Journal of Applied Mathematics and Physics, 2018, 6, 389-404
http://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Chen's Inequalities for Submanifolds in (κ, μ)-Contact Space Form with a Semi-Symmetric Non-Metric Connection

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How to cite this paper: Ahmad, A., Shahzad, F. and Li, J. (2018) Chen's Inequalities for Submanifolds in (κ, μ)-Contact Space Form with a Semi-Symmetric Non-Metric Connection. *Journal of Applied Mathematics and Physics*, **6**, 389-404.
https://doi.org/10.4236/jamp.2018.62037

Received: January 12, 2018
Accepted: February 24, 2018
Published: February 27, 2018

Abstract

In this paper, we obtain Chen's inequalities in (κ, μ)-contact space form with a semi-symmetric non-metric connection. Also we obtain the inequalites for Ricci and K-Ricci curvatures.

Keywords

(κ, μ)-Contact Space Form, Semi-Symmetric Non-Metric Connection, Chen's Inequalities, Ricci Curvature

1. Introduction

In 1924, Friedmann and Schouten [1] introduced the idea of a semi-symmetric connection on a differentiable manifold. A linear connection $\bar{\nabla}$ on a differentiable manifold M is said to be semi-symmetric connection if the torsion tensor $\bar{T}$ of the connection $\bar{\nabla}$ satisfies

$$\bar{T}(\bar{X}, \bar{Y}) = \phi(\bar{Y})\bar{X} - \phi(\bar{X})\bar{Y}$$

where ϕ is a 1-form.

In 1932, Hayden [2] introduced the notion of a semi-symmetric metric connection on a Riemannian manifold (M, g). A semi-symmetric connection $\bar{\nabla}$ is said to be semi-symmetric metric connection if

$$\bar{\nabla} g = 0.$$

Yano [3] studied some properties of a Riemannian manifold endowed with a semi-symmetric metric connection. Submanifolds of a Riemannian manifold with a semi-symmetric metric connection were studied by Nakao [4].

After a long gap, the study of semi-symmetric connection $\bar{\nabla}$ satisfying

$$\bar{\nabla} g \neq 0 \tag{1}$$

was initiated by Prvanovic [5] with the name Pseudo-metric semi-symmetric connection, and was just followed by Smaranda and Andonie [6].

A semi-symmetric connection $\bar{\nabla}$ is said to be a semi-symmetric non-metric connection if it satisfies the condition Equation (1).

In 1992, Agashe and Chafle [7] introduced a semi-symmetric non-metric connection $\bar{\nabla}$ on a Riemannian manifold (M,g) which is given by

$$\bar{\nabla}_{\bar{X}}\bar{Y}=\bar{\nabla}'_{\bar{X}}\bar{Y}+\phi(\bar{X})(\bar{Y})$$

where $\bar{\nabla}'$ is Riemannian connection on M. They give the relation between the curvature tensor of the manifold with respect to the semi-symmetric non-metric connection and the Riemannian connection. They also proved that the projective curvature tensors of the manifold with respect to these connections are equal to each other.

In 2000, Sengupta, De, and Binh [8] gave another type of semi-symmetric non-metric connection. Özgür [9] studied properties of submanifolds of a Reiemannian manifold with the semi-symmetric non-metric connection.

On the other hand, one of the basic problem in submanifold theory is to find the simple relationship between the intrinsic and extrinsic invariants of a submanifold. Chen [10] [11] [12], established inequalities in this respect, called Chen inequalities. And many geometers studied similar problems for different submanifolds in various ambient space, see [13] [14] [15] [16] [17].

Motivated by [7] [21] and [22], we have studied Chen's inequalities for submanifolds in (κ,μ)-contact space form with a semi-symmetric non-metric connection. The paper is organized as follows. In Section 2, we give a brief introduction about semi-symmetric non-metric connection, (κ,μ)-contact space, Chen invarants. In Section 3, for submanifolds in (κ,μ)-contact space form with a semi-symmetric non-metric connection we establish the Chen first inequality and Chen Ricci inequalities by using algebraic lemmas.

2. Preliminaries

Let N^{n+p} be an $(n+p)$-dimensional Riemannian manifold and $\bar{\nabla}$ is a linear connection on N^{n+p}. If the torsion tensor

$$\bar{T}(\bar{X},\bar{Y})=\bar{\nabla}_{\bar{X}}\bar{Y}-\bar{\nabla}_{\bar{Y}}\bar{X}-[\bar{X},\bar{Y}]$$

for any vector fields $\bar{X}$ and $\bar{Y}$ on N^{n+p} satisfies $\bar{T}(\bar{X},\bar{Y})=\phi(\bar{Y})\bar{X}-\phi(\bar{X})\bar{Y}$ for a 1-form ϕ, then the connection $\bar{\nabla}$ is called a semi-symmetric connection.

Let g be a Riemannian metric on N^{n+p}. If $\bar{\nabla}g=0$, then $\bar{\nabla}$ is called a semi-symmetric metric connection on N^{n+p}. If $\bar{\nabla}g\neq 0$, then $\bar{\nabla}$ is called a semi-symmetric non-metric connection on N^{n+p}.

Following [7], a semi-symmetric symmetric non-metric connection $\bar{\nabla}$ on N^{n+p} is given by

$$\bar{\nabla}_{\bar{X}}\bar{Y}=\bar{\nabla}'_{\bar{X}}\bar{Y}+\phi(\bar{Y})\bar{X}$$

for any $\bar{X},\bar{Y}\in\mathcal{X}(N^{n+p})$, where $\bar{\nabla}'$ denotes the Levi-civita connection with respect to the Riemannian metric g and ϕ is a 1-form. Denote by $U = \Phi^{\#}$, *i.e.*, the dual vector field U is defined by $g(U, \bar{X}) = \phi(\bar{X})$, for any vector field $\bar{X}$ on N^{n+p}.

Let M^n be an n-dimensional submanifold of N^{n+p} with the semi-symmetric connection $\bar{\nabla}$ and the Levi-Civita connection $\bar{\nabla}'$. On M^n we consider the induced semi-symmetric connection denoted by ∇ and the induced Levi-Civita connection denoted by ∇'. The Gauss formula with respect to ∇ and ∇' can be written as

$$\bar{\nabla}_X Y = \nabla_X Y + \delta(X,Y),\ \bar{\nabla}'_{\bar{X}} Y = \nabla'_X Y + \delta'(X,Y),\ \forall X, Y \in \mathcal{X}(M^n),$$

where δ' is the second fundamental form of M^n and δ is a $(0,2)$-tensor on M^n. According to [18], we know $\delta = \delta'$.

