbol{S}_T - K_T)^+\right\}$. Constraints 3) and 4) require $E\{\boldsymbol{S}_t\} = \exp(rt)S_0$ where S_t is the price of the asset at time *t*, S_0 is the price at time $t = 0$, and *r* is the risk-free rate. Given that $\boldsymbol{S}_t$ is a geometric Brownian motion process derived from a stochastic process $\boldsymbol{R}_{t,n}$ with mean μn and variance $\sigma^2 n$, *i.e.*, the 1-day returns are normally distributed with drift rate μ and variance σ^2, then the expectation of $\boldsymbol{S}_t$ is given by

$$E\{\boldsymbol{S}_t\} = S_0 \exp\left(\mu t + \sigma^2 t/2\right) \tag{2}$$

for the time origin chosen such that $t = n$.

Since by assumption $\boldsymbol{S}_t$ is geometric Brownian motion process and since the distribution of $\boldsymbol{R}_{t,n}$ is an *n*-fold convolution of the 1-day return distribution, then the constraint $E\{\boldsymbol{S}_t\} = \exp(rt)S_0$ specifies the drift rate for $\boldsymbol{S}_t$ and the *n*-fold convolution dictates the shape and the variance of the distribution of $\boldsymbol{R}_{t,n}$, which then dictates the shape and variance of $\boldsymbol{S}_t$. From Equation (2) and constraints 3) and 4), $\boldsymbol{S}_t$ must have a drift rate $\mu = r - \sigma^2/2$ and from the *n*-fold convolution, $\boldsymbol{R}_{t,n}$ must be normally distributed (by assumption the 1-day return is normally distributed) with a variance of $\sigma^2 n$, and $\sigma^2 n = \sigma^2 t$ when the time origin is chosen such that $t = n$.

The value of a stock is expected to drift at the rate of $\alpha = r + (\alpha - r)$ where $\alpha - r$ is called the risk premium and r is the risk-free rate.

2.2. Risk-Neutral Pricing

Assume that $\boldsymbol{S}_T = S_0 \exp(\xi)$ where $\xi = \boldsymbol{R}_{T,T}$ is the *T*-day return and that the *T*-day return is normally distributed with mean $\alpha' T$ and variance $\sigma^2 T$. Thus $E\{\boldsymbol{S}_T\} = \exp(\alpha T)S_0$ where $\alpha = \alpha' + \sigma^2/2$.

The price of a European call option with strike price K_T is then [10] [11], assuming normal statistics for the returns with support $[-\infty, +\infty]$,

$$\begin{aligned} C_0 = {} & S_0 \mathrm{e}^{(\alpha - r)T} \int_{\ln\left(\frac{K_T}{S_0}\right)}^{\infty} \exp\left(-\frac{\left(\xi - \left(\alpha + 0.5\sigma^2\right)T\right)^2}{2\sigma^2 T}\right) \frac{\mathrm{d}\xi}{\sigma\sqrt{2\pi T}} \\ & - \mathrm{e}^{-rT} K_T \int_{\ln\left(\frac{K_T}{S_0}\right)}^{\infty} \exp\left(-\frac{\left(\xi - \left(\alpha - 0.5\sigma^2\right)T\right)^2}{2\sigma^2 T}\right) \frac{\mathrm{d}\xi}{\sigma\sqrt{2\pi T}}. \end{aligned} \tag{3}$$

The difference between the Black-Scholes option pricing formula and Equation (3) rests in the α in Equation (3). If one sets $\alpha = r$ in Equation (3), then one obtains the Black-Scholes formula. The Black-Scholes formula is obtained by using the concept of risk-neutral pricing, wherein one essentially sets the risk premium, $\alpha - r$, equal to zero. Mathematically this approach can not be correct. Setting $\alpha - r = 0$ violates the no arbitrage condition that $C_T = E\left\{\left(\boldsymbol{S}_T - K_T\right)^+\right\}$. The distribution for $\boldsymbol{S}_t$ is centred about $S_0 \exp(\alpha t)$ (*i.e.*, the mean of the pdf for the *n*-day return is $\alpha' n = \left(\alpha - \sigma^2/2\right)n$). Arbitrarily setting the mean of the distribution will change the value of the expectation $E\left\{\left(\boldsymbol{S}_T - K_T\right)^+\right\}$ and hence mis-price European call options. In general, for small *T*, $(\alpha - r)T \ll \sigma\sqrt{T}$ and the error introduced by arbitrarily setting the mean of the distribution will be small. Application of Girsanov's theorem in risk-neutral pricing is discussed in Sec. 3.

A series expansion in α of C_0 about the risk-neutral value $\alpha = r$ shows that risk-neutral pricing underestimates the price of a call option when $\alpha > r$. $C_0\big|_\alpha$ is the cost at time $t = 0$ of a European call option whereas $C_0\big|_{\alpha=r}$ is the cost of a European call option using risk-neutral pricing (*i.e.*, using the same assumptions that yield the Black-Scholes option pricing formula).

$$C_0\big|_\alpha = C_0\big|_{\alpha=r} + \left(S_0\left(f\left(d_1\right)+1-F\left(d_1\right)\right)-\mathrm{e}^{-rT}K_T f\left(d_2\right)\right)(\alpha-r)T + O\left((\alpha-r)^2 T^2\right) \tag{4}$$

with

$$d_1 = \frac{\ln\left(K_T/S_0\right)-\left(r+\sigma^2/2\right)T}{\sqrt{T}} \tag{5}$$

$$d_2 = d_1 + \sigma^2\sqrt{T} \tag{6}$$

and $f(a)$ and $F(a)$ the pdf and CDF for a normal distribution with a mean of zero and a variance of σ^2. See Equation (8) for definitions of $f(a)$ and $F(a)$.

