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On Quaternionic 3 *CR*-Structure and Pseudo-Riemannian Metric

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Abstract

A *CR*-structure on a $2n+1$-manifold gives a conformal class of Lorentz metrics on the Fefferman S^1-bundle. This analogy is carried out to the *quarternionic conformal* 3-*CR structure* (a generalization of quaternionic *CR*-structure) on a $4n+3$-manifold *M*. This structure produces a conformal class $[g]$ of a pseudo-Riemannian metric *g* of type $(4n+3,3)$ on $M\times S^3$. Let $(\mathrm{PSp}(n+1,1),S^{4n+3})$ be the geometric model obtained from the projective boundary of the complete simply connected quaternionic hyperbolic manifold. We shall prove that *M* is locally modeled on $(\mathrm{PSp}(n+1,1),S^{4n+3})$ if and only if $(M\times S^3,[g])$ is conformally flat (*i.e.* the Weyl conformal curvature tensor vanishes).

Keywords

Conformal Structure, Quaternionic *CR*-Structure, *G*-Structure, Conformally Flat Structure, Weyl Tensor, Integrability, Uniformization, Transformation Groups

1. Introduction

This paper concerns a geometric structure on $(4n+3)$-manifolds which is related with *CR*-structure and also quaternionic *CR*-structure (cf. [1] [2]). Given a quaternionic *CR*-structure $\{\omega_\alpha\}_{\alpha=1,2,3}$ on a $4n+3$-manifold *M*, we have proved in [3] that the associated endomorphism J_α on the $4n$-bundle D naturally extends to a complex structure $\bar{J}_\alpha$ on $\ker\omega_\alpha$. So we obtain 3 *CR*-structures on *M*. Taking into account this fact, we study the following geometric structure on $(4n+3)$-manifolds globally.

A hypercomplex 3 *CR*-structure on a $(4n+3)$-manifold *M* consists of (positive definite) 3 pseudo-Hermitian structures $\{\omega_\alpha, J_\alpha\}_{\alpha=1,2,3}$ on M which satisfies that

1) $\mathrm{D}=\bigcap_{\alpha=1}^{3}\ker\omega_\alpha$ is a $4n$-dimensional subbundle of TM such that $\mathrm{D}+[\mathrm{D},\mathrm{D}]=TM$.

2) Each J_γ coincides with the endomorphism $(d\omega_\beta \mid \mathrm{D})^{-1}\circ(d\omega_\alpha \mid \mathrm{D}):\mathrm{D}\to\mathrm{D}$ $((\alpha,\beta,\gamma)\sim(1,2,3))$ such that $\{J_1,J_2,J_3\}$ constitutes a hypercomplex structure on D.

We call the pair $(\mathrm{D},\{J_1,J_2,J_3\})$ also a hypercomplex 3 *CR*-structure if it is represented by such pseudo-Hermitian structures on *M*. A quaternionic *CR*-structure is an example of our hypercomplex 3 *CR*-structure. As Sasakian 3-structure is equivalent with quaternionic *CR*-structure, Sasakian 3-structure is also an example. Especially the $4n+3$-dimensional standard sphere S^{4n+3} is a hypercomplex 3 *CR*-manifold. The pair $(\mathrm{PSp}(n+1,1),S^{4n+3})$ is the spherical homogeneous model of hypercomplex 3 *CR*-structure in the sense of Cartan geometry (cf. [4]). First we study the properties of *hypercomplex* 3 *CR-structure*. Next we introduce a *quaternionic* 3 *CR*-structure on *M* in a local manner. In fact, let D be a $4n$-dimensional subbundle endowed with a quaternionic structure Q on a $(4n+3)$-manifold *M*. The pair (D,Q) is called quaternionic 3 *CR*-structure if the following conditions hold:

1) $\mathrm{D}+[\mathrm{D},\mathrm{D}]=TM$;

2) *M* has an open cover $\{U_i\}_{i\in\Lambda}$ each U_i of which admits a hypercomplex 3 *CR*-structure $\left(\omega_\alpha^{(i)},J_\alpha^{(i)}\right)_{\alpha=1,2,3}$ such that:

a) $\mathrm{D}\mid U_i=\bigcap_{\alpha=1}^{3}\ker\omega_\alpha^{(i)}$;

b) Each hypercomplex structure $\left\{J_1^{(i)},J_2^{(i)},J_3^{(i)}\right\}_{i\in\Lambda}$ on $\mathrm{D}\mid U_i$ generates a quaternionic structure Q on D.

A $4n+3$-manifold equipped with this structure is said to be a quaternionic 3 *CR*-manifold. A typical example of a quaternionic 3 *CR*-manifold but not a hypercomplex 3 *CR*-manifold is a *quaterninic Heisenberg nilmanifold*. In this paper, we shall study an *invariant* for quaternionic 3 *CR*-structure on $(4n+3)$-manifolds.

Theorem A. *Let* $(M,\{\mathrm{D},Q\})$ *be a quaternionic* 3 *CR-manifold. There exists a pseudo-Riemannian metric g of type* $(4n+3,3)$ *on* $M\times S^3$. *Then the conformal class* $[g]$ *is an invariant for quaternionic* 3 *CR-structure.*

As well as the spherical quaternionic 3 *CR* homogeneous manifold S^{4n+3}, we have the pseudo-Riemannian homogeneous manifold $S^{4n+3}\times S^3$ which is a two-fold covering of the pseudo-Riemannian homogeneous manifold $(S^{4n+3}\times_{\mathbb{Z}_2}S^3,g^0)$. The pair $(\mathrm{PSp}(n+1,1)\times\mathrm{SO}(3),S^{4n+3}\times_{\mathbb{Z}_2}S^3)$ is a subgeometry of conformally flat pseudo-Riemannian homogeneous geometry $(\mathrm{PO}(4n+4,4),S^{4n+3}\times_{\mathbb{Z}_2}S^3)$ where $\mathrm{PSp}(n+1,1)\times\mathrm{SO}(3)\le\mathrm{PO}(4n+4,4)$.

Theorem B. *A quaternionic* 3 *CR-manifold M is spherical* (*i.e. locally modeled on* $(\mathrm{PSp}(n+1,1),S^{4n+3})$) *if and only if the pseudo-Riemannian* *manifold* $(M \times S^3, g)$ *is conformally flat, more precisely it is locally modeled on* $(\mathrm{PSp}(n+1,1)\times \mathrm{SO}(3), S^{4n+3}\times_{\mathbb{Z}_2} S^3)$.

We have constructed a conformal invariant on $(4n+3)$-dimensional pseudo-conformal quaternionic *CR* manifolds in [3]. We think that the Weyl conformal curvature of our new pseudo-Riemannian metric obtained in Theorem A is theoretically the same as this invariant in view of Uniformization Theorem B. But we do not know whether they coincide.

Section 2 is a review of previous results and to give some definition of our notion. In Section 3 we prove the conformal equivalence of our pseudo-Riemannian metrics and prove Theorem A. In Section 4 first we relate our spherical 3 *CR*-homogeneous model $(\mathrm{PSp}(n+1,1), S^{4n+3})$ and the conformally flat pseudo-Riemannian homogeneous model $(\mathrm{PSp}(n+1,1)\times \mathrm{SO}(3), S^{4n+3,3})$. We study properties of 3-dimensional lightlike groups with respect to the pseudo-Riemannian metric g^0 of type $(4n+3,3)$ on $S^{4n+3}\times S^3$. We apply these results to prove Theorem B.

2. Preliminaries

Let $(M, \{\omega_\alpha, J_\alpha\}_{\alpha=1,2,3})$ be a (4*n* + 3)-dimensional hypercomplex 3 *CR*-manifold. Put $(\omega_\alpha, J_\alpha) = (\omega, J)$ for one of α's. By the definition, $(M, \{\omega, J\})$ is a *CR*-manifold. Let $C^{2n+2,0}(M)$ be the canonical bundle over M (*i.e.* the $\mathbb{C}$-line bundle of complex $(2n+2,0)$-forms). Put $C(M) = C^{2n+2,0}(M) - \{0\}/\mathbb{R}^*$ which is a principal bundle: $\mathrm{S}^1 \to C(M) \xrightarrow{p} M$. Compare [[5], Section 2.2]. Fefferman [6] has shown that $C(M)$ admits a Lorentz metric *g* for which the Lorentz isometries S^1 induce a *lightlike* vector field. We recognize the following definition from *pseudo-Riemannian geometry*.