Let $\bar{R}$ and $\bar{R}'$ denote the curvature tensor with respect to $\bar{\nabla}$ and $\bar{\nabla}'$ respectively. We also denote the curvature tensor R and R' associated with ∇ and ∇' repectively. From [7].

$$\bar{R}(X,Y,Z,W) = \bar{R}'(X,Y,Z,W) + S(X,Z)g(Y,W) - S(Y,Z)g(X,W) \quad (2)$$

for all $X, Y, Z, W \in \mathcal{X}(M^n)$, where S is a $(0,2)$-tensor field defined by

$$S(X,Y) = (\bar{\nabla}'_X \phi)Y - \phi(X)\phi(Y),\ \forall X, Y, Z, W \in \mathcal{X}(M^n).$$

Denote by λ the trace of S.

Decomposing the vector field U on M uniquely into its tangent and normal components U^T and $U^{\perp}$, respectively, we have $U = U^T + U^{\perp}$. For any vector field X, Y, Z, W on M, the gauss equation with respect to the semi-symmetric non-metric connection is (see [18])

$$\begin{aligned} \bar{R}(X,Y,Z,W) &= R(X,Y,Z,W) + g(\delta(X,Z), \delta(Y,W)) \\ &\quad - g(\delta(X,W), \delta(Y,Z)) + g(U^{\perp}, \delta(Y,Z))g(X,W) \\ &\quad - g(U^{\perp}, \delta(X,Z))g(Y,W). \end{aligned} \quad (3)$$

In N^{n+p} we can choose a local orthonormal frame $\{e_1, e_2, \cdots, e_n, e_{n+1}, \cdots, e_{n+p}\}$ such that $\{e_1, e_2, \cdots, e_n\}$ are tangent to M^n. Setting $\delta_{ij}^r = g(\delta(e_i, e_j), e_r)$, then the squared lenght of δ is given by

$$\|\delta\|^2 = \sum_{i,j=1}^{n} g(\delta(e_i, e_j)) = \sum_{r=n}^{n+p} \sum_{i,j=1}^{n} (\delta_{ij}^r)^2$$

The mean curvature vector of M^n associated to ∇' is $H' = \frac{1}{n}\sum_{i=1}^{n} \delta'(e_i, e_i)$. The mean curvature vector of M^n associated to ∇ is defined by $H = \frac{1}{n}\sum_{i=1}^{n} \delta(e_i, e_i)$.

Let $\pi \subset T_p M^n$ be a 2-plane section for any $p \in M^n$ and $K(\pi)$ the sectional curvature of M^n associated to the semi-symmetric non-metric connection ∇. The scalar curvature τ associated to the semi-symmetric non-metric connection ∇ at p is defined by

$$\tau(p) = \sum_{1 \le i < j \le n} K(e_i \wedge e_j) \quad (4)$$

Let L_k be a *k*-plane section of T_pM^n and $\{e_1, e_2, \cdots, e_k\}$ any orthonormal basis of L_k. The scalar curvature $\tau(k)$ of L_k associated to the semi-symmetric connection ∇' is given by

$$\tau(L_k) = \sum_{1 \le i < j \le k} \kappa(e_i \wedge e_j) \tag{5}$$

We denote by $(\inf K)(p) = \inf\{K(\pi) \mid \pi \subset T_pM^n, dim\pi = 2\}$. In [12] Chen introduced the first Chen invariant $\delta_m(p) = \tau(p) - (\inf K)(p)$, which is certainly an intrinsic character of M^n.

Suppose L is a *k*-plane section of T_pM and X is a unit vector in L, we choose an orthonormal basis $\{e_1, e_2, \cdots, e_k\}$ of L, such that $e_1 = X$. The Ricci curvature Ric_p of L at X associated to the semi-symmetric metric connection ∇' is given by

$$Ric_L(X) = \kappa_{12} + \kappa_{13} + \cdots + \kappa_{1k} \tag{6}$$

where $\kappa_{ij} = \kappa(e_i \wedge e_j)$. The $Ric_L(X)$ is called a *K*-Ricci curvature. For each integer *k*, $2 \le k \le n$, the Riemannian invariant θ_k on M^n is defined by

$$\theta_k(p) = \left(\frac{1}{k-1}\right)\inf_{L,X}\{Ric_L(X)\}, \ p \in M^n \tag{7}$$

where L is a *k*-plane section in T_pM^n and X is a unit vector in L [19].

Recently, T. Konfogiorgos intoduced the notion of (k,μ)-contact space form [20], which contains the well known class of sasakian space forms for $\kappa = 1$. Thus it is worthwhile to study relationships between intrinsic and extrinsic invariants of submanifolds in a (k,μ)-contact space form with a semi-symmetric non-metric connection $\bar{\nabla}'$.

A $(2m+1)$-dimentional differntiable manifold $\hat{M}$ is called an almost contact metric manifold if there is an almost contact metric structure (φ, ξ, η, g) consisting of a $(1,1)$ tensor field φ, a vector field ξ, a 1-form η and a compatible Riemannian metric g satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \ \eta(\xi) = 1, \ \varphi\xi = 0, \ \eta \circ \phi = 0$$

$$g(X, \varphi Y) = -g(\varphi X, Y), \quad g(X, \xi) = \eta(X) \tag{8}$$

$\forall X, Y \in \mathcal{X}(\hat{M})$. An almost contact metric structure becomes a contact metric structure if $d\eta = \Phi$, where $\Phi(X,Y) = g(X, \varphi Y)$ is the fundamental 2-form of $\hat{M}$.

In a contact metric manifold $\hat{M}$, the $(1,1)$-tensor field h defined by $2h = \mathcal{L}_\xi \varphi$ is symmetric and satisfies

$$h\xi = 0, \quad h\varphi + \varphi h = 0, \quad \bar{\nabla}'\xi = -\varphi - \varphi h, \quad trace(h) = trace(\varphi h) = 0$$

The (k,μ)-nullity distribution of a contact metric manifold $\hat{M}$ is a distribution

$$N(k,\mu): p \to N_p(k,\mu) = \{Z \in T_p\hat{M} \mid \hat{R}(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y] + \mu[g(Y,Z)hX - g(X,Z)hY]\}$$

where k and μ are constants. If $\xi \in N(k,\mu)$, $\hat{M}$ is called a (k,μ)-contact metric manifold. Since in a (k,μ)-contact metric manifold one has $h^2=(k-1)\varphi^2$, therefore $k\le 1$ and if $k=1$ then the structure is Sasakian.

