

6R Robot Inverse Solution Algorithm Based on Quaternion Matrix and Groebner Base

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Abstract

This article proposes a new algorithm of quaternion and dual quaternion in matrix form. It applies quaternion in special cases of rotated plane, transforming the sine and cosine of the rotation angle into matrix form, then exporting flat quaternions base in two matrix form. It establishes serial 6R manipulator kinematic equations in the form of quaternion matrix. Then five variables are eliminated through linear elimination and application of lexicographic Groebner base. Thus, upper bound of the degree of the equation is determined, which is 16. In this way, a 16-degree equation with single variable is obtained without any extraneous root. This is the first time that quaternion matrix modeling has been used in 6R robot inverse kinematics analysis.

Keywords

6R Robot, Quaternion Matrix, Groebner Base, Inverse Kinematics Analysis

1. Introduction

This article uses quaternion matrix to solve the problem of 6R robot inverse kinematics in the three-dimensional space. In fact, this problem has been solved in 1986 [1], through Closed Displacement Equation established by Homogeneous Transforms Matrix. It makes use of the product of the closed equation to conduct linear combination and then gets a new series of related equations. Finally, it performs linear elimination according to the linear combination among equations and does final elimination by the resultant, obtaining the 16-degree equation with single variable in the end. The kinematic model of 6R robot can also be built using Dual quaternion or Double quaternion, but there are some problems existed. When Qiao Shuguang [2] used Double quaternion to solve the problem, 8 extraneous roots appear. In order to eliminate the extraneous roots, four $1 + t^2$ common factors were extracted in Dixon Resultant. Thus, a 16-degree equation

with single variable is obtained. Huang Xiguang [3] and Gan Dongming [4] transformed the sine and cosine of one rotation angle into plural form instead of extracting common factors, but numerical calculation is needed. Because only in that way, ± 12 degree equation can be transformed to ± 8 degree equation, and then a 16-degree equation with single variable can be obtained. Even though they have obtained the single-variable equation, the deduction process depends on numerical computation lacking rigorous proof. A new algorithm using quaternion and dual quaternion in matrix form is put forward in this article. It applies quaternion in special cases of rotated plane, transforming the sine and cosine of the rotation angle into matrix form, then exporting flat quaternion base in two matrixes form. It establishes serial 6R manipulator kinematic equations in the form of quaternion matrix. Then five variables are eliminated through linear elimination and application of lexicographic Groebner base. Thus, upper bound of the degree of the equation is determined, which is 16. In this way, a 16-degree equation with single variable is obtained without any extraneous root. This is the first time that quaternion matrix [5] [6] modeling has been used in 6R robot inverse kinematics analysis.

2. Homogeneous Transformations in the Form of Quaternion Matrix

A dual quaternion can be expressed as:

$$\hat{q} = ai + bj + ck + d + a_0i\varepsilon + b_0j\varepsilon + c_0k\varepsilon + d_0\varepsilon \tag{1}$$

which the last four part can be written as

$$a_0i\varepsilon + b_0j\varepsilon + c_0k\varepsilon + d_0\varepsilon = \begin{bmatrix} d_0 + c_0i & a_0 + b_0i \\ -a_0 + b_0i & d_0 - c_0i \end{bmatrix} \varepsilon$$

which the part in the bracket can be transformed to matrix form using the method above, so the matrix form of dual quaternion can be expressed as:

$$\begin{aligned} \hat{q} &= ai + bj + ck + d + a_0i\varepsilon + b_0j\varepsilon + c_0k\varepsilon + d_0\varepsilon \\ &= \begin{bmatrix} d + ci & a + bi \\ -a + bi & d - ci \end{bmatrix} + \begin{bmatrix} d_0 + c_0i & a_0 + b_0i \\ -a_0 + b_0i & d_0 - c_0i \end{bmatrix} \varepsilon \end{aligned}$$

Using D-H method to build robot coordinate system, the relationship of two adjacent joint coordinate systems $i-1$ and i , can be described by the parameters in **Figure 1**.