Definition 1. *In general if* S^1 *induces a lightlike vector field with respect to a Lorentz metric of a Lorentz manifold, then* S^1 *is said to be a lightlike group acting as Lorentz isometries. Similarly if each generator* S^1 *of* S^3 *is chosen to be a lightlike group, then we call* S^3 *also a lightlike group.*

We recall a construction of the Fefferman-Lorentz metric from [5] (cf. [6]). Let ξ be the Reeb vector field for (ω, J). The circle S^1 generates the vector field T on $C(M)$. Define dt to be a 1-form on $C(M)$ such that

$$dt(\mathrm{T}) = 1,\ dt(V) = 0\ \left({}^\forall V \in TM\right). \tag{2.1}$$

In [[5], (3.4) Proposition] J. Lee has shown that there exists a unique real 1-form σ on $C(M)$. The explicit form of σ is obtained from [[5], (5.1) Theorem] in this case:

$$\sigma = \frac{1}{2n+3}\left(dt + i\omega_\alpha^\alpha - \frac{i}{2}h^{\alpha\bar{\beta}}dh_{\alpha\bar{\beta}} - \frac{1}{2(2n+2)}R\omega\right). \tag{2.2}$$

Here 1-forms $\{\omega_\alpha^\beta, \tau_\beta\}$ are connection forms of ω such that

$$\begin{aligned} d\omega &= ih_{\alpha\beta}\omega^\alpha \wedge \omega^{\bar{\beta}}, \\ d\omega^\alpha &= \omega^\beta \wedge \omega_\beta^\alpha + \omega \wedge \tau^\alpha. \end{aligned} \tag{2.3}$$

The function R is the Webster scalar curvature on M. Note from (2.2)

$$d\sigma=\frac{1}{2n+3}\left(id\omega_{\alpha}^{\alpha}-\frac{1}{2(2n+2)}Rd\omega-\frac{1}{2(2n+2)}dR\wedge\omega\right). \tag{2.4}$$

Normalize dt so that we may assume $\sigma(\mathrm{T})=1$. Let $\sigma\odot\omega$ denote the symmetric 2-form defined by $\sigma\cdot\omega+\omega\cdot\sigma$. Since $\omega(\mathrm{T})=0$, it follows $\sigma\odot\omega(\mathrm{T},\mathrm{T})=0$. The Fefferman-Lorentz metric for (ω,J) on $C(M)$ is defined by

$$g(X,Y)=\sigma\odot\omega(X,Y)+d\omega(JX,Y). \tag{2.5}$$

Here $T(C(M))=\langle\mathrm{T}\rangle\oplus\langle\xi\rangle\oplus\ker\omega$. Since ξ is the Reeb field, $d\omega(JX,\xi)=0$. As $[\ker\omega,\mathrm{T}]=0$, $d\omega(JX,T)=0$ $\left({}^{\forall}X\in\ker\omega\right)$. On the other hand, $J(\{\mathrm{T},\xi\})=0$ by the definition. We have

$$g(\xi,\mathrm{T})=1,\ g(\mathrm{T},\mathrm{T})=0. \tag{2.6}$$

Thus g becomes a Lorentz metric on $C(M)$ in which S^1 is a lightlike group.

Theorem 2 ([5]). *If* $\omega'=u\omega$, *then* $g'=ug$.

3. Hypercomplex 3 *CR*-Structure

Our strategy is as follows: first we construct a pseudo-Riemannian metric locally on each neighborhood of $M\times S^3$ by Condition I below and then sew these metrics on each intersection to get a globally defined pseudo-Riemannian metric on $M\times S^3$ using Theorem 4. (See the proof of Theorem A.)

Suppose that $\left(M,\{\omega_\alpha,J_\alpha\}_{\alpha=1,2,3}\right)$ is a hypercomplex 3 *CR*-manifold of dimension $(4n+3)$. Put $\omega=\omega_1 i+\omega_2 j+\omega_3 k$. It is an $\mathrm{Im}\mathbb{H}$-valued 1-form annihilating D. In general, there is no canonical choice of ω annihilating D. In [[3], Lemma 1.3] we observed that if ω' is another $\mathrm{Im}\mathbb{H}$-valued 1-form annihilating D, then

$$\omega'=\lambda\omega\bar{\lambda} \tag{3.1}$$

for some $\mathbb{H}$-valued function λ on M. (Here $\bar{\lambda}$ is the quaternion conjugate.) If we put $\lambda=\sqrt{u}a$ for a positive function u and $a\in\mathrm{Sp}(1)$, then $\omega'=ua\omega\bar{a}$ such that the map $z\mapsto az\bar{a}$ $(z\in\mathbb{H})$ represents a matrix function $A\in\mathrm{SO}(3)$. If $\{J'_\alpha\}_{\alpha=1,2,3}$ is a hypercomplex structure on D for ω', then they are related as $[J'_1J'_2J'_3]=[J_1J_2J_3]A$.

For each (ω_α,J_α), we obtain a unique real 1-form σ_α on $C(M)$ from Section 2 (cf. (2.2)). First of all we construct a pseudo-Riemannian metric on $M\times S^3$. In general $C(M)$ is a nontrivial principal S^1-bundle. It is the trivial bundle when we restrict to a neighborhood. So for our use we assume:

Condition I. $C(M)$ *is trivial as bundle, i.e.* $C(M)=M\times S^1$.

We construct a 1-form σ_α on $M\times S^3$ $(\alpha=1,2,3)$ as follows. Let $\mathrm{T}_\alpha,\mathrm{T}_\beta,\mathrm{T}_\gamma$ generate $\{e^{i\theta}\}_{\theta\in\mathbb{R}}$, $\{e^{j\theta}\}_{\theta\in\mathbb{R}}$, $\{e^{k\theta}\}_{\theta\in\mathbb{R}}$ of S^3 respectively. Obtained as in (2.2), we have σ_α's on each $C(M)=M\times S^1$ such that

$$\sigma_\alpha(\mathrm{T}_\alpha)=1, \sigma_\beta(\mathrm{T}_\beta)=1, \sigma_\gamma(\mathrm{T}_\gamma)=1.$$

We then extend σ_α to $M\times S^3$ by setting

$$\sigma_\alpha(\mathrm{T}_\beta)=\sigma_\alpha(\mathrm{T}_\gamma)=0 \tag{3.2}$$

Since $[\mathrm{T}_\beta,\mathrm{T}_\gamma]=2\mathrm{T}_\alpha$ on TS^3, $d\sigma_\alpha(\mathrm{T}_\beta,\mathrm{T}_\gamma)=-\frac{1}{2}\sigma_\alpha([\mathrm{T}_\beta,\mathrm{T}_\gamma])=-1=-2\sigma_\beta\wedge\sigma_\gamma(\mathrm{T}_\beta,\mathrm{T}_\gamma)$. Note that for any $p\in M$,

$$d\sigma_\alpha+2\sigma_\beta\wedge\sigma_\gamma=0 \text{ on } \{p\}\times S^3\ \left((\alpha,\beta,\gamma)\sim(1,2,3)\right). \tag{3.3}$$

On the other hand, we recall the following from [[3], Lemma 4.1].

Proposition 3. *The following hold*:

$$d\omega_1(J_1X,Y)=d\omega_2(J_2X,Y)=d\omega_3(J_3X,Y)\ \left({}^\forall X,Y\in\mathrm{D}\right).$$

In particular $g^{\mathrm{D}}=d\omega_\alpha\circ J_\alpha$ is a positive definite invariant symmetric bilinear form on D;

$$g^{\mathrm{D}}(X,Y)=g^{\mathrm{D}}(J_\alpha X,J_\alpha Y).$$

Choose a frame field $\{X_1,\cdots,X_{4n}\}$ on D such that $J_\alpha X_j=X_{\alpha n+j}$ $(j=1,\cdots,n)$ with $d\omega_\alpha(J_\alpha X_j,X_k)=\delta_{jk}$. Let θ^i be the dual frame to X_i $(i=1,\cdots,4n)$ such that

$$d\omega_\alpha(J_\alpha X,Y)=\sum_{i=1}^{4n}\theta^i(X)\cdot\theta^i(Y)\ \left({}^\forall X,Y\in\mathrm{D}\right). \tag{3.4}$$

Let ξ_α be the Reeb field for ω_α respectively. There is a decomposition $T(M\times S^3)=TM\oplus\{\mathrm{T}_\alpha,\mathrm{T}_\beta,\mathrm{T}_\gamma\}=\{\xi_1,\xi_2,\xi_3\}\oplus\mathrm{D}\oplus\{\mathrm{T}_\alpha,\mathrm{T}_\beta,\mathrm{T}_\gamma\}$.