$C_0\big|_{\alpha=r} < C_0\big|_\alpha$ when $\alpha > r$ since the expansion is only valid for $S_T > K_T$, which follows from the definition $C_T = E\left\{\left(\boldsymbol{S}_T - K_T\right)^+\right\}$. Thus the risk-neutral pricing underestimates the value of the call option and gives, on average, an advantage to the option buyer when $\alpha > r$. The converse holds. The seller has, on average, an advantage under risk-neutral pricing when $\alpha < r$. Neither party has, on average, an advantage when the true value of α is used to price an option. The best estimate of a true value is typically the sample mean.

Table 1 gives pricing of European call options for $T = 30/365$ years, $r = 1\%$ per annum, $\sigma = 0.01\sqrt{365}$ as measured over one-year, and $S_0 = 50.00$ for geometric Brownian motion with Gaussian increments. Note that risk-neutral pricing (*i.e.*, the Black-Scholes formula) underestimates the price of a European call option when $\alpha > r$ and overestimates the price of a European call option when $\alpha < r$.

The "success" of risk-neutral pricing owes to the fact that the magnitude of the random fluctuations are typically significantly greater than the magnitude of the risk premium, *i.e.*, $\sigma\sqrt{T} \gg \left|(\alpha-r)T\right|$, and thus the random fluctuations

Table 1. The costs of European options, C_0, for various strike prices and values of α, given $r = 1\%$ per annum, $S_0 = \$50.00$, $T = 30/365$ years, and $\sigma = 0.01\sqrt{30}$. The column labelled $\alpha = r$ corresponds to prices obtained via risk-neutral pricing, *i.e.*, by the Black-Scholes formula.

K_T	$\alpha = r/2$	$\alpha = r$	$\alpha = 4r$	$\alpha = 8r$
46.00	4.089	4.108	4.225	4.382
48.00	2.369	2.385	2.483	2.616
50.00	1.102	1.113	1.178	1.268
52.00	0.393	0.398	0.430	0.475
54.00	0.105	0.107	0.118	0.134

obscure the drift. Presumably fluctuations about the mean value are described well by the shape and scale parameters of the distribution and one should use the best available estimate of the location parameter of the distribution (*i.e.*, the mean drift rate, α) to price an option.

Consider a normal distribution with mean μ and variance σ^2. If $\mu \ll \sigma$, then to an error of $< \mu/\sigma$, μ can be ignored. This can be verified from a series expansion of the CDF

$$F(a-\mu) \approx F(a) - \mu f(a)\left(1 + \frac{a\mu}{2\sigma^2} + \frac{\left(a^2-\sigma^2\right)\mu^2}{6\sigma^4}\right). \tag{7}$$

Note that $f(a)$ has a factor of σ^{-1}, that $F(a)$ is the cumulative density function (CDF), and that $f(x)$ is the zero mean pdf with variance σ^2:

$$F(a) = \int_{-\infty}^{a} f(x)\,\mathrm{d}x = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{a} \exp\left(-x^2/\left(2\sigma^2\right)\right)\mathrm{d}x. \tag{8}$$

The standard deviation σ is a measure of the width of the distribution whereas the mean μ shifts the curve left or right. For a broad curve, small shifts left or right make a small difference.

In an Ito calculus formalism, risk-neutral pricing is explained as

$$\begin{aligned}\mathrm{d}\boldsymbol{S}_t &= \left(\alpha-\sigma^2/2\right)\boldsymbol{S}_t\mathrm{d}t + \sigma\boldsymbol{S}_t\mathrm{d}\boldsymbol{W}_t \\ &= \left(r-\sigma^2/2\right)\boldsymbol{S}_t\mathrm{d}t + (\alpha-r)\boldsymbol{S}_t\mathrm{d}t + \sigma\boldsymbol{S}_t\mathrm{d}\boldsymbol{W}_t \\ &= \left(r-\sigma^2/2\right)\boldsymbol{S}_t\mathrm{d}t + \sigma\boldsymbol{S}_t\left(\mathrm{d}\boldsymbol{W}_t + \frac{\alpha-r}{\sigma}\mathrm{d}t\right) \\ &= \left(r-\sigma^2/2\right)\boldsymbol{S}_t\mathrm{d}t + \sigma\boldsymbol{S}_t\mathrm{d}\boldsymbol{W}_t'\end{aligned} \tag{9}$$