In **Figure 1**, a new coordinate system can be derived by the coordinate system $i-1$ rotating on its axis Z with angle θ_i and transforming s_i along the same axis. Similarly, by rotating this new coordinate system on its axis X with angle α_i and transforming a_i along the same X , the coordinate system i can be obtained. This kinematics can be described in matrix form like this

$$\begin{aligned} &Q_{i-1}^z(\theta_i, s_i) Q_{i-1}^x(\alpha_{i-1}, a_{i-1i}) \\ Q_{i-1}^z(\theta_i, s_i) &= Z_{i-1} + \varepsilon Z_{i-1}^0 = Z_{i-1} + \varepsilon S_{i-1} Z_{i-1} / 2 \end{aligned}$$

in which:

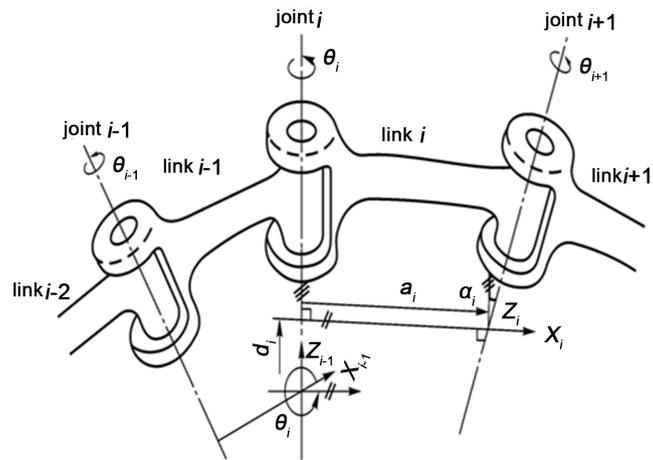


Figure 1. D-H coordinate transform.

$$\begin{aligned}
 Z_{i-1} &= \left\{ 0, 0, \sin \frac{\theta}{2} K, \cos \frac{\theta}{2} \right\} \\
 &\Rightarrow \begin{pmatrix} X & 0 \\ 0 & \bar{X} \end{pmatrix}, \\
 X &= \cos \frac{\theta}{2} + i \sin \frac{\theta}{2} = e^{i\frac{\theta}{2}}, \\
 \bar{X} &= \cos \frac{\theta}{2} - i \sin \frac{\theta}{2} = e^{-i\frac{\theta}{2}} \\
 s &= \{0, 0, s_1, 0\} \\
 &\Rightarrow \begin{pmatrix} is_1 & 0 \\ 0 & -is_1 \end{pmatrix}.
 \end{aligned}$$

So

$$Q_{i-1}^z(\theta_i, s_i) = \begin{bmatrix} e^{i\frac{\theta_i}{2}} & 0 \\ 0 & e^{-i\frac{\theta_i}{2}} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} is_1 e^{i\frac{\theta_i}{2}} & 0 \\ 0 & is_1 e^{-i\frac{\theta_i}{2}} \end{bmatrix} \varepsilon$$

among which the latter one is a conversion related to x -axis.

$$Q_{i-1}^x(\alpha_{i-1i}, a_{i-1i}) = X_{i-1} + \varepsilon X_{i-1}^0 = X_{i-1} + \varepsilon A_{i-1} X_{i-1} / 2,$$

in which

$$X_{i-1} = \begin{bmatrix} \cos \frac{\alpha}{2} & \sin \frac{\alpha}{2} \\ -\sin \frac{\alpha}{2} & \cos \frac{\alpha}{2} \end{bmatrix}$$

$$A_{i-1} = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$$

$$Q_{i-1}^x(\alpha_{i-1i}, a_{i-1i}) = \begin{bmatrix} \cos \frac{\alpha}{2} & \sin \frac{\alpha}{2} \\ -\sin \frac{\alpha}{2} & \cos \frac{\alpha}{2} \end{bmatrix} + \frac{1}{2} \varepsilon \begin{bmatrix} -a \sin \frac{\alpha}{2} & a \cos \frac{\alpha}{2} \\ -a \cos \frac{\alpha}{2} & -a \sin \frac{\alpha}{2} \end{bmatrix}$$