As before let $\sigma\odot\omega=\sum_{\alpha=1}^3(\sigma_\alpha\cdot\omega_\alpha+\omega_\alpha\cdot\sigma_\alpha)$ be a symmetric 2-form. Define a pseudo-Riemannian metric on $M\times S^3$ by

$$\begin{aligned} g(X,Y)&=\sum_{\alpha=1}^{3}\left(\sigma_\alpha(X)\cdot\omega_\alpha(Y)+\omega_\alpha(X)\cdot\sigma_\alpha(Y)\right)+d\omega_\alpha(J_\alpha X,Y)\\ &=\sigma\odot\omega(X,Y)+\sum_{i=1}^{4n}\theta^i\cdot\theta^i(X,Y). \end{aligned} \tag{3.5}$$

As in (2.6) it follows that $g(\xi_\alpha,\mathrm{T}_\alpha)=1$, $g(\mathrm{T}_\alpha,\mathrm{T}_\alpha)=0$. If we note $\sigma_\alpha(\xi_\alpha)\neq 0$, letting $\eta_\alpha=\xi_\alpha-\sigma_\alpha(\xi_\alpha)\mathrm{T}_\alpha$, it follows $g(\eta_\alpha,\eta_\alpha)=0$. So

$$\begin{bmatrix} g(\eta_\alpha,\eta_\alpha) & g(\eta_\alpha,\mathrm{T}_\alpha)\\ g(\mathrm{T}_\alpha,\eta_\alpha) & g(\mathrm{T}_\alpha,\mathrm{T}_\alpha)\end{bmatrix}=\begin{bmatrix}0&1\\1&0\end{bmatrix}$$

$(\alpha=1,2,3)$. As $g\,|\,\mathrm{D}=g^{\mathrm{D}}$ is positive definite from Proposition 3, g is a pseudo-Riemannian metric of type $(4n+4,3)$ on $M\times S^3$.

Theorem 4. *Let g' be the pseudo-Riemannian metric on $M\times S^3$ corresponding to another* $\mathrm{Im}\mathbb{H}$*-valued* 1*-form ω' on M representing* (D,Q), *i.e.* $\omega'=ua\omega\bar{a}$ $(a\in\mathrm{Sp}(1),u>0)$, *then* $g'=u\cdot g$.

We divide a proof according to whether $\omega'=u\omega$ or $\omega'=a\omega\bar{a}$.

Proposition 5. *If* $\omega'=u\omega$, *then* $g'=u\cdot g$.

Proof. (Existence.) Suppose $\omega'=u\omega$. We show the existence of such a 1-form

σ' for ω'. Let $\{\mathrm{T}_\alpha,\xi_\alpha,X_1,\cdots,X_{4n}\}_{\alpha=1,2,3}$ be the frame on $M\times S^3$ for ω. Then ω' determines another frame $\{\mathrm{T}'_\alpha,\xi'_\alpha,X'_1,\cdots,X'_{4n}\}$. Since each T'_α generates the same S^1 as that of T_α, note

$$\mathrm{T}_\alpha=\mathrm{T}'_\alpha\ \left(\alpha=1,2,3\right). \tag{3.6}$$

Let $\{X_i\}_{i=1,\cdots,4n}$ be the frame on D. Then the Reeb field ξ'_α for each ω'_α is described as

$$\xi_\alpha=u\cdot\xi'_\alpha+x_1^{(\alpha)}\sqrt{u}X'_1+\cdots+x_{4n}^{(\alpha)}\sqrt{u}X'_{4n}\ \left(\alpha=1,2,3\right). \tag{3.7}$$

$\left(\exists\, x_i^{(\alpha)}\in\mathbb{R},\ i=1,\cdots,n\right)$. As $u\cdot d\omega=d\omega'$ on D and $g^{\mathrm{D}}\left(X,Y\right)=g^{\mathrm{D}}\left(J_\alpha X,J_\alpha Y\right)$ from Proposition 3, there exists a matrix $B=\left(b_i^k\right)\in\mathrm{Sp}\left(n\right)$ such that

$$X_i=\sqrt{u}\sum_{k=1}^{4n}b_i^kX'_k. \tag{3.8}$$

Two frames $\{\mathrm{T}_\alpha,\xi_\alpha,X_1,\cdots,X_{4n}\}$, $\{\mathrm{T}'_\alpha,\xi'_\alpha,X'_1,\cdots,X'_{4n}\}$ give the coframes $\{\omega_\alpha,\theta^1,\cdots,\theta^{4n},\sigma_\alpha\}$, $\{\omega'_\alpha,\theta'^1,\cdots,\theta'^{4n},\sigma'_\alpha\}$ on $M\times S^3$ respectively. Then the above Equations (3.6), (3.7), (3.8) determine the relations between coframes:

$$\begin{aligned}&\omega'_\alpha=u\cdot\omega_\alpha\ \left(\alpha=1,2,3\right),\\&\theta'^i=\sqrt{u}\sum_{j=1}^{4n}b_j^i\theta^j+\sqrt{u}x_i^{(1)}\cdot\omega_1+\sqrt{u}x_i^{(2)}\cdot\omega_2+\sqrt{u}x_i^{(3)}\cdot\omega_3,\end{aligned} \tag{3.9}$$

Moreover if we put

$$\begin{aligned}\sigma'_\alpha=\sigma_\alpha-&\left(\sum_{j=1}^{4n}\left(\sum_{i=1}^{4n}b_j^ix_i^{(\alpha)}\right)\theta^j+\frac{1}{2}\sum_{i=1}^{4n}x_i^{(\beta)}x_i^{(\alpha)}\cdot\omega_\beta\right.\\&\left.+\frac{1}{2}\sum_{i=1}^{4n}x_i^{(\gamma)}x_i^{(\alpha)}\cdot\omega_\gamma\right)-\frac{1}{2}\sum_{i=1}^{4n}\left|x_i^{(\alpha)}\right|^2\omega_\alpha,\end{aligned} \tag{3.10}$$

then (3.15) and (3.10) show that

$$\left(\omega'_1,\omega'_2,\omega'_3,\theta'^1,\cdots,\theta'^{4n},\sigma'_1,\sigma'_2,\sigma'_3\right)=\left(\omega_1,\omega_2,\omega_3,\theta^1,\cdots,\theta^{4n},\sigma_1,\sigma_2,\sigma_3\right)\mathrm{P}$$

for which

$$\mathrm{P}=\left(\begin{array}{c|c|ccc} & \sqrt{u}x^{(1)} & \frac{-\left|x^{(1)}\right|^2}{2} & \frac{-x^{(1)}\cdot x^{(2)}}{2} & \frac{-x^{(1)}\cdot x^{(3)}}{2}\\ u\mathrm{I}_3 & \sqrt{u}x^{(2)} & \frac{-x^{(2)}\cdot x^{(1)}}{2} & \frac{-\left|x^{(2)}\right|^2}{2} & \frac{-x^{(2)}\cdot x^{(3)}}{2}\\ & \sqrt{u}x^{(3)} & \frac{-x^{(3)}\cdot x^{(1)}}{2} & \frac{-x^{(3)}\cdot x^{(2)}}{2} & \frac{-\left|x^{(3)}\right|^2}{2}\\ \hline 0 & \sqrt{u}B & -B^tx^{(1)} & -B^tx^{(2)} & -B^tx^{(3)}\\ \hline 0 & 0 & & \mathrm{I}_3 & \end{array}\right).$$

If I_{4n}^3 is a symmetric matrix defined by

$$\mathrm{I}_{4n}^3=\left(\begin{array}{c|ccc|c} 0 & 0 & \cdots & 0 & \mathrm{I}_3\\ \hline 0 & & & & 0\\ \vdots & & \mathrm{I}_{4n} & & \vdots\\ 0 & & & & 0\\ \hline \mathrm{I}_3 & 0 & \cdots & 0 & 0 \end{array}\right), \tag{3.11}$$

it is easily checked that $\mathrm{P}\mathrm{I}^3_{4n}\,{}^t\mathrm{P} = u \cdot \mathrm{I}^3_{4n}$.