where $\mathrm{d}\boldsymbol{W}_t = \boldsymbol{W}(t+\mathrm{d}t) - \boldsymbol{W}(t)$ is an increment of Brownian motion (or Weiner process $\boldsymbol{W}(t)$ ([15], p. 79) with $E\{\mathrm{d}\boldsymbol{W}_t\} = 0$, $E\left\{\mathrm{d}\boldsymbol{W}_t^2\right\} = \mathrm{d}t$, and $E\{\mathrm{d}\boldsymbol{W}_t\mathrm{d}\boldsymbol{W}_s\} = 0, s \neq t$. $\mathrm{d}\boldsymbol{W}_t'$ is for a transformed process with non-zero mean such that $E\{\mathrm{d}\boldsymbol{W}_t'\} = (\alpha-r)/\sigma$. Provided that $(\alpha-r)/\sigma$ is negligible, then the process that underlies $\mathrm{d}\boldsymbol{W}_t'$ is to a good approximation equal to the process that underlies $\mathrm{d}\boldsymbol{W}_t$ and the values of C_T as calculated under the processes that yield $\mathrm{d}\boldsymbol{W}_t$ and $\mathrm{d}\boldsymbol{W}_t'$ are essentially the same. If $(\alpha-r)/\sigma$ is not negligible, then the risk premium needs to be known to calculate the value of an option $C_T = E\left\{\left(\boldsymbol{S}_T - K_T\right)^+\right\}$ and risk-neutral pricing is not possible. It should be noted that $\mathrm{d}\boldsymbol{W}_t'$ strictly speaking is not a Brownian motion: $E\{\mathrm{d}\boldsymbol{W}_t'\} \neq 0$ and $E\{\mathrm{d}\boldsymbol{W}_t'\mathrm{d}\boldsymbol{W}_s'\} \neq \mathrm{d}t\delta_{t,s}$. Girsanov's theorem is discussed in Sec. 3.

The solution to Equation (9) is not $\boldsymbol{S}_t = S_0\exp(rt)$ since $E\{\mathrm{d}\boldsymbol{W}_t'\} \neq 0$. The solution to Equation (9) is, with $\alpha' = \alpha - \sigma^2/2$,

$$\boldsymbol{S}_t = S_0\exp\left(\int_0^t \alpha' + \sigma\boldsymbol{w}(\tau)\,\mathrm{d}\tau\right) \tag{10}$$

which follows from the solution, in a Langevin formalism [16] [17], of the equivalent stochastic differential equation to Equation (9):

$$\frac{\mathrm{d}\boldsymbol{S}(t)}{\mathrm{d}t} = \alpha'\boldsymbol{S}(t) + \sigma\boldsymbol{S}(t)\boldsymbol{w}(t) \tag{11}$$

where $\boldsymbol{S}_t = \boldsymbol{S}(t)$ and $\boldsymbol{w}(t)$ is a zero mean stochastic process that in a limit is delta function correlated. In the limit, $\boldsymbol{w}(t)$ is a white noise and the Wiener process $\boldsymbol{W}(t) = \int_0^t \boldsymbol{w}(\tau)\mathrm{d}\tau$, $\mathrm{d}\boldsymbol{W}_t = \boldsymbol{W}(t+\mathrm{d}t) - \boldsymbol{W}(t) = \boldsymbol{w}(t)\mathrm{d}t$ ([15], p. 79). Equation (11) is equivalent to the first line of Equation (9). In the Langevin formalism, the equation for the development in time of the average value of $\boldsymbol{S}(t)$, $\overline{S(t)}$, is

$$\frac{\mathrm{d}\overline{S(t)}}{\mathrm{d}t} = \left(\alpha' + \frac{\sigma^2}{2}\right)\overline{S(t)} = \alpha \overline{S(t)} \tag{12}$$

with solution

$$\overline{S(t)} = S_0 \exp\left(\int_0^t \alpha' \mathrm{d}\tau + \frac{\sigma^2}{2} t\right) = S_0 \exp\left(\int_0^t \alpha \mathrm{d}\tau\right). \tag{13}$$

In the event that α is a constant, $\overline{S(t)} = S_0 \exp(\alpha t)$. The mean value of $\boldsymbol{S}_t$ (or $\boldsymbol{S}(t)$, $\boldsymbol{S}(t) = \boldsymbol{S}_t$) drifts at the rate α when the time development of $\boldsymbol{S}_t$ (or $\boldsymbol{S}(t)$) is described by Equation (9) in an Ito formulation or equivalently by Equation (11) in a Langevin formulation.

Note that the $\sigma^2/2$ contribution to the drift arises from averaging over an ensemble of realizations of the stochastic process $\boldsymbol{w}(t)$ (c.f. Equations (2), (10), and (13)). Care must be employed in obtaining and in interpreting results within the Ito formalism. One could attempt, in the Langevin picture, to hide the risk premium $\alpha - r$ in $\boldsymbol{w}(t)$, as was attempted in Equation (9) to justify risk-neutral pricing. However, one would experience a similar difficulty. The transformed $\boldsymbol{w}(t)$ would not be zero mean and would not be delta function correlated in the limit, and knowledge of $\alpha - r$ would still be required to price the option unless $(\alpha - r)/\sigma$ is negligible.

Langevin equations are first order differential equations with noise driving terms. The Langevin equations should be interpreted as integral equations ([16], p. 172; [17], Ch. 10.2; [15], p. 79). Average values found by the Langevin approach are identical to solutions found by Ito's calculus ([17], p. 189). The Langevin approach has the advantage that transformations obey the usual rules of calculus. Ito's lemma is not applied for each transformation in the Langevin approach ([17], pp. 189, 282).