The sectional curvature $\hat{K}(X,\varphi X)$ of a plane section spanned by a unit vector orthogonal to ξ is called a φ-sectional curvature. If the (k,μ)-contact metric manifold $\hat{M}$ has constant φ-sectional curvature *C*, then it is called a (k,μ)-contact space form and it is denoted by $\hat{M}(C)$. The curvature tensor of $\hat{M}(C)$ is given by [20].

$$\begin{aligned}\bar{R}'(X,Y)Z=&\frac{c+3}{4}\{g(Y,Z)X-g(x,z)y\}+\frac{c+3-4k}{4}\{\eta(X)\eta(Z)Y\\&-\eta(Y)\eta(Z)X+g(X,Z)\eta(Y)\xi-g(Y,Z)\eta(X,\xi)\}\\&+\frac{c-1}{4}\{2g(X,\varphi Y)\varphi Z+g(\varphi X,\varphi Z)\varphi Y-g(\varphi Z)\varphi X\}\\&+\frac{1}{2}\{g(hY,Z)hX-g(hX,Z)hY+g(\varphi hX,Z)\varphi hY\\&-g(\varphi hY,Z)\varphi hX\}-g(X,Z)hY+g(Y,Z)hX+\eta(X)\eta(Z)hY\\&-\eta(Y)\eta(Z)hX-g(hX,Z)Y+g(hY,Z)X-g(hY,Z)\eta(X)\xi\\&+g(hX,Z)\eta(Y)\xi+\mu\{\eta(Y)\eta(Z)hX-\eta(X)\eta(Z)hY\\&+g(hY,Z)\eta(X)\xi-g(hX,Z)\eta(Y)\xi\}\end{aligned}\tag{9}$$

$\forall X,Y,Z\in\mathcal{X}(\hat{M})$, Where $c+2k=-1=k-\mu$ if $k<1$.

For a vector field *X* on a submanifold *M* of a (k,μ)-contact form $\hat{M}(C)$, Let PX be the tangential part of φX. Thus, P is an endomorphism of the tangent bundle of *M* and satisfies $g(X,PY)=-g(PX,Y)$ for $X,Y\in\mathcal{X}(\hat{M})$. $(\varphi h)^T X$ and $h^T X$ are the tangential parts of φhX and hX, respectively. Let $\{e_1,e_2,\cdots,e_n\}$ be an orthonormal basis of T_pM. We set $\|\vartheta\|^2=\sum_{i,j=1}^n g(e_i,\vartheta e_j)^2$, $\vartheta\in\{P,(\vartheta h)^T,h^T\}$. Let $\pi\subset T_pM$ be a 2-plane section spanning by an orthonormal basis $\{e_1,e_2\}$. Then $\beta(\pi)$ given by

$$\beta(\pi)=\langle e_1,Pe_2\rangle^2$$

is a real number in $[0,1]$, which is independent of the choice of orthonormal basis $\{e_1,e_2\}$. Put $\gamma(\pi)=(\eta(e_1))^2+(\eta(e_2))^2$

$$\theta(\pi)=\eta(e_1)^2 g(h^Te_2,e_2)+\eta(e_2)^2 g(h^Te_1,e_1)-2\eta(e_1)\eta(e_2)g(h^Te_1,e_2)$$

Then $\gamma(\pi)$ and $\theta(\pi)$ are also real numbers and do not depend on the choice of orthonormal basis $\{e_1,e_2\}$, of course, $\gamma(\pi)\in[0,1]$

3. Chen's First Inequality

For submanifold of a (k,μ)-contact space form endowed with a semi-symmetric non-matric connection, we establish th following optimal inequality relating the scalar curvature and the squared mean curvature, which will be called Chen first inequality. We recall the following lemma.

Lemma 3.1 ([22]) *Let* $f(x_1,x_2,\cdots,x_n)$ *for* $(n\ge 3)$ *be a function in* R^n *defined by*

$$f\left(x_1,x_2,\cdots,x_n\right)=\left(x_1+x_2\right)\sum_{i=3}^{n}x_i+\sum_{3\le i<j\le n}x_ix_j.$$

If $x_1+x_2+\cdots+x_n=(n-1)\varepsilon$, then we have

$$f\left(x_1,x_2,\cdots,x_n\right)\le\frac{(n-1)(n-2)}{2}\varepsilon^2$$

with the equality holding if and only if $x_1+x_2=\cdots=x_n=\varepsilon.$

Theorem 3.1 *Let M ba an n-dimensional* $(n\ge 3)$ *submanifold of a* $(2m+1)$*-dimensional* (k,μ)*-contact form* $\hat{M}(C)$ *endowed with a semi-symmetric non-metric connection* $\bar{\nabla}'$ *such that* $\xi\in TM$ *. Then, for each* 2*-plane section* $\pi\subset T_pM$ *. We have,*

$$\begin{aligned}\tau(p)-K(\pi)\le{}&\frac{n^2(n-2)}{2(n-1)}\|H\|^2+\frac{1}{8}n(n-3)(c+3)+(n-1)k\\&+\frac{3(c-1)}{8}\left[\|P\|^2-2\beta(\pi)\right]+\frac{1}{4}(c+3-4k)\gamma(\pi)\\&-(\mu-1)\theta(\pi)-\frac{1}{2}\left[2trace\left(h|_\pi\right)+det\left(h|_\pi\right)-det\left(\varphi h|_\pi\right)\right]\\&+(\mu+n-2)trace\left(h^T\right)+\frac{1}{4}\left[\left\|(\varphi h)^T\right\|^2-\left\|h^T\right\|^2-\left(trace(\varphi h)^T\right)^2\right.\\&\left.+\left(trace\left(h^T\right)\right)^2\right]-\frac{n(n-1)}{2}\varphi(H)-\frac{n-1}{2}\lambda+\Omega\end{aligned}\tag{10}$$