Letting $\omega' = (\omega'_1, \omega'_2, \omega'_3)$ and $\sigma' = (\sigma'_1, \sigma'_2, \sigma'_3)$, we define a pseudo-Riemannian metric

$$g' = \sigma' \odot \omega' + \sum_{i=1}^{4n} \theta'^i \cdot \theta'^i. \tag{3.12}$$

Then a calculation shows

$$\begin{aligned} g' &= \sum_{\alpha=1}^{3} \left(\sigma'_\alpha \cdot \omega'_\alpha + \omega'_\alpha \cdot \sigma'_\alpha\right) + \sum_{i=1}^{4n} \theta'^i \cdot \theta'^i \\ &= \left(\omega', \theta'^1, \cdots, \theta'^{4n}, \sigma'\right) \mathrm{I}^3_{4n}\,{}^t\left(\omega', \theta'^1, \cdots, \theta'^{4n}, \sigma'\right) \\ &= \left(\omega, \theta^1, \cdots, \theta^{2n}, \sigma\right) \mathrm{P}\mathrm{I}^3_{4n}\,{}^t\mathrm{P}\,{}^t\left(\omega, \theta^1, \cdots, \theta^{2n}, \sigma\right) \\ &= u \cdot \left(\omega, \theta^1, \cdots, \theta^{2n}, \sigma\right) \mathrm{I}^3_{4n}\,{}^t\left(\omega, \theta^1, \cdots, \theta^{2n}, \sigma\right) \\ &= u\left(\sum_{\alpha=1}^{3} \left(\sigma_\alpha \cdot \omega_\alpha + \omega_\alpha \cdot \sigma_\alpha\right) + \sum_{i=1}^{4n} \theta^i \cdot \theta^i\right) = u \cdot g. \end{aligned} \tag{3.13}$$

(Uniqueness.) We prove the above σ' is uniquely determined with respect to ω'. Let $\mathcal{F} = \{\omega_\alpha, \theta^1, \cdots, \theta^{4n}, \theta^{4n+1}, \theta^{4n+2}\}$ be the coframe for ω_α where $\theta^{4n+1} = \omega_\beta, \theta^{4n+2} = \omega_\gamma$. We have a Fefferman-Lorentz metric on $M \times S^1$ from (3.5) and (3.4) under Condition I:

$$\begin{aligned} g_\alpha &= \sigma_\alpha \odot \omega_\alpha + \frac{1}{3} d\omega_\alpha \circ J_\alpha \\ &= \sigma_\alpha \odot \omega_\alpha + \frac{1}{3}\left(\sum_{i=1}^{4n} \theta^i \cdot \theta^i + \omega_\beta \cdot \omega_\beta + \omega_\gamma \cdot \omega_\gamma\right). \end{aligned} \tag{3.14}$$

(We take the coefficient $\frac{1}{3}$ for our use.) When $\omega'_\alpha = u\omega_\alpha$, the coframe $\mathcal{F}$ will be transformed into a coframe $\mathcal{F}' = \{\omega'_\alpha, \theta'^1_\alpha, \cdots, \theta'^{4n}_\alpha, \theta'^{4n+1}_\alpha, \theta'^{4n+2}_\alpha\}$ such as

$$\begin{aligned} \theta'^i_\alpha &= \sqrt{u} \sum_j c^i_{\alpha j} \theta^j + \sqrt{u}\, y^i_\alpha \omega_\alpha, \\ \theta'^{4n+1}_\alpha &= \sqrt{u}\,\theta^{4n+1} = \sqrt{u}\,\omega_\beta, \\ \theta'^{4n+2}_\alpha &= \sqrt{u}\,\theta^{4n+2} = \sqrt{u}\,\omega_\gamma, \end{aligned} \tag{3.15}$$

$$\left({}^\exists y^i_\alpha \in \mathbb{R}, {}^\exists \left(c^i_{\alpha j}\right) \in \mathrm{Sp}(n), i, j = 1, \cdots, n\right).$$

If g'_α is the corresponding metric on $M \times S^1$, then $g'_\alpha = u g_\alpha$ by Theorem 2 and there exists a unique 1-form $\tilde{\sigma}_\alpha$ such that

$$\begin{aligned} g'_\alpha &= \tilde{\sigma}_\alpha \odot \omega'_\alpha + \frac{1}{3}\left(\sum_{i=1}^{4n} \theta'^i_\alpha \cdot \theta'^i_\alpha + \theta'^{4n+1}_\alpha \cdot \theta'^{4n+1}_\alpha + \theta'^{4n+2}_\alpha \cdot \theta'^{4n+2}_\alpha\right) \\ &= \tilde{\sigma}_\alpha \odot \omega'_\alpha + \frac{1}{3}\left(\sum_{i=1}^{4n} \theta'^i_\alpha \cdot \theta'^i_\alpha + u\omega_\beta \cdot \omega_\beta + u\omega_\gamma \cdot \omega_\gamma\right). \end{aligned} \tag{3.16}$$

If we sum up this equality for $\alpha = 1, 2, 3$;

$$\begin{aligned} g'_1 + g'_2 + g'_3 &= \tilde{\sigma} \odot \omega' + \frac{1}{3} \sum_{\alpha, i} \theta'^i_\alpha \cdot \theta'^i_\alpha + \frac{2}{3} u\left(\omega_\alpha \cdot \omega_\alpha + \omega_\beta \cdot \omega_\beta + \omega_\gamma \cdot \omega_\gamma\right) \\ &= ug_1 + ug_2 + ug_3 \\ &= u\left(\sigma \odot \omega + \sum_{i=1}^{4n} \theta^i \cdot \theta^i + \frac{2}{3}\left(\omega_\alpha \cdot \omega_\alpha + \omega_\beta \cdot \omega_\beta + \omega_\gamma \cdot \omega_\gamma\right)\right), \end{aligned}$$

which yields

$$\tilde{\sigma}\odot\omega' + \frac{1}{3}\sum_{\alpha=1}^{3}\sum_{i=1}^{4n}\theta_\alpha'^i\cdot\theta_\alpha'^i = u\left(\sigma\odot\omega + \sum_{i=1}^{4n}\theta^i\cdot\theta^i\right) = ug. \tag{3.17}$$

Compared this with (3.13) it follows

$$\sigma' = \tilde{\sigma},\ \text{i.e.}\ \sigma'_\alpha = \tilde{\sigma}_\alpha\ (\alpha = 1,2,3). \tag{3.18}$$

By uniqueness of $\tilde{\sigma}_\alpha$, σ'_α defined by (3.10) is a unique real 1-form with respect to ω'.

Next put $\tilde{\omega} = a\cdot\omega\cdot\bar{a}$. The conjugate $z\mapsto az\bar{a}$ $\left({}^\forall z\in\mathbb{H}\right)$ represents a matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix} \in \mathrm{SO}(3)$. Then it follows

$$\tilde{\omega} = [\omega_1,\omega_2,\omega_3]A\begin{bmatrix} i\\ j\\ k\end{bmatrix} \tag{3.19}$$

By our definition, a hypercomplex structure $\{J_1,J_2,J_3\}$ on D satisfies that $(d\omega_\beta\mid\mathrm{D})^{-1}\circ(d\omega_\alpha\mid\mathrm{D}) = J_\gamma$ $(\alpha,\beta,\gamma)\sim(1,2,3)$. A new hypercomplex structure on D is described as

$$\begin{pmatrix}\tilde{J}_1\\ \tilde{J}_2\\ \tilde{J}_3\end{pmatrix} = {}^tA\begin{pmatrix}J_1\\ J_2\\ J_3\end{pmatrix}. \tag{3.20}$$

Differentiate (3.19) and restrict to D (in fact, $d\tilde{\omega} = a\cdot d\omega\cdot\bar{a}$ on D), using Proposition 3, a calculation shows

$$\begin{aligned} d\tilde{\omega}_\alpha(X,Y) &= -a_{1\alpha}g^{\mathrm{D}}(J_1X,Y) + a_{2\alpha}g^{\mathrm{D}}(J_2X,Y) + a_{3\alpha}g^{\mathrm{D}}(J_3X,Y)\\ &= -g^{\mathrm{D}}\left(\left(a_{1\alpha}J_1 + a_{2\alpha}J_2 + a_{3\alpha}J_3\right)X,Y\right) = -g^{\mathrm{D}}\left(\tilde{J}_\alpha X,Y\right),\end{aligned}$$

$$d\tilde{\omega}_\alpha\left(\tilde{J}_\alpha X,Y\right) = g^{\mathrm{D}}(X,Y)\ (\alpha = 1,2,3). \tag{3.21}$$

In particular, we have $(d\tilde{\omega}_\beta\mid\mathrm{D})^{-1}\circ(d\tilde{\omega}_\alpha\mid\mathrm{D}) = \tilde{J}_\gamma$ $(\alpha,\beta,\gamma)\sim(1,2,3)$.

Proposition 6. *If* $\tilde{\omega} = a\omega\bar{a}$, *then* $\tilde{g} = g$.

Proof. Let $\tilde{g}(X,Y) = \tilde{\sigma}\odot\tilde{\omega}(X,Y) + d\tilde{\omega}_\alpha\left(\tilde{J}_\alpha X,Y\right)$. Since $\tilde{\sigma}_\alpha$ is uniquely determined by $\tilde{\omega}_\alpha$ and $\tilde{\omega} = [\omega_1,\omega_2,\omega_3]A = \omega A$ from (3.19), it implies that

$$\tilde{\sigma} = [\sigma_1,\sigma_2,\sigma_3]A = \sigma A. \tag{3.22}$$

Note that

$$\begin{aligned}\tilde{\sigma}\odot\tilde{\omega} &= \sum_{\alpha=1}^{3}\left(\tilde{\sigma}_\alpha\cdot\tilde{\omega}_\alpha + \tilde{\omega}_\alpha\cdot\tilde{\sigma}_\alpha\right) = \sigma A{}^tA{}^t\omega + \omega A{}^tA{}^t\sigma\\ &= \sigma{}^t\omega + \omega{}^t\sigma = \sigma\odot\omega.\end{aligned} \tag{3.23}$$

By (3.21),

$$\tilde{g} = \tilde{\sigma}\odot\tilde{\omega} + d\tilde{\omega}_\alpha\circ\tilde{J}_\alpha = \sigma\odot\omega + g^{\mathrm{D}} = g.$$

***Proof of Theorem* 4.** Suppose $\omega' = \lambda\omega\bar{\lambda} = u\tilde{\omega}$ where $\tilde{\omega} = a\omega\bar{a}$. It follows from Proposition 5 that $g' = u\tilde{g}$. By Proposition 6, we have $\tilde{g} = g$ and hence $g' = ug$. This finishes the proof under Condition I.