3. Girsanov's Theorem

Girsanov's theorem [18] provides for a multiplicative transformation that alters the shape of the pdf. This allows the mean of a random variable to be set to an arbitrary value, without using an additive correction to the original problem or pdf ([19], pp. 36-39). In option pricing, the goal behind the multiplicative transformation is an economic one: to find a risk-neutral measure to ensure no risk-free arbitrages in pricing ([19], p. 39). The multiplicative transformation must decrease probabilities for values greater than the desired mean and increase probabilities for values that are less than the desired mean.

3.1. Probability Measures

A pdf defines a probability measure *P*. If the pdf for a random variable $\boldsymbol{x}$ is $f_x(x)$, then

$$P_x(A)=\int_A f_x(\xi)\,\mathrm{d}\xi. \tag{14}$$

If $A(x)$ is a small neighbourhood of a specific outcome *x*, then $P_x(A(x))=\mathrm{d}P_x(x)=f_x(x)\mathrm{d}x$. For $G(\boldsymbol{x})$ a function of the random variable $\boldsymbol{x}$, then the expectation of $G(\boldsymbol{x})$ over $\boldsymbol{x}$ is

$$\mathrm{E}_x\{G(\boldsymbol{x})\}=\int G(\xi)\,\mathrm{d}P_x(\xi)=\int G(\xi)f_x(\xi)\,\mathrm{d}\xi. \tag{15}$$

Two probability measures *P* and *Q* are said to be equivalent if, for any set *A* in the probability space, $P(A)>0$ AND $Q(A)>0$ ([15], p. 245). This means that all events that are possible under the measure *P* are possible under the measure *Q*, and vice versa. If for some set *B*, $P(B)=0$ and $Q(B)>0$, then the event *B* is impossible in measure *P* whereas event *B* is possible in measure *Q*, and the two measures *P* and *Q* are not equivalent. The possibilities that not-equivalent measures describe are different.

3.2. Transformed Stochastic Process [19]

Consider a stochastic process $\boldsymbol{x}(t)$ (for a stochastic process, $\boldsymbol{x}(t)$ is a random variable for each point in time *t*) that has a drift rate μ and is driven by a Wiener process $\boldsymbol{W}(t)$, such that, e.g., $\mathrm{d}\boldsymbol{x}(t)=\mu\boldsymbol{x}(t)\mathrm{d}t+\sigma\boldsymbol{x}(t)\mathrm{d}\boldsymbol{W}(t)$. Define ([19], pp. 36-39 and pp. 210-214)

$$\boldsymbol{Z}=\exp\left(-(\mu-r)\boldsymbol{x}(t)-\frac{1}{2}(\mu-r)^2\right). \tag{16}$$

$\tilde{P}$ and *P* are equivalent probability measures and are related by ([19], pp. 36-39 and pp. 210-214)

$$\tilde{P}=\int \boldsymbol{Z}\,\mathrm{d}P \tag{17}$$

where $\tilde{P}$ is the equivalent risk-neutral measure. $\boldsymbol{Z}=\mathrm{d}\tilde{P}/\mathrm{d}P$ is called the Radon-Nikodým derivative.

If $\mathrm{d}P$ is the unit-normal distribution (*i.e.*, the underlying pdf is normally distributed with zero mean and standard deviation of unity, $\mathrm{d}P\sim N(0,1)$) then

$$\tilde{P}(A)=\int_A\frac{1}{\sqrt{2\pi}}\exp\left(-(\mu-r)x-\frac{1}{2}(\mu-r)^2\right)\exp\left(-x^2/2\right)\mathrm{d}x \tag{18}$$

and the equivalent risk-neutral measure for this example, for a small neighbourhood $A(x)$ near x, is

$$\lim_{A(x)\to\mathrm{d}x}\tilde{P}(A)=\mathrm{d}\tilde{P}(x)=\frac{1}{\sqrt{2\pi}}\exp\left(-\left(x-(\mu-r)\right)^2/2\right)\mathrm{d}x. \tag{19}$$

Clearly $\boldsymbol{Z}$ is selected to give the desired mean to the equivalent measure. In the example given here, $\mathrm{d}\tilde{P}(x)$ is a unit normal distribution with a mean of $\mu-r$. A change of variable starts the transformation of the process

$\mathrm{d}\boldsymbol{W}(t)+(\mu-r)\mathrm{d}t/\sigma$ to a zero mean process $\mathrm{d}\tilde{\boldsymbol{W}}(t)$.

The transformed process for the example given is $\mathrm{d}\tilde{\boldsymbol{x}}(t)=r\tilde{\boldsymbol{x}}(t)\mathrm{d}t+\sigma\tilde{\boldsymbol{x}}(t)\mathrm{d}\tilde{\boldsymbol{W}}(t)$ where $r'=r+\sigma^2/2$ could be the risk-free rate and $\mathrm{d}\tilde{\boldsymbol{W}}(t)$ is a Gaussian increment at time t of a Brownian motion. In the equivalent measure $\mathrm{d}\tilde{P}$, $\tilde{\boldsymbol{x}}(t)$ is a geometric Brownian motion that increases on average at the risk-free rate r': $\mathrm{E}\{\tilde{\boldsymbol{x}}(t)\}=\tilde{x}(0)\exp(r't)$.