The equality in (10) holds at $p\in M$ if and only if there exits an orthonormal basis $\{e_1,e_2,\cdots,e_n\}$ of T_pM and an orthonormal basis $\{e_{n+1},\cdots,e_{2m+1}\}$ of $T_p^\perp M$ such that (a) $\pi=span\{e_1,e_2\}$ and (b) the forms of shape operators $A_r\equiv A_{e_r}$, $r=n+1,\cdots,2m+1$

$$A_{n+1}=\begin{pmatrix}\delta_{11}^{n+1}&0&0\\0&\delta_{22}^{n+1}&0\\0&0&\left(\delta_{11}^{n+1}+\delta_{22}^{n+1}\right)I_{n-2}\end{pmatrix}$$

$$A_r=\begin{pmatrix}\delta_{11}^{r}&\delta_{12}^{r}&0\\\delta_{12}^{r}&-\delta_{11}^{r}&0\\0&0&0_{n-2}\end{pmatrix}$$

Proof. Let $\pi\subset T_pM$ be a 2-plane section. We choose an orthonormal basis $\{e_1,e_2,\ldots,e_n\}$ for T_pM and $\{e_{n+1},\cdots,e_{2m+1}\}$ for $T_p^\perp M$ such that $\pi=Span\{e_1,e_2\}$. Setting $X=W=e_i$, $Y=Z=e_j$, $i\ne j$, $i,j=1,\cdots,n$. And using (2), (3) and (9) we get

$$\begin{aligned}R_{ijji}={}&\frac{c+3}{4}+\frac{c+3-4k}{4}\left\{-\eta(e_i)^2-\eta(e_j)^2\right\}+\frac{c-1}{4}\left\{3g\left(e_i,\varphi e_j\right)^2\right\}\\&+\frac{1}{2}\left\{g\left(e_i,\varphi he_j\right)^2-g\left(e_i,he_j\right)^2+g\left(e_i,he_i\right)g\left(e_j,he_j\right)\right.\\&\left.-g\left(e_i,\varphi he_i\right)g\left(e_j,\varphi he_j\right)\right\}+g\left(e_i,he_i\right)+2\eta(e_i)\eta(e_j)g\left(e_i,he_j\right)\\&-g\left(he_i,e_i\right)\eta(e_j)^2-g\left(he_j,e_j\right)\eta(e_i)^2+g\left(he_j,e_j\right)\end{aligned}$$

$$
\begin{aligned}
&+\mu\left\{g\left(he_i,e_i\right)\eta\left(e_j\right)^2+g\left(he_j,e_j\right)\eta\left(e_i\right)^2-2\eta\left(e_i\right)\eta\left(e_j\right)g\left(e_i,he_j\right)\right\}\\
&-\varphi\left(\delta\left(e_j,e_j\right)\right)-S\left(e_j,e_j\right)-g\left(\delta\left(e_i,e_j\right),\delta\left(e_j,e_i\right)\right)\\
&+g\left(\delta\left(e_i,e_i\right),\delta\left(e_j,e_j\right)\right)
\end{aligned}\tag{11}
$$

From (11) we get

$$
\begin{aligned}
\tau=&\frac{1}{8}\left\{n(n-1)(c+3)+3(c-1)\|p\|^2-2(n-1)(c+3-4k)\right\}\\
&+\frac{1}{4}\left\{\left\|(\varphi h)^T\right\|^2-\left\|h^T\right\|^2-\left(trace(\varphi h)^T\right)^2+\left(trace\left(h^T\right)\right)^2\right\}\\
&+(\mu+n-1)trace\left(h^T\right)+\sum_{r=n+1}^{2m+1}\sum_{1\le i<j\le n}\left[\delta_{ii}^r\delta_{jj}^r-\left(\delta_{ij}^r\right)^2\right]\\
&-\frac{n(n-1)}{2}\phi(H)-\frac{n-1}{2}\lambda.
\end{aligned}\tag{12}
$$

where $\phi(H)=\frac{1}{n}\sum_{i=1}^{n}\phi\left(\delta\left(e_i,e_i\right)\right)=g\left(U^{\perp},H\right)$. On the other hand, using (11) we have

$$
\begin{aligned}
R_{1212}=&\frac{1}{4}\{c+3+3(c-1)\beta(\pi)-(c+3-4k)\gamma(\pi)\}\\
&+\frac{1}{2}\left\{det\left(h|_{\pi}\right)-det\left(\varphi h|_{\pi}\right)\right\}+trace\left(h|_{\pi}\right)-\theta(\pi)+\mu\theta(\pi)\\
&+\sum_{r=n+1}^{2m+1}\left[\delta_{11}^r\delta_{22}^r-\left(\delta_{12}^r\right)^2\right]-\phi\left(\delta\left(e_2,e_2\right)\right)-S\left(e_2,e_2\right)\\
=&\frac{1}{4}\{c+3+3(c-1)\beta(\pi)-(c+3-4k)\gamma(\pi)+4(\mu-1)\theta(\pi)\}\\
&+\frac{1}{2}\left\{det\left(h|_{\pi}\right)-det\left(\varphi h|_{\pi}\right)+2trace\left(h|_{\pi}\right)\right\}+\sum_{r=n+1}^{2m+1}\left[\delta_{11}^r\delta_{22}^r-\left(\delta_{12}^r\right)^2\right]-\Omega,
\end{aligned}\tag{13}
$$

where Ω is denoted by $\phi\left(\delta\left(e_2,e_2\right)\right)+S\left(e_2,e_2\right)=\Omega$.