Proof of Theorem A

Proof. Let $(M, \{\mathrm{D}, Q\})$ be a quaternionic 3 *CR*-manifold. Then *M* has an open cover $\{U_i\}_{i\in\Lambda}$ where each U_i admits a hypercomplex 3 *CR*-structure $\left(\omega_\alpha^{(i)}, J_\alpha^{(i)}\right)_{\alpha=1,2,3}$. Put $\omega^{(i)} = \omega_1^{(i)} i + \omega_2^{(i)} j + \omega_3^{(i)} k$ which is an $\mathrm{Im}\mathbb{H}$-valued 1-form on U_i. Since we may assume that U_i is homeomorphic to a ball (*i.e.* contractible), Condition I is satisfied for each U_i, *i.e.* $C(U_i) = U_i \times S^1$. Then we have a pseudo-Riemannian metric $g^{(i)} = \sum_{\alpha=1}^{3} \sigma_\alpha^{(i)} \odot \omega_\alpha^{(i)} + d\omega_\alpha^{(i)} \circ J_\alpha^{(i)}$ on $U_i \times S^3$ for $\omega^{(i)}$ by Theorem 4. Suppose $U_i \cap U_j \neq \varnothing$. By condition a) of 2) (cf. Introduction), $\mathrm{D} \mid U_i \cap U_j = \ker \omega^{(i)} \mid U_i \cap U_j = \ker \omega^{(j)} \mid U_i \cap U_j$. Then by the equivalence (3.1) there exists a function $\lambda = \sqrt{u}a$ defined on $U_i \cap U_j$ such that

$$\omega^{(j)} = \lambda \cdot \omega^{(i)} \cdot \bar{\lambda} = ua\omega^{(i)}\bar{a} \text{ on } U_i \cap U_j. \tag{3.24}$$

It follows from Theorem 4 that $g^{(j)} = ug^{(i)}$ on $U_i \cap U_j$. We may put $u = u^{ji}$ which is a positive function defined on $U_i \cap U_j$. By construction, it is easy to see that $u^{ki} = u^{kj} u^{ji}$ on $U_i \cap U_j \cap U_k \neq \varnothing$. This implies that $\{u\}_{i,j\in\Lambda}$ defines a 1-cocycle on *M*. Since $\mathbb{R}^+$ is a fine sheaf as the germ of local continuous functions, note that the first cohomology $H^1(\mathcal{U}, \mathbb{R}^+) = 0$. (Here $\mathcal{U}$ is a chain complex of covers running over all open covers of *M*.) Therefore there exists a local function $\{f\}_{i,j\in\Lambda}$ defined on each U_i such that $\delta f(j,i) = u^{ji}$, *i.e.* $f_i \cdot f_j^{-1} = u^{ji}$ on $U_i \cap U_j$. We obtain that

$$f_j \cdot g^{(j)} = f_i \cdot g^{(i)} \text{ on } (U_i \cap U_j) \times S^3.$$

Then we may define

$$g \mid U_i \times S^3 = f_i \cdot g^{(i)}. \tag{3.25}$$

so that *g* is a globally defined pseudo-Riemannian metric on $M \times S^3$. If another family $\{\omega'^i\}_{i\in\Lambda}$ represents the same quaternionic 3 *CR*-structure (D, Q), then the same argument shows that $g' = ug$ on $M \times S^3$ for some positive function. Hence the conformal class $[g]$ is an invariant for quaternionic 3 *CR*-structure. In particular, the Weyl curvature tensor $W(g)$ is also an invariant. This completes the proof of Theorem A.

4. Model Geometry and Transformations

We introduce *spherical* 3 *CR-homogeneous model* $\left(\mathrm{PSp}(n+1,1), S^{4n+3}\right)$ and *conformally flat pseudo-Riemannian homogeneous model* $\left(\mathrm{PSp}(n+1,1) \times \mathrm{SO}(3), S^{4n+3,3}\right)$ equipped with *pseudo-Riemannian metric* g^0 of type $(4n+3,3)$ and then characterize the *lightlike subgroup* in $\mathrm{PSp}(n+1,1) \times \mathrm{SO}(3)$.

4.1. Pseudo-Riemannian Metric g^0

Let us start with the quaternionic vector space $\mathbb{H}^{n+2}$ endowed with the Hermitian form:

$$\langle z,w\rangle=\bar{z}_1w_1+\cdots+z_{n+1}w_{n+1}-\bar{z}_{n+2}w_{n+2}\quad\left(z,w\in\mathbb{H}^{n+2}\right). \tag{4.1}$$

The q-cone is defined by

$$V_0=\left\{z\in\mathbb{H}^{n+2}-\{0\}\mid\langle z,z\rangle=0\right\}. \tag{4.2}$$

When $\mathbb{H}^{n+2}$ is viewed as the real vector space $\mathbb{R}^{4n+8}$, $\mathrm{O}(4n+4,4)$ denotes the full subgroup of $\mathrm{GL}(4n+8,\mathbb{R})$ preserving the bilinear form $\mathrm{Re}\langle,\rangle$. Consider the commutative diagrams below. The image of the pair $(\mathrm{O}(4n+4,4),V_0)$ by the projection $P_{\mathbb{R}}$ is the homogeneous model of conformally flat pseudo-Riemannian geometry $\left(\mathrm{PO}(4n+4,4),S^{4n+3,3}\right)$ in which $S^{4n+3,3}=P_{\mathbb{R}}(V_0)$ is diffeomorphic to a quotient manifold $S^{4n+3}\times_{\mathbb{Z}_2}S^3$. The identification $\mathbb{H}^{n+2}=\mathbb{R}^{4n+8}$ gives a natural embedding $\mathrm{Sp}(n+1,1)\cdot\mathrm{Sp}(1)\to\mathrm{O}(4n+4,4)$ which results a special geometry $\left(\mathrm{PSp}(n+1,1)\times\mathrm{SO}(3),S^{4n+3,3}\right)$ from $\left(\mathrm{PO}(4n+4,4),S^{4n+3,3}\right)$.

As usual, the image of $\left(\mathrm{Sp}(n+1,1)\cdot\mathrm{Sp}(1),V_0\right)$ by $P_{\mathbb{H}}$ is spherical quarternionic 3 CR-geometry $\left(\mathrm{PSp}(n+1,1),S^{4n+3}\right)$.

$$\begin{tikzcd}
\mathbb{R}^* \arrow[d] \arrow[dr] & & & \\
\mathbb{H}^* \arrow[r] \arrow[d] & \mathbb{H}^{n+2}-\{0\} \arrow[rr, "P_{\mathbb{H}}"] \arrow[dr, "P_{\mathbb{R}}"'] & & \mathbb{HP}^{n+1} \\
\mathrm{SO}(3) \arrow[rr] & & \mathbb{RP}^{4n+7} \arrow[ur, "P"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
V_0 \arrow[rr, "P_{\mathbb{H}}"] \arrow[dr, "P_{\mathbb{R}}"'] & & S^{4n+3} \\
& S^{4n+3,3} \arrow[ur, "P"'] &
\end{tikzcd} \tag{4.3}$$

We describe a pseudo-Riemannian metric g^0 on $S^{4n+3,3}=S^{4n+3}\times_{\mathbb{Z}_2}S^3$. Let $S^{4n+3}\times S^3$ be the product of unit spheres. For $(z,w)\in S^{4n+3}\times S^3$, $|z|^2-|w|^2=1-1=0$ so $S^{4n+3}\times S^3\subset V_0$. Then $P_{\mathbb{R}}(V_0)=S^{4n+3,3}$ induces a 2-fold covering $P_{\mathbb{R}}:S^{4n+3}\times S^3\to S^{4n+3,3}$ for which $P_*:T\left(S^{4n+3}\times S^3\right)\to TS^{4n+3,3}$ is an isomorphism.