3. Closed-Form Kinematics Equations in Term of Double Quaternion Matrix

3.1. Mathematical Modeling

Inverse solution of position of 6R robot in three dimensions is that given the position and gesture and parameter s_p, a_p, α_i , find the input angles $(\theta_i (i = 1, 2, \dots, 6))$ of every joints. 6R serial mechanism (Figure 2) has six joints and it needs 6 three-dimension movements described in Figure 1. According to basic theories of quaternion matrix, the mathematical model of inverse solution of kinematics position of 6R serial mechanism in three dimensions can be expressed in matrix form:

$$Q_1^z(\theta_1, s_1)Q_1^x(\alpha_{12}, a_{12})Q_2^z(\theta_2, s_2)Q_2^x(\alpha_{23}, a_{23}) \cdots Q_6^z(\theta_6, s_6)Q_6^x(\alpha_{61}, a_{61}) = M \quad (2)$$

θ_p, s_p, α_i and a_i in the equations are structural parameters of 6R robot manipulate arm. $Q_i^z(\theta_i, s_i)$ and $Q_i^x(\alpha_i, a_i)$ are quaternion matrix transformations that rotate and move along z-axis and x-axis respectively. M is the expression in quaternion matrix form of the position and gesture of the end of the robot. For general 6R robot, unknown variables are $\theta_i (i = 1, 2, \dots, 6)$ and other variable are determined by the structure of robot which are known. Our goal is to find out the six angles. We use T_i and A_i to represent $Q_i^z(\theta_i, s_i)$ and $Q_i^x(\alpha_i, a_i)$. As Qiao [2] said that two of the six variables on the left of Equation (2) can be moved to the right and then it can be solved by Dixon resultant. Here, we move three of six variables on the left to the right, then Equation (2) becomes:

$$T_1 A_1 T_2 A_2 T_3 = M A_3^{-1} T_4^{-1} A_4^{-1} T_5^{-1} A_5^{-1} T_6^{-1} \quad (3)$$

Assume that the left and right part in equation is T_7 and T_8 respectively, which are quaternions in matrix form. Using Mathematic 6.0, we get:

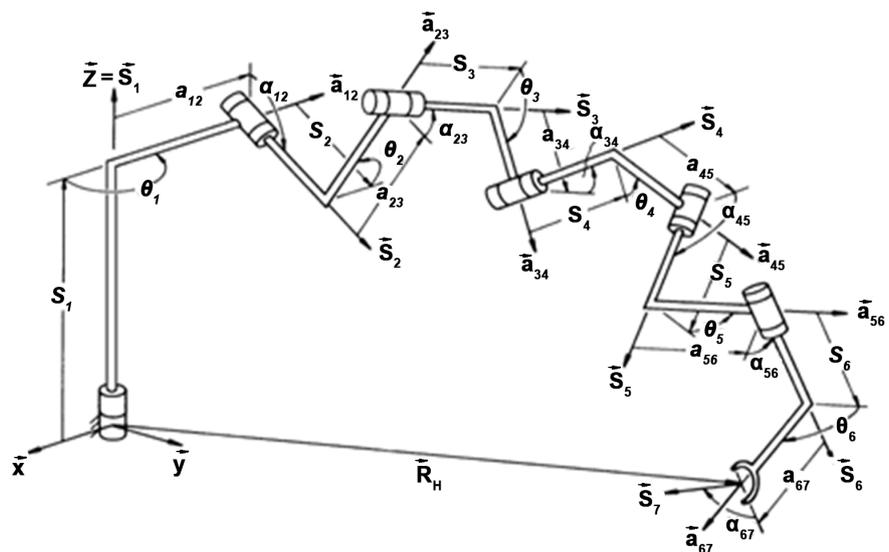


Figure 2. A general 6R manipulator.