From (12) and (13). It follows that

$$
\begin{aligned}
\tau-K(\pi)=&\frac{1}{8}n(n-3)(c+3)+\left(n_1\right)k+\frac{3(c-1)}{8}\left[\|P\|^2-2\beta(\pi)\right]\\
&+\frac{1}{4}(c+3-4k)\gamma(\pi)-(\mu-1)\varphi(\pi)-\frac{1}{2}\left\{2trace\left(h|_{\pi}\right)\right.\\
&\left.+det\left(h|_{\pi}\right)-det\left(\varphi h|_{\pi}\right)\right\}+(\mu+n-2)trace\left(h^T\right)\\
&+\frac{1}{4}\left\{\left\|(\varphi h)^T\right\|^2-\left\|h^T\right\|^2-\left(trace(\varphi h)^T\right)^2+\left(trace\left(h^T\right)\right)^2\right\}\\
&+\sum_{r=n+1}^{2m+1}\left[\left(\delta_{11}^r+\delta_{22}^r\right)\sum_{3\le i\le n}\delta_{ii}^r+\sum_{3\le i<j\le n}\delta_{ii}^r\delta_{jj}^r-\sum_{3\le j\le n}\left(\delta_{1j}^r\right)^2\right.\\
&\left.-\sum_{2\le i<j\le n}\left(\delta_{ij}^r\right)^2\right]-\frac{n(n-1)}{2}\phi(H)-\frac{n-1}{2}\lambda+\Omega\\
\le&\frac{1}{8}n(n-3)(c+3)+(n-1)k+\frac{3(c-1)}{8}\left[\|P\|^2-2\beta(\pi)\right]\\
&+\frac{1}{4}(c+3-4k)\gamma(\pi)-\mu(\mu-1)\theta(\pi)-\frac{1}{2}\left\{2trace\left(h|_{\pi}\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
&+det\left(h|_\pi\right)-det\left(\varphi h|_\pi\right)\Big\}+(\mu+n-2)trace\left(h^T\right)\\
&+\frac{1}{4}\left\{\left\|(\varphi h)\right\|^2-\left\|h^T\right\|^2-\left(trace(\varphi h)^T\right)^2+\left(trace\left(h^T\right)\right)^2\right\}\\
&+\sum_{r=n+1}^{2m+1}\left[\left(\delta_{11}^r+\delta_{22}^r\right)\sum_{3\le i\le n}\delta_{ii}^r+\sum_{3\le i<j\le n}\delta_{ii}^r\delta_{jj}^r\right]\\
&-\frac{n(n-1)}{2}\phi(H)-\frac{n-1}{2}\lambda+\Omega
\end{aligned}
\tag{14}
$$

Let us consider the following problem:

$$
\max\left\{f_r\left(\delta_{11}^r,\cdots,\delta_{nn}^r\right)=\left(\delta_{11}^r+\delta_{22}^r\right)\sum_{3\le i\le n}\delta_{ii}^r+\sum_{3\le i<j\le n}\delta_{ii}^r\delta_{jj}^r \mid \delta_{11}^r+\cdots+\delta_{nn}^r=k^r\right\}
$$

where k^r is a real constant.

From lemma 3.1, We know

$$
f_r\le\frac{n-2}{2(n-1)}\left(k^r\right)^2 \tag{15}
$$

with the equality holding if and only if

$$
\delta_{11}^r+\delta_{22}^r=\delta_{ii}^r=\frac{k^r}{n-1},\, i=3,\cdots,n \tag{16}
$$

From (14) and (15), we have

$$
\begin{aligned}
\tau-K(\pi)=&\frac{n^2(n-2)}{2(n-1)}\|H\|^2+\frac{1}{8}n(n-3)(c+3)+(n-1)k\\
&+\frac{3(c-1)}{8}\left[\|P\|^2-2\beta(\pi)\right]+\frac{1}{4}(c+3-4k)\gamma(\pi)-(\mu-1)\theta(\pi)\\
&-\frac{1}{2}\left[2trace\left(h|_\pi\right)+det\left(h|_\pi\right)-det\left(\varphi h|_\pi\right)\right]+(\mu+n-2)trace\left(h^T\right)\\
&+\frac{1}{4}\left[\left\|(\varphi h)^T\right\|^2-\left\|h^T\right\|^2-\left(trace(\varphi h)^T\right)^2+\left(trace\left(h^T\right)\right)^2\right]\\
&-\frac{n(n-1)}{2}\phi(H)-\frac{n-1}{2}\lambda+\Omega
\end{aligned}
$$

If the equality in (10) holds, then the inequalities given by (14) and (15) become equalities. In this case we have

$$
\sum_{2\le i\le n}\left(\delta_{1i}^r\right)^2=0,\ \sum_{2\le i<j\le n}\left(\delta_{ij}^r\right)^2=0,\ \forall r.
$$

$$
\delta_{11}^r+\delta_{22}^r=\delta_{ii}^r,\ 3\le i\le n,\ \forall r.
$$

From [18] we know $\delta'=\delta$. So choose a suitable orthonormal basis, the shape operators take the desired forms.

The converse is easy to follow.

For a Sasakian space form $\hat{M}(c)$, we have $\kappa=1$ and $h=0$. So using Theorem 3.1, we have the following corollary.

Corollary 3.1 *Let M be an n-dimensional* $(n\ge 3)$ *submanifold in a sasakian space form* $\hat{M}(c)$ *endowed with a semi-symmetric non-metric connection such that* $\xi\in TM$ *. Then, for each point* $p\in M$ *and each plane section*

$\pi \subset T_pM$, *we have*

$$\begin{aligned} \tau - K(\pi) \le & \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{1}{8}n(n-3)(c+3)+(n-1) \\ & + \frac{3(c-1)}{8}\left[\|P\|^2 - 2\beta(\pi)\right] + \frac{c-1}{4}\gamma(\pi) \\ & - \frac{n(n-1)}{2}\phi(H) - \frac{n-1}{2}\lambda + \Omega \end{aligned} \tag{17}$$

If U is a tangent vector field to *M*, then the equality in (17) holds at $p \in M$ if and only there exists an orthonormal basis $\{e_1, e_2, \cdots, e_n\}$ of T_pM and orthonormal basis $\{e_{n+1}, \cdots, e_{2m+1}\}$ of $T_p^{\perp}M$ such that

$$\pi = Span\{e_1, e_2\}$$

and the forms of shape operators $A_r \equiv A_{e_r}$, $r = n+1, \cdots, 2m+1$, become

$$A_{n+1} = \begin{pmatrix} \delta_{11}^{n+1} & 0 & 0 \\ 0 & \delta_{22}^{n+1} & 0 \\ 0 & 0 & \left(\delta_{11}^r + \delta_{22}^r\right) I_{n-2} \end{pmatrix},$$

$$A_r = \begin{pmatrix} \delta_{11}^r & \delta_{12}^r & 0 \\ \delta_{12}^r & -\delta_{11}^r & 0 \\ 0 & 0 & 0_{n-2} \end{pmatrix}.$$

Since in case of non-Sasakian (κ, μ)-contact space form, we have $\kappa < 1$, thus $c = -2\kappa - 1$ and $\mu = \kappa + 1$. Putting these values in (17), we can have a direct corollary to Theorem 3.1.