3.3. Analysis

Gardiner ([15], p. 245) writes "can show that two stochastic differential equations can be considered *equivalent* if their noise terms are the same even if their drift terms are different". See Equations (20) below for examples of stochastic differential equations with different drift and noise terms.

Gardiner ([15], p. 246) writes "the **possible** sample paths from the two equations are identical, but depending on the choice of measure for the underlying driving process $V(t)$ the **relative frequency of the paths is different**". Emphasis added. All paths are available in the two measures P and $\tilde{P}$, but different paths and hence end points are emphasized in the two measures.

Gardiner ([15], p. 246) writes that "the stochastic differential equations

$$\mathrm{d}\boldsymbol{x}(t)=a(t)\mathrm{d}t+b(t)\mathrm{d}\boldsymbol{W}(t)$$

$$\mathrm{d}\boldsymbol{y}(t)=f(t)\mathrm{d}t+g(t)\mathrm{d}\boldsymbol{W}(t) \tag{20}$$

are equivalent if $b(t)=g(t)$. *This result is Girsanov's theorem.*"

Gardiner ([15], p. 247) shows simulations of a stochastic differential equation. The simulations for the same noise but different drifts look qualitatively indistinguishable whereas the simulation with different noise looks different than the other simulations with same noise but different drifts: "it is quite credible that either could be a simulation of the other equation." Qualitatively indistinguishable appears to mean the rms noises about the local trend lines appear to be similar.

This result is not surprising. The solution to Equation (9) or to Equation (11) in a Langevin approach is given by Equation (10) and can be recast as

$$\frac{\boldsymbol{S}_t}{S_0\exp\left(\int_0^t\alpha'\mathrm{d}\tau\right)}=\exp\left(\int_0^t\sigma\boldsymbol{w}(\tau)\mathrm{d}\tau\right)=\boldsymbol{S}_t'. \tag{21}$$

$\boldsymbol{S}_t'$ is a scaled version of $\boldsymbol{S}_t$ and is determined solely by the noise $\boldsymbol{w}(t)$, $\boldsymbol{w}(t)\mathrm{d}t=\mathrm{d}\boldsymbol{W}(t)$. Provided that the drift α' is not too large compared to σ, then any $\boldsymbol{S}_t$ with the same σ but different drift should look similar. In this case, one is using the higher frequency noise as a fiducial to compare observations of $\boldsymbol{S}_t$ for $0\le t\le T$ for different $\boldsymbol{w}(t)$ and/or α'.

Figure 1 presents simulations of $\boldsymbol{S}_t$ for 200 time steps (200 days) for $\alpha=r$ and for $\alpha=8r$, with $r=1\%$ per annum. The smooth lines are the expected values. All simulations used $\sigma=0.01$. Note the similarity between the simulations with different drifts. Note also the difference between the simulations and the expected means for the different drift rates: a simulation with $\alpha = r$ (red curves) gives values that are less than the values obtained with $\alpha = 8r$ (blue curves).

Figure 2 presents simulations of S_t for 1000 time steps (1000 days) for $\alpha = r$ and for $\alpha = 8r$, with $r = 1\%$ per annum. The smooth lines are the