$$\begin{aligned}
 T_7 &= U_1 + \varepsilon U_2 = \begin{bmatrix} u_{11} & u_{13} \\ u_{12} & u_{14} \end{bmatrix} + \varepsilon \begin{bmatrix} u_{21} & u_{23} \\ u_{22} & u_{24} \end{bmatrix} \\
 T_8 &= V_1 + \varepsilon V_2 = \begin{bmatrix} v_{11} & v_{13} \\ v_{12} & v_{14} \end{bmatrix} + \varepsilon \begin{bmatrix} v_{21} & v_{23} \\ v_{22} & v_{24} \end{bmatrix}
 \end{aligned}
 \tag{4}$$

Since $u_{1i} = v_{1i}$ and $u_{2i} = v_{2i}$ ($i = 1, 2, 3, 4$) and we assume that $e^{i\theta_j/2} = x_j, j = 1, 2, \dots, 6$, we can obtain eight equations below:

$$\begin{aligned}
 &a_{11}x_1x_2x_3 + a_{12}x_1x_2x_3^{-1} + a_{13}x_1x_2^{-1}x_3 + a_{14}x_1^{-1}x_2x_3 \\
 &+ a_{15}x_1^{-1}x_2^{-1}x_3 + a_{16}x_1x_2^{-1}x_3^{-1} + a_{17}x_1^{-1}x_2x_3^{-1} + a_{18}x_1^{-1}x_2^{-1}x_3^{-1} \\
 &= b_{11}x_4x_5x_6 + b_{12}x_4x_5x_6^{-1} + b_{13}x_4^{-1}x_5x_6 + b_{14}x_4x_5^{-1}x_6 \\
 &+ b_{15}x_4x_5^{-1}x_6^{-1} + b_{16}x_4^{-1}x_5^{-1}x_6 + b_{17}x_4^{-1}x_5x_6^{-1} + b_{18}x_4^{-1}x_5^{-1}x_6^{-1} \quad (i = 1, 2, \dots, 8)
 \end{aligned}
 \tag{5}$$

where a_i and b_i are known parameters determined by the structure of mechanical arm.

3.2. Eliminate x_4, x_5 and x_6

Assume the right end of the eight Equations in (5) to be:

$$\begin{aligned}
 y_1 &= x_4x_5x_6; y_2 = x_4x_5x_6^{-1}; y_3 = x_4x_5^{-1}x_6; \\
 y_5 &= x_4^{-1}x_5x_6; y_4 = x_4x_5^{-1}x_6^{-1}; y_6 = x_4^{-1}x_5^{-1}x_6; \\
 y_7 &= x_4^{-1}x_5x_6^{-1}; y_8 = x_4^{-1}x_5^{-1}x_6^{-1} \\
 x_4x_5x_6 - y_1 &= 0; x_4x_5 - x_6y_2 = 0; x_4x_6 - x_5y_3 = 0; \\
 x_5x_6 - x_4y_4 &= 0; x_4 - x_5x_6y_5 = 0; x_6 - x_4x_5y_6 = 0; \\
 x_5 - x_4x_6y_7 &= 0; 1 - x_4x_5x_6y_8 = 0.
 \end{aligned}
 \tag{6}$$

Solve y_i and we can get eight equations below:

$$\begin{aligned}
 &c_{11}x_1x_2x_3 + c_{12}x_1x_2x_3^{-1} + c_{13}x_1x_2^{-1}x_3 + c_{14}x_1^{-1}x_2x_3 + c_{15}x_1x_2^{-1}x_3^{-1} \\
 &+ c_{16}x_1^{-1}x_2x_3^{-1} + c_{17}x_1^{-1}x_2^{-1}x_3 + c_{18}x_1^{-1}x_2^{-1}x_3^{-1} = y_i \quad (i = 1, 2, \dots, 8)
 \end{aligned}
 \tag{7}$$

where c_i are known parameters determined by the structure of mechanical arm.