Corollary 3.2 *Let Let M be an n-dimensional* $(n \ge 3)$ *submanifold in a non-Sasakian* (κ, μ) *-contact space form* $\hat{M}(c)$ *with a semi-symmetric non-metric connection such that* $\xi \in TM$ *. Then, for each point* $p \in M$ *and each plane section* $\pi \subset T_pM$ *, When* $c = -2k - 1$, $\mu = k + 1$ *we have*

$$\begin{aligned} \tau - K(\pi) = & \frac{n^2(n-2)}{2(n-1)}\|H\|^2 - \frac{1}{4}n(n-3)(k-1) + (n-1)k - \frac{3}{4}(k+1)\|P\|^2 \\ & + \frac{1}{2}\left[3(k+1)\beta(\pi) - (3k-1)\gamma(\pi) - 2k\theta(\pi)\right] \\ & - \frac{1}{2}\left[2trace\left(h|_{\pi}\right) + det\left(h|_{\pi}\right) - det\left(\varphi h|_{\pi}\right)\right] + (k+n-1)trace\left(h^T\right) \\ & + \frac{1}{4}\left[\left\|(\varphi h)^T\right\|^2 - \left\|h^T\right\|^2 - \left(trace(\varphi h)^T\right)^2 + \left(trace\left(h^T\right)\right)^2\right] \\ & - \frac{n(n-1)}{2}\phi(H) - \frac{n-1}{2}\lambda + \Omega \end{aligned} \tag{18}$$

If U is a tangent vector field to *M*, then the equality in (18) holds at $p \in M$ if and only there exists an orthonormal basis $\{e_1, e_2, \cdots, e_n\}$ of T_pM and orthonormal basis $\{e_{n+1}, \cdots, e_{2m+1}\}$ of $T_p^{\perp}M$ such that

$$\pi = Span\{e_1, e_2\}$$

and the forms of shape operators $A_r \equiv A_{e_r}$, $r = n+1, \cdots, 2m+1$, become

$$A_{n+1}=\begin{pmatrix}\delta_{11}^{n+1} & 0 & 0\\ 0 & \delta_{22}^{n+1} & 0\\ 0 & 0 & \left(\delta_{11}^{r}+\delta_{22}^{r}\right)I_{n-2}\end{pmatrix},$$

$$A_r=\begin{pmatrix}\delta_{11}^{r} & \delta_{12}^{r} & 0\\ \delta_{12}^{r} & -\delta_{11}^{r} & 0\\ 0 & 0 & 0_{n-2}\end{pmatrix}.$$

4. Ricci Curvature and *K*-Ricci Curvatures

In this section, we establish inequality between Ricci curvature and the squared mean curvature for submanifolds in a (κ,μ)-contact space form with a semi-symmetric non-metric connection. This inequality is called Chen-Ricci inequality [19].

First we give a lemma as following. First we give a lemma as following.

Lemma 4.1 *([22]) Let* $f(x_1,x_2,\cdots,x_n)$ *be a function in* R^n *defined by*

$$f(x_1,x_2,\cdots,x_n)=x_1\sum_{i=1}^{n}x_i.$$

If $x_1+x_2+\cdots+x_n=2\varepsilon$, then we have

$$f(x_1,x_2,\cdots,x_n)\le\varepsilon^2,$$

with the equality holding if and only if $x_1+x_2+\cdots+x_n=\varepsilon$.

Theorem 4.1 *Let M be an n-dimensional* $(n\ge 2)$ *submanifold of a* $(2m+1)$*-dimensional* (κ,μ)*-contact space form* $\hat{M}(c)$ *endowed with a semi-symmetric non-metric connection such that* $\xi\in TM$. *Then for each point* $p\in M$,

1) For each unit vector X in T_pM, we have

$$\begin{aligned}Ric(X)\le{}&\frac{n^2}{4}\|H\|^2+\frac{(n-1)(c+3)}{4}+\frac{3(c-1)}{4}\|PX\|^2\\&-\frac{c+3-4\kappa}{4}\left[1+(n-2)\eta(X)^2\right]+\frac{1}{2}\Big[\left\|(\varphi hX)^T\right\|^2-\left\|(hX)^T\right\|^2\\&+g(hX,X)trace\left(h^T\right)-g(\varphi hX,X)trace\left((\varphi h)^T\right)\Big]\\&+(\mu+n-3)g(X,hX)+\left[1+(\mu-1)\eta(X)^2\right]trace\left(h^T\right)\\&-n\phi(H)+\phi(\delta(X,X))-\lambda+S(X,X).\end{aligned}\tag{19}$$

2) If $H(p)=0$, a unit tangent vector $X\in T_pM$ satisfies the equality case of (19) if and only if $X\in N(p)=\left\{X\in T_pM\mid (X,Y)=0,\forall Y\in T_pM\right\}$.

3) The equality of (19) holds identically for all unit tangent vectors if and only if

either

1) $n\ne 2$, $\delta_{ij}^{r}=0, i,j=1,2,\cdots,n; r=n+1,\cdots,2m+1,$

or

2) $n=2$, $\delta_{11}^{r}=\delta_{22}^{r}, \delta_{12}^{r}=0, r=3,\cdots,2m+1.$

Proof. (1) Let $X\in T_pM$ be an unit vector. We choose an orthonormal basis

$e_1,\cdots,e_n,e_{n+1},\cdots,e_{2m+1}$ such that $e_1,\cdots,e_n$ are tangential to M at p with $e_1 = X$.

Using (11), we have

$$\begin{aligned}
Ric(X) = {} & \frac{(n-1)(c+3)}{4} - \frac{c+3-4\kappa}{4}\left[1+(n-2)\eta(X)^2\right] + \frac{3(c-1)}{4}\|PX\|^2 \\
& + \frac{1}{2}\left[\left\|(\varphi hX)^T\right\|^2 - \left\|(hX)^T\right\|^2 + g(hX,X)\,trace\left(h^T\right)\right. \\
& \left. - g(\varphi hX,X)\,trace\left((\varphi h)^T\right)\right] + (\mu+n-3)\,g(X,hX) \\
& + \left(1-\eta(X)^2+\mu\eta(X)^2\right)trace\left(h^T\right) - n\phi(H) \\
& + \phi(\delta(X,X)) - \lambda + S(X,X) + \sum_{r=n+1}^{2m+1}\sum_{i=2}^{n}\left[\delta_{11}^r\delta_{ii}^r - \left(\delta_{1i}^r\right)^2\right] \\
\le {} & \frac{(n-1)(c+3)}{4} + \frac{3(c-1)}{4}\|PX\|^2 - \frac{c+3-4k}{4}\left[1+(n-2)\eta(X)^2\right] \\
& + \frac{1}{2}\left[\left\|(\varphi hX)^T\right\|^2 - \left\|(hX)^T\right\|^2 - g(\varphi hX,X)\,trace(\varphi h)^T\right. \\
& \left. + g(hX,X)\,trace\left(h^T\right)\right] + (\mu+n-3)\,g(X,hX) \\
& + \left(1-\eta(X)^2+\mu\eta(X)^2\right)trace\left(h^T\right) - n\phi(H) \\
& + \phi(\delta(X,X)) - \lambda + S(X,X) + \sum_{r=n+1}^{2m+1}\sum_{i=2}^{n}\delta_{11}^r\delta_{ii}^r.
\end{aligned} \tag{20}$$