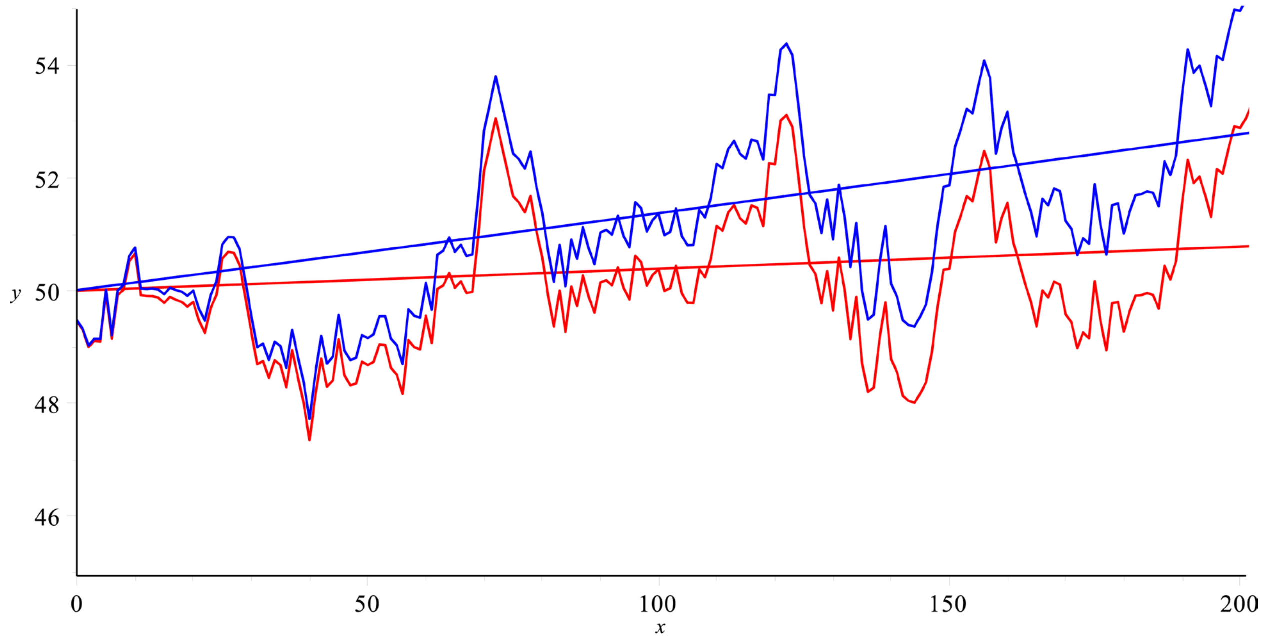

Figure 1. Simulations of S_t for $\alpha = r = 1\%$ per annum (red curves) and for $\alpha = 8\%$, $r = 1\%$ (blue curves) with $\boldsymbol{S}_t$ given by Equation (10). The smooth curves are the expected values $E\{\boldsymbol{S}_t\} = \overline{S_t}$, Equation (13). All calculations in this figure used $\sigma = 0.01$.

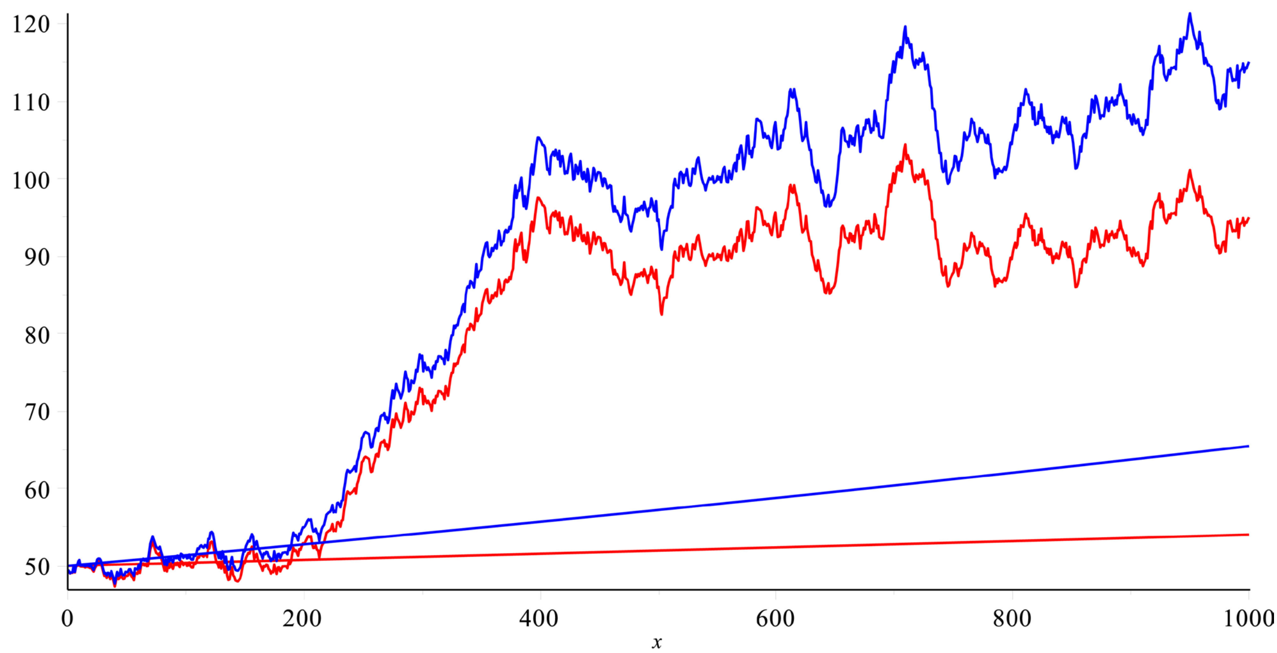

Figure 2. Simulations of S_t for $\alpha = r = 1\%$ per annum (red curves) and for $\alpha = 8\%$, $r = 1\%$ (blue curves) with $\boldsymbol{S}_t$ given by Equation (10). The smooth curves are the expected values $E\{\boldsymbol{S}_t\} = \overline{S_t}$, Equation (13). All calculations in this figure used $\sigma = 0.01$.

expected values. All simulations used $\sigma = 0.01$. **Figure 1** shows the first 201 points of **Figure 2**. The sharp rise in S_t owes to the sequence of random draws. A different seed for the pseudo random number generator gives drastically different results.

Figure 2 is included to show the drift possible (observe the trend from $t = 200$ to $t = 400$) owing to random draws. The noise $\boldsymbol{w}(t)$ had a mean of 0.06 over the interval $t = 1$ to $t = 1000$; 0.026 over $t = 1$ to $t = 200$; 0.30 from $t = 200$ to $t = 400$; and, −0.007 from $t = 400$ to $t = 1000$.

The simulations for the same noise but different drifts look qualitatively indistinguishable, as pointed out by Gardiner ([15], p. 247).

Gardiner ([15], p. 247) writes:

> "Girsanov's theorem is now the justification for use of the drift rate r instead of μ in the valuation of options using the risk-neutral procedure. The noise term is identical for both cases, and in the case we can say that the two processes can be seen as arising from the choice of a different probability measure to the same set of sample paths. In some sense it can be shown that this is a rigorously justifiable procedure [10.11], although not everyone would accept that. However, the use of change of measure is now an accepted part of the procedure for valuing options and other derivatives when one goes beyond the simple geometric Brownian motion picture."