Let $x\in S^{4n+3}\times S^3$ where we put $P_{\mathbb{R}}(x)=[x]$. Choose $y\in S^{4n+3}\times S^3$ such that $\langle x,y\rangle=1$. Denote by $\{x,y\}^{\perp}$ the orthogonal complement in $\mathbb{H}^{n+2}$ with respect to $\langle,\rangle$. As $T_xV_0=\left\{Z\in\mathbb{H}^{n+2}\mid\mathrm{Re}\langle x,Z\rangle=0\right\}$, it follows $T_xV_0=y\,\mathrm{Im}\,\mathbb{H}\oplus x\mathbb{H}\oplus\{x,y\}^{\perp}\subset\mathbb{H}^{n+2}$ such that

$$T_x\left(S^{4n+3}\times S^3\right)=y\,\mathrm{Im}\,\mathbb{H}\oplus x\,\mathrm{Im}\,\mathbb{H}\oplus\{x,y\}^{\perp}.$$

In particular, $T_xV_0=x\mathbb{R}\oplus T_x\left(S^{4n+3}\times S^3\right)$. Note that this decomposition does not depend on the choice of points $x'\in[x]$ and y' with $\langle x',y'\rangle=1$. (see [3], Theorem 6.1]). We define a pseudo-Riemannian metric on $S^{4n+3,3}$ to be

$$g^0_{[x]}\left(P_{\mathbb{R}*}X,P_{\mathbb{R}*}Y\right)=\mathrm{Re}\langle X,Y\rangle\quad\left({}^{\forall}X,Y\in T_x\left(S^{4n+3}\times S^3\right)\right). \tag{4.4}$$

Noting $\mathrm{Re}\langle ya, ya\rangle = \mathrm{Re}\langle xa, xa\rangle = 0$, $\mathrm{Re}\langle xa, ya\rangle = 1$ $\left({}^{\forall} a \in \mathrm{Sp}(1)\right)$ and $\mathrm{Re}\langle , \rangle|_{\{x,y\}^{\perp}}$ is positive definite, $g^0_{[x]}$ is a pseudo-Riemannian metric of type $(4n+3,3)$ at each $[x] \in S^{4n+3,3}$.

4.2. Conformal Group $\mathrm{O}(4n+4,4)$

It is known more or less but we need to check that $\mathrm{O}(4n+4,4)$ acts on $S^{4n+3} \times S^3$ as conformal transformations with respect to $\mathrm{Re}\langle , \rangle$ and so does $\mathrm{PO}(4n+4,4)$ on $\left(S^{4n+3,3}, g^0\right)$.

For any $h \in \mathrm{O}(4n+4,4)$, $\langle hx, hx\rangle = \langle x, x\rangle = 0$ so $hx \in V_0$. However hx does not necessarily belong to $S^{4n+3} \times S^3$. Normalized hx, there is $x' \in S^{4n+3} \times S^3$ such that $(hx)\lambda = x'$ for some $\lambda \in \mathbb{R}^+$. Note $[hx] = P_{\mathbb{R}}(hx) = P_{\mathbb{R}}(x')$. If $R_\lambda : \mathbb{H}^{n+2} \to \mathbb{H}^{n+2}$ is the right multiplication defined by $R_\lambda(z) = z\lambda$, then there is the commutative diagram:

$$
\begin{array}{ccc}
T_{x'}V_0 & \searrow P_{\mathbb{R}*} & \\
R_{\lambda*}\uparrow & & T_{[hx]}S^{4n+3,3} \\
T_{hx}V_0 & \nearrow P_{\mathbb{R}*} &
\end{array}
$$

in which $R_*(h_*X) = (h_*X)\lambda \in T_{x'}V_0$. As $T_{x'}V_0 = x'\mathbb{R} \oplus T_{x'}\left(S^{4n+3} \times S^3\right)$, we have $(h_*X)\lambda = x'\mu + X'$ for some $\mu \in \mathbb{R}$, $X' \in T_{x'}\left(S^{4n+3} \times S^3\right)$. Since $P_*\left(T_x\mathbb{R}^*\right) = P_*(x\mathbb{R}) = 0$ and $P_{\mathbb{R}} : \left(\mathrm{O}(4n+4,4), V_0\right) \to \left(\mathrm{PO}(4n+4,4), S^{4n+3,3}\right)$ is equivariant, it follows

$$h_*P_{\mathbb{R}*}(X) = P_{\mathbb{R}*}(h_*X) = P_{\mathbb{R}*}\left((h_*X)\lambda\right) = P_{\mathbb{R}*}(x'\mu + X') = P_{\mathbb{R}*}(X').$$

Similarly $h_*P_{\mathbb{R}*}(Y) = P_{\mathbb{R}*}(Y')$ for $(h_*Y)\lambda = x'\nu + Y'$ for some $\nu \in \mathbb{R}$, $Y' \in T_{x'}\left(S^{4n+3} \times S^3\right)$. As $\mathrm{Re}\langle x', X'\rangle = \mathrm{Re}\langle x', Y'\rangle = 0$, a calculation shows

$$
\begin{aligned}
& g^0_{[hx]}\left(h_*P_{\mathbb{R}*}(X), h_*P_{\mathbb{R}*}(Y)\right) \\
&= g^0_{[hx]}\left(P_{\mathbb{R}*}(X'), P_{\mathbb{R}*}(Y')\right) = \mathrm{Re}\langle X', Y'\rangle \\
&= \mathrm{Re}\langle x'\mu + X', x'\nu + Y'\rangle = \mathrm{Re}\langle (h_*X)\lambda, (h_*Y)\lambda\rangle \\
&= \lambda^2 \mathrm{Re}\langle h_*X, h_*Y\rangle = \lambda^2 \mathrm{Re}\langle X, Y\rangle \\
&= \lambda^2 g^0_{[x]}\left(P_{\mathbb{R}*}(X), P_{\mathbb{R}*}(Y)\right).
\end{aligned}
$$

Hence $h \in \mathrm{O}(4n+4,4)$ acts as conformal transformation with respect to g^0.

4.3. Conformal Subgroup $\mathrm{Sp}(n+1,1)\cdot\mathrm{Sp}(1)$

Let (I, J, K) be the standard hypercomplex structure on $\mathbb{H}^{n+2}$ defined by

$$Iz = -zi, Jz = -zj, Kz = -zk.$$

Put $Q = \mathrm{span}(I, J, K)$ as the associated quaternionic structure. Then $\mathrm{Re}\langle , \rangle$ leaves invariant Q. The full subgroup of $\mathrm{O}(4n+4,4)$ preserving Q is isomorphic to $\mathrm{Sp}(n+1,1)\cdot\mathrm{Sp}(1)$, *i.e.* the intersection of $\mathrm{O}(4n+4,4)$ with $\mathrm{GL}(n+2,\mathbb{H})\cdot\mathrm{GL}(1,\mathbb{H})$.

Let $\rho: S^3 \to \mathrm{O}(4n+4,4)$ be a faithful representation. Then the subgroup $\rho(S^3)$ preserves Q so it is contained in

$$\Big(\overbrace{\mathrm{Sp}(1)\times\cdots\times\mathrm{Sp}(1)}^{n+2}\Big)\cdot\mathrm{Sp}(1) \le \overbrace{\mathrm{SO}(4)\times\cdots\times\mathrm{SO}(4)}^{n+2}$$

which is a subgroup of $\mathrm{SO}(4n+4)\times\mathrm{SO}(4)$.

4.4. Three Dimensional Lightlike Group

Choose $S^1 \le S^3$ and consider a representation restricted to S^1. As we may assume that the semisimple group $\rho(S^3)$ belongs to $(\mathrm{Sp}(1)\times\cdots\times\mathrm{Sp}(1))\cdot\mathrm{Sp}(1)$, this reduces to a faithful representation: $\rho: S^1 \to T^{n+2}\cdot S^1$ such that

$$\rho(t) = \left(\left(\mathrm{e}^{ia_1 t},\cdots,\mathrm{e}^{ia_{n+2}t}\right)\cdot\mathrm{e}^{ibt}\right). \tag{4.5}$$

Here we may assume that $a_i \ge 0$ are relatively prime $(i = 1,\cdots,n+2)$ without loss of generality, and either $b = 0$ or 1. The element $\rho(t)$ acts on $S^{4n+3}\times S^3 \subset V_0$ as

$$\begin{aligned}\rho(t)(z_1,\cdots,z_{n+1},w) &= \left(e^{ia_1t}z_1,\cdots,e^{ia_{n+1}t}z_{n+1},e^{ia_{n+2}t}w\right)\cdot e^{-ibt}\\ &= \left(e^{ia_1t}z_1e^{-ibt},\cdots,e^{ia_{n+1}t}z_{n+1}e^{-ibt},e^{ia_{n+2}t}we^{-ibt}\right)\end{aligned} \tag{4.6}$$

where $|z_1|^2+\cdots+|z_{n+1}|^2-|w|^2=0$ for $(z,w)=(z_1,\cdots,z_{n+1},w)\in V_0$. If X is the vector field induced by $\rho(S^1)$ at (z,w), then it follows

$$X = (ia_1z_1,\cdots,ia_{n+1}z_{n+1},ia_{n+2}w)-(z_1ib,\cdots,z_{n+1}ib,wib). \tag{4.7}$$