Let us consider the function $f_r : R^n \to R$, defined by

$$f_r\left(\delta_{11}^r,\delta_{22}^r,\cdots,\delta_{nn}^r\right) = \sum_{i=2}^{n}\delta_{11}^r\delta_{ii}^r.$$

We consider the problem

$$\max\left\{f_r \mid \delta_{11}^r + \cdots + \delta_{nn}^r = k^r\right\},$$

where k^r is a real constant. From lemma 4.1, we have

$$f_r \le \frac{k^r}{4}. \tag{21}$$

With equality holding if and only if

$$\delta_{11}^r = \sum_{i=2}^{n}\delta_{ii}^r = \frac{k^r}{2}. \tag{22}$$

From (20) and (21) we get

$$\begin{aligned}
Ric(X) \le {} & \frac{n^2}{4}\|H\|^2 + \frac{(n-1)(c+3)}{4} + \frac{3(c-1)}{4}\|PX\|^2 \\
& - \frac{c+3-4\kappa}{4}\left[1+(n-2)\eta(X)^2\right] + \frac{1}{2}\left[\left\|(\varphi hX)^T\right\|^2 - \left\|(hX)^T\right\|^2\right. \\
& \left. - g(\varphi hX,X)\,trace(\varphi h)^T + g(hX,X)\,trace\left(h^T\right)\right] \\
& + (\mu+n-3)\,g(X,hX) + \left[1+(\mu-1)\eta(X)^2\right]trace\left(h^T\right) \\
& - n\phi(H) + \phi(\delta(X,X)) - \lambda + S(X,X).
\end{aligned}$$

2) For a unit vector $X \in T_pM$, if the equality case of (19) holds, from (20), (21) and (22) we have

$$\delta_{1i}^r = 0, i \neq 1, \forall r.$$

$$\delta_{11}^r + \delta_{22}^r + \cdots + \delta_{nn}^r = 2\delta_{11}^r, \forall r.$$

Since $H(p) = 0$, we know

$$\delta_{11}^r = 0, \forall r.$$

So we get

$$\delta_{1j}^r = 0, \forall r.$$

i.e. $X \in N(p)$

The converse is trivial.

3) For all unit vector $X \in T_pM$, the equality case of (19) holds if and only if

$$2\delta_{ii}^r = \delta_{11}^r + \cdots + \delta_{nn}^r, i = 1, \cdots, n; r = n+1, \cdots, 2m+1.$$

$$\delta_{ij}^r = 0, i \neq j, r = n+1, \cdots, 2m+1.$$

Thus we have two cases, namely either $n \neq 2$ or $n = 2$.

In the first case we

$$\delta_{ij}^r = 0, i, j = 1, \cdots, n; r = n+1, \cdots, 2m+1.$$

In the second case we have

$$\delta_{11}^r = \delta_{22}^r, \delta_{12}^r = 0, r = 3, \cdots, 2m+1.$$

The converse part is straightforward.

Corollary 4.1 *Let M be an n-dimensional* $(n \geq 2)$ *submanifold in a Sasakian space form* $\hat{M}(c)$ *endowed with a semi-symmetric non-metric connection such that* $\xi \in TM$. *Then for each point* $p \in M$, *For each unit vector X in* T_pM, *for* $k = 1$, $h = 0$ *we have*

$$Ric(X) \leq \frac{n^2}{4}\|H\|^2 + \frac{(n-1)(c+3)}{4} + \frac{3(c-1)}{4}\|PX\|^2 - \frac{c-1}{4}\left[1 + (n-2)\eta(X)^2\right]$$
$$- n\phi(H) + \phi(\delta(X, X)) - \lambda + S(X, X).$$

Corollary 4.2 *Let M be an n-dimensional* $(n \geq 2)$ *submanifold in a non-Sasakian space form* $\hat{M}(c)$ *endowed with a semi-symmetric non-metric connection such that* $\xi \in TM$. *Then for each point* $p \in M$, *For each unit vector* $X \in T_pM$, $\forall p \in M$, *we have*

$$\begin{aligned} Ric(X) \leq{} & \frac{n^2}{4}\|H\|^2 + \frac{(n-1)(1-\kappa)}{2} - \frac{3(\kappa+1)}{2}\|PX\|^2 \\ & + \frac{3\kappa - 1}{2}\left[1 + (n-2)\eta(X)^2\right] + \frac{1}{2}\Big[\left\|(\varphi hX)^T\right\|^2 - \left\|(hX)^T\right\|^2 \\ & - g(\varphi hX, X)\,trace(\varphi h)^T + g(hX, X)\,trace(h^T)\Big] \\ & + (\kappa + n - 2)g(hX, X) + \left[1 + \kappa\eta(X)^2\right]trace(h^T) \\ & - n\phi(H) + \phi(\delta(X, X)) - \lambda + S(X, X). \end{aligned}$$

Theorem 4.2 *Let M be an n-dimensional* $(n \geq 3)$ *submanifold in a* $(2m+1)$

-dimensional (κ,μ) *-contact space form* $\hat{M}(c)$ *endowed with a semi-symmetric non-metric connection such that* $\xi \in TM$ *. Then we have*

$$n(n-1)\|H\|^2 \ge n(n-1)\Theta_k(p) - \frac{1}{4}\Big\{n(n-1)(c+3)+3(c-1)\|P\|^2 - 2(n-1)(c+3-4k)\Big\} - \frac{1}{2}\Big\{\|(\varphi h)^T\|^2 - \|h^T\|^2 - \big(trace(\varphi h)^T\big)^2 + \big(trace(h^T)\big)^2\Big\} - 2\big[\mu+(n-1)\big]trace(h^T) + n(n-1)\phi(H) + (n-1)\lambda.$$