Gardiner does not seem to be a true believer, and seems resigned to a deeply engrained status quo.

From [20]:

> "The relation (5) has been called by Cox-Ingersoll-Ross (1981) the "Local Expectation Hypothesis", a terminology which has led to some confusion. Note that the equilibrium process has not been changed, it is the same under both measures P and $\tilde{P}$. Girsanov's Theorem allows us to replace the relation (4) through the equivalent simpler relation (5). In particular, no assumption has been made about the existence of risk-neutral investors. In a real economy neither a "representative" nor a "risk-neutral" investor will exist, since both assumptions would prevent the existence of a (stable) equilibrium. The great advantage of the representation (5) under the (martingale) measure $\tilde{P}$ is that we do not have to know anything about the individual expectations P and the investors' attitude towards risk.
> In summary: $\boldsymbol{x}_t$ has not been changed. It is the same equilibrium price process as under P, but in simpler representation under $\tilde{P}$. $\tilde{P}$ is called the "equivalent risk-neutral measure" or the "P-equivalent martingale measure". ([20], p. 72)

It would appear that $\boldsymbol{x}_t$ has been changed. The possible paths are the same, but more weight (probability) has been placed on lower yielding paths (assuming the risk premium is greater than zero). From [19]:

> "After the change of measure, we are still considering the same set of stock

> price paths, but we have shifted the probability on them. If $\alpha > r$, as it normally is, then the change of measure puts more probability on the paths with lower return so that the overall mean rate of return is reduced from α to *r*." ([19], p. 217)
>
> "In finance, the change from the actual to the risk-neutral probability measure changes the distribution of asset prices without changing the asset prices themselves, ..." ([19], p. 37)

Altering the distribution changes the problem, unless the alteration is undone by an inverse transformation to return to the original frame of reference. Essentially, the risk-neutral approach appears to be to multiply one term on one side of an equation by *Z*. This is not a valid mathematical approach. Consider $x = -\alpha x + a$ with solution $x = a/(1+\alpha)$. Now solve $x = -\alpha x + Za$. The solutions for x are not the same unless $Z = 1$.

In the development of the risk-neutral measure $\mathrm{d}\boldsymbol{W}_t'$ in Equation (9), it is considered that $\mathrm{d}\boldsymbol{W}_t'$ is a Weiner process (Brownian motion). Strictly speaking, Brownian motion is a zero mean process. $\mathrm{d}\boldsymbol{W}_t'$ is not a zero mean process. One might wish to apply a coordinate transformation such that $\mathrm{d}\boldsymbol{W}_t'$ is a zero mean Brownian motion in the transformed frame of reference. However, one must remember that one is working in a transformed coordinate system, and provide a reverse transformation at the end to obtain the answer in the original frame of reference. This is similar to the problem of relative motion.

Consider an airplane that can cruise at *v* km/hour with respect to the air mass in which the airplane is embedded (*i.e.*, the local air) and assume that the local air is moving relative to the ground. If one is interested in the location of the plane after a given time, one can solve the problem by using a coordinate system that is embedded in the moving air. Relative to this coordinate system the plane has travelled a distance $d = vt$ in time *t*. The choice of coordinate system makes the problem look simpler. However, to know the location of the plane relative to a reference coordinate system on the ground such as an airport, one must know the relationship between the reference coordinate system and the moving coordinate system. In a similar manner, to find the value of an asset or an option, one must know the risk premium.

3.4. A Thought Experiment

Consider pricing a very long lived option, one so long lived that the mean value αt is $> rt + 4\sigma_t$. With 99.99% certainty (normal statistics are assumed in this work), the value of the underlying $\boldsymbol{S}_t$ will be $> rt$. Does it make sense to use risk-neutral pricing and force $E\{\boldsymbol{S}_t\} = S_0 \exp(rt)$? In this case, the approximation that $\alpha t \ll \sigma_t$ does not hold, and it appears that risk-neutral pricing would not be a reasonable approach.

3.5. Apply Girsanov's Theorem

Let us examine the justification for risk-neutral pricing, which is presented as

Equation (9). As before, define a risk premium, move it to the noise term $\boldsymbol{W}_t$, and absorb the risk premium in $\boldsymbol{W}_t'$.

$$\begin{aligned}\mathrm{d}\boldsymbol{S}_t &= \left(\alpha-\sigma^2/2\right)\boldsymbol{S}_t\mathrm{d}t+\sigma\boldsymbol{S}_t\mathrm{d}\boldsymbol{W}_t \\ &= \left(r-\sigma^2/2\right)\boldsymbol{S}_t\mathrm{d}t+\left(\alpha-r\right)\boldsymbol{S}_t\mathrm{d}t+\sigma\boldsymbol{S}_t\mathrm{d}\boldsymbol{W}_t \\ &= \left(r-\sigma^2/2\right)\boldsymbol{S}_t\mathrm{d}t+\sigma\boldsymbol{S}_t\left(\mathrm{d}\boldsymbol{W}_t+\frac{\alpha-r}{\sigma}\mathrm{d}t\right) \\ &= \left(r-\sigma^2/2\right)\boldsymbol{S}_t\mathrm{d}t+\sigma\boldsymbol{S}_t\mathrm{d}\boldsymbol{W}_t'.\end{aligned} \tag{22}$$