Proposition 7. *If* $\rho: S^1 \to T^{n+2}\cdot S^1$ *is a faithful lightlike* 1*-parameter group, then it has either one of the forms*:

$$\begin{aligned}&\rho(t) = \left(e^{it},\cdots,e^{it}\right) \le \Big(\overbrace{\mathrm{Sp}(1)\times\cdots\times\mathrm{Sp}(1)}^{n+2}\Big) \le \mathrm{Sp}(n+1,1)\cdot\{1\},\\ &\rho(t) = (1,\cdots,1)\cdot e^{it} \le \{1\}\cdot\mathrm{Sp}(1) \le \mathrm{Sp}(n+1,1)\cdot\mathrm{Sp}(1).\end{aligned} \tag{4.8}$$

Proof. **Case (i)** $b = 0$. $X = (ia_1z_1,\cdots,ia_{n+1}z_{n+1},ia_{n+2}w)$ from (4.7) so that $\langle X,X\rangle = a_1^2|z_1|^2+\cdots+a_{n+1}^2|z_{n+1}|^2-a_{n+2}^2|w|^2 = (a_1^2-a_{n+2}^2)|z_1|^2+\cdots+(a_{n+1}^2-a_{n+2}^2)|z_{n+1}|^2$. Since $\mathrm{Re}\langle X,X\rangle = 0$ and we assume $a_i \ge 0$, it follows

$$a_1 = a_{n+2},\cdots,a_{n+1} = a_{n+2}.$$

As a_i's are relatively prime, this implies

$$a_1 = \cdots = a_{n+1} = a_{n+2} = 1.$$

As a consequence $\rho(t) = \left(e^{it},\cdots,e^{it}\right) \le \mathrm{Sp}(n+1,1)\cdot\{1\}$. In this case note that $T_x\left(S^{4n+3}\times S^3\right) = \mathrm{Im}\,\mathbb{H}y \oplus \mathrm{Im}\,\mathbb{H}x \oplus \{x,y\}^{\perp}$ such that $\langle x,y\rangle \in \mathbb{R}^*$.

Case (ii) $b = 1$. It follows from (4.7) that

$$X = (ia_1z_1,\cdots,ia_{n+1}z_{n+1},ia_{n+2}w)-(z_1i,\cdots,z_{n+1}i,wi).$$

Put $Y = (ia_1z_1,\cdots,ia_{n+1}z_{n+1},ia_{n+2}w)$, $W = (z_1i,\cdots,z_{n+1}i,wi) = xi$ such that $X = Y - W$ and $\langle W,W\rangle = \bar{i}\langle x,x\rangle i = 0$. Calculate

$$
\begin{aligned}
&\langle Y,Y\rangle = a_1^2|z_1|^2+\cdots+a_{n+1}|z_{n+1}|^2-a_{n+2}^2|w|^2,\\
&\langle Y,W\rangle = a_1\bar{z}_1\bar{i}z_1 i+\cdots+a_{n+1}\bar{z}_{n+1}\bar{i}z_{n+1}i-a_{n+2}\bar{w}\bar{i}wi,\\
&\mathrm{Re}\langle Y,W\rangle = a_1|z_1|^2+\cdots+a_{n+1}|z_{n+1}|^2-a_{n+2}|w| = \mathrm{Re}\langle W,Y\rangle.
\end{aligned} \tag{4.9}
$$

This shows

$$
\begin{aligned}
&\mathrm{Re}\langle X,X\rangle = \mathrm{Re}\langle Y-W,Y-W\rangle\\
&= \mathrm{Re}\langle Y,Y\rangle - 2\,\mathrm{Re}\langle Y,W\rangle + \mathrm{Re}\langle W,W\rangle = \mathrm{R}\langle Y,Y\rangle - 2\,\mathrm{Re}\langle Y,W\rangle\\
&= \left(a_1^2-2a_1\right)|z_1|^2+\cdots+\left(a_{n+1}^2-2a_{n+1}\right)|z_{n+1}|^2-\left(a_{n+2}^2-2a_{n+2}\right)|w|^2\\
&= \left(\left(a_1^2-2a_1\right)-\left(a_{n+2}^2-2a_{n+2}\right)\right)|z_1|^2+\cdots+\left(\left(a_{n+1}^2-2a_{n+1}\right)-\left(a_{n+2}^2-2a_{n+2}\right)\right)|z_{n+1}|^2\\
&= \left(\left(a_1-1\right)^2-\left(a_{n+2}-1\right)^2\right)|z_1|^2+\cdots+\left(\left(a_{n+1}-1\right)^2-\left(a_{n+2}-1\right)^2\right)|z_{n+1}|^2.
\end{aligned}
$$

Thus

$$
\left(a_1-1\right)^2=\left(a_{n+2}-1\right)^2,\cdots,\left(a_{n+1}-1\right)^2=\left(a_{n+2}-1\right)^2. \tag{4.10}
$$

On the other hand, we may assume in general

$$
\begin{aligned}
&a_1=\cdots=a_k=0.\\
&a_{k+1}-1\le 0,\cdots,a_l-1\le 0.\\
&a_{l+1}-1\ge 0,\cdots,a_{n+1}-1\ge 0.
\end{aligned}
$$

(ii-1). Suppose $a_{n+2}-1\ge 0$. As $0<a_j\le 1$ for $k+1\le j\le l$, it implies $a_{k+1}=\cdots=a_l=1$. Since $\left(a_{k+1}-1\right)^2=\left(a_{n+2}-1\right)^2$ from (4.10), it follows $a_{n+2}=1$. Again from (4.10), $\left(a_j-1\right)^2=0$ and so $a_j=1$ $\left(l+1\le j\le n+1\right)$. Note that $a_i\ne 0$ because $\left(a_i-1\right)^2=\left(a_{n+2}-1\right)^2=0$. Thus $a_1=a_2=\cdots=a_{n+2}=1$. This implies $\rho(t)=\left(e^{it},\cdots,e^{it}\right)\cdot e^{it}$.

(ii-2). Suppose $a_{n+2}-1<0$. In this case $a_{n+2}=0$. By (4.10), it follows that ${}^{\forall}a_i\ne 0$ and $a_1=\cdots=a_l=1$, $a_i=2$ $\left(l+1\le i\le n+1\right)$. Thus $\rho(t)=\left(1,\cdots,1,e^{i2t},\cdots,e^{i2t},1\right)\cdot e^{it}$. This contradicts that nonzero a_i's $\left(1\le i\le n+1\right)$ are relatively prime.

(ii-3). Suppose $a_{n+2}-1<0$ and $a_1=a_2=\cdots=a_{n+1}=0$. Again $a_{n+2}=0$ and so $\rho(t)=(1,\cdots,1)\cdot e^{it}$.

To complete the proof of the proposition we prove the following. Put $x=(z,w)=\left(z_1,\cdots,z_{n+1},w\right)\in S^{4n+3}\times S^3\subset V_0$ such that $\langle x,x\rangle=0$.

Lemma 8. Case (ii-1) *does not occur.*

Proof. It follows from (4.7) that

$$
X=\left(iz_1,\cdots,iz_{n+1},iw\right)-\left(z_1 i,\cdots,z_{n+1}i,wi\right)=ix-xi. \tag{4.11}
$$

Put $x=p+jq$ $(p,q\in\mathbb{C}^{n+2})$. Then $X=2kq$. As $\langle X,X\rangle=0$ implies $\langle q,q\rangle=0$. On the other hand, the equation

$$
0=\langle x,x\rangle=(\langle p,p\rangle+\langle q,q\rangle)-2j\langle\overline{p},q\rangle
$$

shows $\langle p,p\rangle+\langle q,q\rangle=0,\langle\overline{p},q\rangle=0$. Note that if $S^{2n+1}\times S^1$ is the canonical subset in $S^{4n+3}\times S^3$, then $\langle p,p\rangle=0$ if and only if $p\in S^{2n+1}\times S^1$. Since X is a nontrivial vector field on $S^{4n+3} \times S^3$, there is a point x in the open subset $S = S^{4n+3} \times S^3 \setminus S^{2n+1} \times S^1$ such that $\langle p, p \rangle \neq 0$ and thus $\langle X, X \rangle \neq 0$ on *S*, which contradicts that *X* is a lightlike vector field.