Proof. Let $\{e_1,\cdots,e_n\}$ be an orthonormal basis of T_pM. We denote by $L_{i1,\cdots,ik}$ the *k*-plane section spanned by $e_{i1},\cdots,e_{ik}$. From (5) and (6), it follows that

$$\tau(L_{i1},\cdots,e_{ik}) = \frac{1}{2}\sum_{i\in\{i1,\cdots,ik\}} Ric_{L_{i1,\cdots,ik}}(e_i) \tag{23}$$

and

$$\tau(p) = \frac{1}{C_{n-2}^{k-2}} \sum_{1\le i1<\cdots<ik\le n} \tau(L_{i1,\cdots,ik}). \tag{24}$$

Combining (7), (23) and (24), we obtain

$$\tau(p) \ge \frac{n(n-1)}{2}\Theta_k(p). \tag{25}$$

We choose an orthonormal basis $\{e_1,\cdots,e_n\}$ of T_pM such that e_{n+1} is in the direction of the mean curvature vector $H(p)$ and $\{e_1,\cdots,e_n\}$ diagnolize the shape operator A_{n+1}. Then the shape operators take the following forms:

$$A_{n+1} = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}, \tag{26}$$

$$traceA_r = 0,\ r = n+2,\cdots,2m+1. \tag{27}$$

From (11), we have

$$2\tau = \frac{1}{4}\Big\{n(n-1)(c+3)+3(c-1)\|P\|^2 - 2(n-1)(c+3-4k)\Big\} + \frac{1}{2}\Big\{\|(\varphi h)^T\|^2 - \|h^T\|^2 - \big(trace(\varphi h)^T\big)^2 + \big(trace(h^T)\big)^2\Big\} + 2\big[\mu+(n-1)\big]trace(h^T) - n(n-1)\varphi(H) - (n-1)\lambda + n^2\|H\|^2 - \|\delta\|^2. \tag{28}$$

Using (26) and (28), we obtain

$$n^2\|H\|^2 = 2\tau + \sum_{i=1}^{n} a_i^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}\big(\delta_{ij}^r\big)^2 - \frac{1}{4}\Big\{n(n-1)(c+3)+3(c-1)\|P\|^2 - 2(n-1)(c+3-4k)\Big\} - \frac{1}{2}\Big\{\|(\varphi h)^T\|^2 - \|h^T\|^2 - \big(trace(\varphi h)^T\big)^2 + \big(trace(h^T)\big)^2\Big\} - 2\big[\mu+(n-1)\big]trace(h^T) + n(n-1)\phi(H) + (n-1)\lambda. \tag{29}$$

On the other hand from (26) and (27), we have

$$\left(n\|H\|\right)^2=\left(\sum a_i\right)^2\le n\sum_{i=1}^{n}a_i^2. \tag{30}$$

From (29) and (30), it follows that

$$\begin{aligned}
&n(n-1)\|H\|^2\\
&\ge 2\tau-\frac{1}{4}\left\{n(n-1)(c+3)+3(c-1)\|P\|^2-2(n-1)(c+3-4k)\right\}\\
&\quad-\frac{1}{2}\left\{\left\|(\varphi h)^T\right\|^2-\left\|h^T\right\|^2-\left(trace(\varphi h)^T\right)^2+\left(trace\left(h^T\right)\right)^2\right\}\\
&\quad-2\left[\mu+(n-1)\right]trace\left(h^T\right)+n(n-1)\phi(H)+(n-1)\lambda+\sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}\left(\delta_{ij}^r\right)^2\\
&\ge 2\tau-\frac{1}{4}\left\{n(n-1)(c+3)+3(c-1)\|P\|^2-2(n-1)(c+3-4k)\right\}\\
&\quad-\frac{1}{2}\left\{\left\|(\varphi h)^T\right\|^2-\left\|h^T\right\|^2-\left(trace(\varphi h)^T\right)^2+\left(trace\left(h^T\right)\right)^2\right\}\\
&\quad-2\left[\mu+(n-1)\right]trace\left(h^T\right)+n(n-1)\phi(H)+(n-1)\lambda
\end{aligned}$$

Using (25), we obtain

$$\begin{aligned}
&n(n-1)\|H\|^2\\
&\ge n(n-1)\Theta_k(p)-\frac{1}{4}\left\{n(n-1)(c+3)+3(c-1)\|P\|^2\right.\\
&\quad\left.-2(n-1)(c+3-4k)\right\}-\frac{1}{2}\left\{\left\|(\varphi h)^T\right\|^2-\left\|h^T\right\|^2-\left(trace(\varphi h)^T\right)^2\right.\\
&\quad\left.+\left(trace\left(h^T\right)\right)^2\right\}-2\left[\mu+(n-1)\right]trace\left(h^T\right)\\
&\quad+n(n-1)\phi(H)+(n-1)\lambda.
\end{aligned}$$

Corollary 4.3 *Let M be an n-dimensional* $(n\ge 3)$ *submanifold in a Sasakian space form* $\hat{M}(c)$ *endowed with a semi-symmetric non-metric connection such that* $\xi\in TM$. *Then for each point* $p\in M$, *For each unit vector* $X\in T_pM$, $\forall p\in M$, *we have*

$$\begin{aligned}
&n(n-1)\|H\|^2\\
&\ge n(n-1)\Theta_k(p)-\frac{1}{4}\left\{n(n-1)(c+3)+3(c-1)\|P\|^2-2(n-1)(c-1)\right\}\\
&\quad+n(n-1)\phi(H)+(n-1)\lambda.
\end{aligned}$$

Corollary 4.4 *Let M be an n-dimensional* $(n\ge 3)$ *submanifold in a non-Sasakian space form* $\hat{M}(c)$ *endowed with a semi-symmetric non-metric connection such that* $\xi\in TM$. *Then for each point* $p\in M$, *For each unit vector* $X\in T_pM$, $\forall p\in M$, *we have*

$$\begin{aligned}
&n(n-1)\|H\|^2\\
&\ge n(n-1)\Theta_k(p)-\frac{1}{2}\left\{n(n-1)(1-\kappa)-3(\kappa+1)\|P\|^2+2(n-1)(3\kappa-1)\right\}\\
&\quad-\frac{1}{2}\left\{\left\|(\varphi h)^T\right\|^2-\left\|h^T\right\|^2-\left(trace(\varphi h)^T\right)^2+\left(trace\left(h^T\right)\right)^2\right\}\\
&\quad-2(k+n)trace\left(h^T\right)+n(n-1)\phi(H)+(n-1)\lambda.
\end{aligned}$$

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