Now $\boldsymbol{W}_t'$ is a non-zero mean noise term, except in the special case that $\alpha-r=0$. Multiply the noise term by $\boldsymbol{Z}/\boldsymbol{Z}$, c.f. Equation (16), to obtain

$$\begin{aligned}\mathrm{d}\boldsymbol{S}_t &= \left(r-\sigma^2/2\right)\boldsymbol{S}_t\mathrm{d}t+\frac{\sigma}{\boldsymbol{Z}}\boldsymbol{S}_t\boldsymbol{Z}\mathrm{d}\boldsymbol{W}_t' \\ &= \left(r-\sigma^2/2\right)\boldsymbol{S}_t\mathrm{d}t+\frac{\sigma}{\boldsymbol{Z}}\boldsymbol{S}_t\mathrm{d}\boldsymbol{W}_t''.\end{aligned} \tag{23}$$

$\boldsymbol{Z}$ is chosen such that the mean value of $\boldsymbol{Z}\mathrm{d}\boldsymbol{W}_t'$ is zero, *i.e.*, $\mathrm{E}\left\{\boldsymbol{Z}\mathrm{d}\boldsymbol{W}_t'\right\}=0$. One could redefine $\sigma/\boldsymbol{Z}$ on the right hand side of Equation (23) to obtain

$$\mathrm{d}\boldsymbol{S}_t = \left(r-\sigma^2/2\right)\boldsymbol{S}_t\mathrm{d}t+\boldsymbol{\sigma}_G\boldsymbol{S}_t\mathrm{d}\boldsymbol{W}_t'', \tag{24}$$

which is a simple-looking equation but it must be remembered that $\boldsymbol{\sigma}_G=\sigma/\boldsymbol{Z}$ is no longer a constant. The dependence of $\boldsymbol{\sigma}_G$ on $\boldsymbol{Z}^{-1}$ undoes the work to create a zero mean noise term $\mathrm{d}\boldsymbol{W}_t''$. Alternatively, one could multiply both sides of Equation (23) by $\boldsymbol{Z}$, follow the rules for transformation of stochastic differential equations and variables ([17], Sec. 10.2-10.4, p. 275), only to find the same result that the work undid the desired result of setting the mean of $\mathrm{d}\boldsymbol{W}_t''$ to zero.

If in Equation (9) or in Equation (22) $\left(\alpha-r\right)/\sigma$ can be ignored, then under this approximation risk-neutral pricing would be accurate.

3.6. A Scaled Approach

Equation (21) suggests an approach to understand risk-neutral pricing of a European call option. Start with the expression for the value of the call option and manipulate to remove the drift owing to the risk premium in $\boldsymbol{S}_T$:

$$\begin{aligned}C_0 &= \exp\left(-rT\right)E\left\{\left(\boldsymbol{S}_T-K_T\right)^+\right\} \\ &= \exp\left(-rT\right)\frac{\exp\left(\int_0^T\left(\alpha-r\right)\mathrm{d}\tau\right)}{\exp\left(\int_0^T\left(\alpha-r\right)\mathrm{d}\tau\right)}E\{(\boldsymbol{S}_T-K_T)^+\} \\ &= \exp\left(-rT\right)\exp\left(\int_0^T\left(\alpha-r\right)\mathrm{d}\tau\right)E\left\{\left(\boldsymbol{S}_T'-K_T'\right)^+\right\}\end{aligned} \tag{25}$$

where both $\boldsymbol{S}_T$ and K_T have been scaled by the same factor to remove drift in $\boldsymbol{S}_T$ owing to the risk premium. The expectation in the last line might be anticipated at first look to be risk neutral. However, K_T' is a function of the risk premium: $K_T'=K_T\times\exp\left(\int_0^T-\left(\alpha-r\right)\mathrm{d}\tau\right)$. In addition, the inverse scaling factor remains in the expression for C_0 as an overall multiplicative factor. The risk premium is thus required to price the option. If the risk premium is negligible, then the scaling factor is approximately unity and risk-neutral pricing of the European call option would be sufficiently accurate. The approach presented in this section is silent on the magnitude of the risk premium that is negligible—one would need to examine the expectation, as was done in Sec. 2.2, to determine the magnitude. The rms fluctuations of the one day returns, σ, is the relevant metric. It is the magnitude of the ratio $(\alpha - r)/\sigma$ that determines what is negligible or not.

4. Conclusions

Risk-neutral pricing of European call options is mathematically an approximation. Provided that $(\alpha - r)/\sigma$ is small, then the drift rate is obscured over short time intervals by random fluctuations and one is justified in ignoring the risk premium $\alpha - r$ in the drift rate. It would seem honest to state this as the rationale behind risk-neutral pricing, rather than appealing to theorems and pretending that risk-neutral pricing is exact.

Risk-neutral pricing underestimates the price of a European call option when $\alpha > r$ and overestimates the price of a European call option when $\alpha < r$.

It is interesting to note that risk-neutral pricing of European call options ignores $\alpha - r$ but takes great care to include $\sigma^2/2$ in the average drift rate. If the risk premium can be ignored, then likely $\sigma^2/2$ can also be ignored. For the S & P 500, on average $\sigma^2/2 \approx 5\times10^{-5}$ whereas the risk premium $\alpha - r \approx 1\times10^{-4}$.

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