In (7), we use lexicographic method to order, so $x_4 > x_5 > x_6$, then we compute the Groebner base. Use algebraic system to perform symbolic operation, then we get 21 bases. However, 11 bases among them are eliminated because of containing x_4, x_5 and x_6 . Other ten lower-degree bases are selected as quadric bases:

$$\begin{aligned}
 -y_6 * y_7 + y_5 * y_8 &= 0; -y_4 * y_6 + y_3 * y_8 = 0; \\
 -y_4 * y_7 + y_2 * y_8 &= 0; -y_2 * y_5 + y_1 * y_7 = 0; \\
 -y_3 * y_5 + y_1 * y_6 &= 0; -y_2 * y_3 + y_1 * y_4 = 0; \\
 -1 + y_4 * y_5 &= 0; -1 + y_3 * y_7 = 0; \\
 -1 + y_1 * y_8 &= 0; -1 + y_2 * y_6 = 0.
 \end{aligned}
 \tag{8}$$

Substitute $y_1 \sim y_8$ into Equation (8) and multiply $x_1^2x_2^2x_3^2$ to both sides, then we obtain 10 equations about x_1^2, x_2^2, x_3^2 whose degrees are no more than 4:

$$d_{i1} + d_{i2}x_1^2 + \dots + d_{i27}x_1^4x_2^4x_3^4 = 0 \quad (i = 1, \dots, 10).
 \tag{9}$$

For (9), assume $x_1^2 = X_1, x_2^2 = X_2, x_3^2 = X_3$ and simplify this equation by

lexicographic Groebner base. Detailed steps are:

Use Mathematica on Intel Pentium IV, 2.93 GHz, RAM 1 G, PC and rank the Groebner bases in (9) according to $X_1 > X_2 > X_3$. This process cost 5.657 seconds and we obtain 11 bases.

Seven of the 11 bases are shown below:

$$e_{i1} + e_{i2}X_1 + e_{i3}X_1^2 + e_{i4}X_2 + e_{i5}X_1X_2 + e_{i6}X_2^2 + e_{i7}X_2^3 \quad (i = 1, 2, \dots, 7). \quad (10)$$

Other four bases are abandoned because they have high degree of X_1 and X_2 . The coefficients e_{ij} in (10) are 4, 3, 1, 3, 2, 2, 1 algebraic expression of X_3 .

The seven Equations in (10) can be transformed into matrix form:

$$N_{7 \times 7} \mathbf{T} = 0 \quad (11)$$

where $\mathbf{T} = [1, X_1, X_1^2, X_2, X_1X_2, X_2^2, X_2^3]^T$.

In order to have solutions, the value of determinant of coefficient must be 0, that is

$$\det(N_{7 \times 7}) = 0. \quad (12)$$

According to the analysis to the degree of X_3 , matrix $N_{7 \times 7}$ has 16 degrees about X_3 . So the degree of expansion is no more than 16.

According to (12), there is no need to extract any common factor and 16-degree input and output equations with single variable X_3 can be obtained:

$$\sum_{i=0}^{16} s_i X_3^i = 0 \quad (13)$$

where s_i are real coefficients determined by input parameters. Solve (13), 16 solutions will be obtained and through:

$$\theta_3 = \ln X_3 / i \quad (14)$$

value of θ_3 is available.

Substitute x_3 to (8) and extract eight equations to build an equation set; consider $X_1, X_1^2, X_2, X_1X_2, X_2^2, X_2^3$ as unknown variables and solve it. Then we can get X_1 and X_2 . Through (14), we can get θ_1 and θ_2 . Input θ_1 and θ_2 to (7), y_i can be solved. Then, by (6) and (14), θ_4, θ_5 and θ_6 are also available.

3.3. Sample Analysis

Use the parameters in example by Q. Shuguang, which the structural parameters are shown in **Table 1**. According to the parameter above, we obtain the position and gesture matrix of the end of the mechanical arm, which has 16 groups of joint angles.

The results of back solution are shown in **Table 2**, in which there are four real solutions and the 14th solution equals to the initial value. In addition, the result in **Table 2** is exactly the same as the result gained by program in [2] [7] and [8]. In this way, this algorithm is correct.

4. Conclusion

Based on research of quaternion in matrix form, we apply it on the modeling of 6R robot. By using lexicographic Groebner base twice, we eliminate 3 variables

Table 1. Structural parameters of 6R robot.

Index i	Moving distance along x -axis a/mm	Rotation angle around x -axis $\alpha_j(^{\circ})$	Moving distance along z -axis s/mm	Rotation angle around z -axis $\theta_j(^{\circ})$
1	100	90	900	80
2	400	-90	100	-16
3	800	45	200	110
4	125	90	300	70
5	200	30	700	-30
6	300	50	300	20

Table 2. 16 groups of solutions of 6R robot.

Index	$\theta_1(^{\circ})$	$\theta_2(^{\circ})$	$\theta_3(^{\circ})$	$\theta_4(^{\circ})$	$\theta_5(^{\circ})$	$\theta_6(^{\circ})$
1	126.904 + 61.1805i	-160.617 - 29.2234i	162.153 + 80.0235i	131.966 - 81.8681i	156.188 - 80.3445i	-10.6735 + 74.01i
2	126.904 - 61.1805i	-160.617 + 29.2234i	162.153 - 80.0235i	131.966 + 81.8681i	156.188 + 80.3445i	-10.6735 - 74.01i
3	-154.594 + 3.76672i	-56.7362 - 33.2051i	-119.269 - 28.7903i	165.725 - 1.57149i	59.3951 + 71.4594i	6.67326 - 40.6869i
4	-154.594 - 3.76672i	-56.7362 + 33.2051i	-119.269 + 28.7903i	165.725 + 1.57149i	59.3951-71.4594i	6.67326 + 40.6869i
5	80	-16	110	70	-30	20
6	72.0137	3.45572	120.785	56.4888	19.0818	-13.9585
7	-3.33344 - 3.61438i	8.78943 - 30.7833i	171.434 + 30.7778i	76.1364 - 72.1063i	3.69214 + 39.4426i	-16.8816 - 54.9133i
8	-3.33344 + 3.61438i	8.78943 + 30.7833i	171.434 - 30.7778i	76.1364 + 72.1063i	3.69214 - 39.4426i	-16.8816 + 54.9133i
9	-69.6973 + 43.8222i	15.5748 - 28.126i	-160.994 - 16.4298i	112.288 - 59.3653	8.20624 + 62.8281	-16.1271 - 50.7677i
10	-69.6973 - 43.8222i	15.5748 + 28.126i	-160.994 + 16.4298ii	112.288 + 59.3653	8.20624-62.8281	-16.1271 + 50.7677i
11	-121.824	52.449	-46.4873	-7.0332	55.0851	-102.853
12	34.4192 + 2.54131i	94.567 - 5.42461i	85.5904 + 5.78504i	140.208 - 6.40323i	109.799 - 10.493i	-33.5871 + 5.78306i
13	34.4192 - 2.54131i	94.567 + 5.42461i	85.5904 - 5.78504i	140.208 + 6.40323i	109.799 + 10.493i	-33.5871 - 5.78306i
14	-110.338	105.204	-67.5107	37.7924	-146.258	58.386
15	-41.4745 + 92.6597i	173.727 + 9.92687i	-48.825 + 311.265i	-79.3546 + 5.61266i	-139.649 + 83.3555i	-139.951 - 275.645i
16	-41.4745 - 92.6597i	173.727 - 9.92687i	-48.825 - 311.265i	-79.3546 - 5.61266i	-139.649 - 83.3555i	-139.951 + 275.645i

and obtain 10 bases in the first time and 11 bases in the second time. Thus, the degree of variables decreases. Using 7 bases among them and resultant elimination, the analytical solution of the 16-degree equation is obtained and the numerical example gets the same result as the literature above. It is a convenient method which can be applied to inverse kinematics analysis of 6R robot in three dimensions, laying a new foundation for real application.

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