4.5. Proof of Theorem B

Applying Proposition 7 to a lightlike group S^3 we obtain:

Corollary 9. *Let* $\rho : S^3 \to \mathrm{O}(4n+4,4)$ *be a faithful representation which preserves the metric* $\mathrm{Re}\langle , \rangle$ *on* V_0. *If* $\rho(S^3)$ *is a lightlike group on* $S^{4n+3} \times S^3$, *then either one of the following holds.*

$$\begin{aligned} &\rho(S^3) = \mathrm{diag}(\mathrm{Sp}(1) \times \cdots \times \mathrm{Sp}(1)) \leq \mathrm{Sp}(n+1,1) \cdot \{1\}, \\ &\rho(S^3) = \{1\} \cdot \mathrm{Sp}(1) \leq \mathrm{Sp}(n+1,1) \cdot \mathrm{Sp}(1). \end{aligned} \tag{4.13}$$

Let $(\mathrm{diag}(\mathrm{Sp}(1) \times \cdots \times \mathrm{Sp}(1)) \cdot \mathrm{Sp}(1), S^{4n+3} \times S^3)$ be as in (4.13). If $f : S^{4n+3} \times S^3 \to S^{4n+3} \times S^3$ is a map defined by $f((z_1, \cdots, z_{n+1}, w)) = (\bar{w}z_1, \cdots, \bar{w}z_{n+1}, \bar{w})$, then for $a \in \mathrm{Sp}(1)$, $b \in \mathrm{Sp}(1)$,

$$f((az_1, \cdots, az_{n+1}, aw\bar{b})) = (b\bar{w}z_1, \cdots, b\bar{w}z_{n+1}, b\bar{w}\bar{a}).$$

So the equivariant diffeomorphism f induces a quotient equivariant diffeomorphism

$$\hat{f} : (\mathrm{Sp}(1), S^{4n+3} \times S^3 / \rho(S^3)) \to (\mathrm{diag}(\mathrm{Sp}(1) \times \cdots \times \mathrm{Sp}(1)), S^{4n+3}). \tag{4.14}$$

We prove Theorem B of Introduction.

Proof. Suppose that the pseudo-Riemannian manifold $(M \times S^3, g)$ is *conformally flat*. Let $\pi = \pi_1(M)$ be the fundamental group and $\tilde{M}$ the universal covering of *M*. By the developing argument (cf. [7]), there is a developing pair:

$$(\rho, \mathrm{Dev}) : (\pi \times S^3, \tilde{M} \times S^3, \tilde{g}) \to (\mathrm{O}(4n+4,4), S^{4n+3} \times S^3, g^0)$$

where Dev is a conformal immersion such that $\mathrm{Dev}^* g^0 = u\tilde{g}$ for some positive function u on $\tilde{M} \times S^3$ and $\rho : \pi \times S^3 \to \mathrm{O}(4n+4,4)$ is a holonomy homomorphism for which Dev is equivariant with respect to ρ.

By Corollary 9, if $\rho(S^3) = \{1\} \cdot \mathrm{Sp}(1) \leq \mathrm{Sp}(n+1,1) \cdot \mathrm{Sp}(1)$, then the normalizer of $\mathrm{Sp}(1)$ in $\mathrm{O}(4n+4,4)$ is isomorphic to $\mathrm{Sp}(n+1,1) \cdot \mathrm{Sp}(1)$. In particular, $\rho(\pi \times S^3) = \rho(\pi) \times \mathrm{Sp}(1) \leq \mathrm{Sp}(n+1,1) \cdot \mathrm{Sp}(1)$ where $\rho(S^3) = \{1\} \cdot \mathrm{Sp}(1)$. We have the commutative diagram:

$$\begin{array}{ccc} S^3 & \xrightarrow{\rho} & \mathrm{Sp}(1) \\ \downarrow & & \downarrow \\ (\pi \times S^3, \tilde{M} \times S^3) & \xrightarrow{(\rho, \mathrm{Dev})} & (\rho(\pi) \times \mathrm{Sp}(1), S^{4n+3} \times S^3) \\ \downarrow & & \downarrow \\ (\pi, \tilde{M}) & \xrightarrow{(\hat{\rho}, \mathrm{dev})} & (\rho(\pi), S^{4n+3}) \end{array} \tag{4.15}$$

where $\rho(\pi) \leq \mathrm{PSp}(n+1,1)$ and dev is an immersion which is $\hat{\rho}$ -

equivariant.

If $\rho(S^3) = \mathrm{diag}(\mathrm{Sp}(1)\times\cdots\times\mathrm{Sp}(1)) \leq \mathrm{Sp}(n+1,1)\cdot\{1\}$ from (4.13), then $\rho(\pi\times S^3) = \rho(S^3)\cdot\rho(\pi) \leq \mathrm{diag}(\mathrm{Sp}(1)\times\cdots\times\mathrm{Sp}(1))\cdot\mathrm{Sp}(1)$. Composed f with Dev, we have an equivariant diffeomorphism $\hat{f}\circ\mathrm{dev}:(\pi,\tilde{M})\to(\rho(\pi),S^{4n+3})$ where $\rho(\pi)\leq\mathrm{diag}(\mathrm{Sp}(1)\times\cdots\times\mathrm{Sp}(1))\leq\mathrm{PSp}(n+1,1)$. In each case taking the developing map either dev of (4.15) or $\hat{f}\circ\mathrm{dev}$, a quaternionic 3 *CR*-manifold M is *spherical, i.e.* uniformized with respect to $(\mathrm{PSp}(n+1,1),S^{4n+3})$.

Conversely recall $\left(\omega^0,\{J_\alpha^0\}_{\alpha=1,2,3}\right)$ is the *standard* quaternionic 3 *CR*-structure on S^{4n+3} equipped with the standard hypercomplex structure $Q^0=\{J_\alpha^0\}_{\alpha=1,2,3}$ on D^0. Suppose that $\left(\omega,\{J_\alpha\}_{\alpha=1,2,3}\right)$ is a *spherical quaternionic* 3 *CR-structure* on M with a quaternionic structure Q, then there exists a developing map $\mathrm{dev}:\tilde{M}\to S^{4n+3}$ such that

$$\mathrm{dev}^*\omega^0=\lambda\tilde{\omega}\bar{\lambda}$$

for some $\mathbb{H}$-valued function λ on $\tilde{M}$ with a lift of quaternionic 3 *CR*-structure $\tilde{\omega}$. In particular, $\mathrm{dev}_*\mathrm{D}=\mathrm{D}^0$ and $\mathrm{dev}_*Q=Q^0$.

Let $\tilde{g}$ be a pseudo-Riemannian metric on $\tilde{M}\times S^3$ for $\tilde{\omega}$ which is a lift of g and ω to $\tilde{M}\times S^3$ respectively. Put $\omega'=\mathrm{dev}^*\omega^0$. Let $\lambda=\sqrt{u}a$ be a function for $u>0$ and $a\in\mathrm{Sp}(1)$ such that

$$\omega'=ua\tilde{\omega}\bar{a}.$$

By the definition, recall $d\omega_\beta^0(J_\gamma^0V,W)=d\omega_\alpha^0(V,W)$ $(^\forall V,W\in\mathrm{D}^0)$. The induced quaternionic structure $\{J'_\alpha\}_{\alpha=1,2,3}$ for $\omega'=\mathrm{dev}^*\omega^0$ is obtained as $d(\mathrm{dev}^*\omega_\beta^0)(J'_\gamma X,Y)=d(\mathrm{dev}^*\omega_\alpha^0)(X,Y)$. Since $d\omega_\beta^0(\mathrm{dev}_*J'_\gamma X,\mathrm{dev}_*Y)=d\omega_\alpha^0(\mathrm{dev}_*X,\mathrm{dev}_*Y)$, taking $V=\mathrm{dev}_*X$, we obtain

$$\mathrm{dev}_*J'_\gamma X=J_\gamma^0\mathrm{dev}_*X\quad(^\forall X\in\mathrm{D}). \tag{4.16}$$

As $\mathrm{dev}_*Q=Q^0=\mathrm{span}(J_\alpha^0,\alpha=1,2,3)$, note that $\{J'_\alpha\}_{\alpha=1,2,3}\in Q$.

On the other hand, let g' be the pseudo-Riemannian metric on $\tilde{M}\times S^3$ for ω', it follows from Theorem 4

$$g'=u\tilde{g}. \tag{4.17}$$

Take the above element $a\in S^3$ and let $\rho:S^3\to S^3$ be a homomorphism defined by $\rho(s)=as\bar{a}$ $(^\forall s\in S^3)$. Define a map $\mathrm{dev}\times\rho:\tilde{M}\times S^3\to S^{4n+3}\times S^3$ which makes the diagram commutative. (Here p is the projection onto the left summand.)

$$\begin{array}{ccc}
S^3 & \xrightarrow{\rho} & S^3 \\
\downarrow & & \downarrow \\
\tilde{M}\times S^3 & \xrightarrow{\mathrm{dev}\times\rho} & S^{4n+3}\times S^3 \\
{\scriptstyle p}\downarrow & & {\scriptstyle p}\downarrow \\
\tilde{M} & \xrightarrow{\mathrm{dev}} & S^{4n+3}
\end{array} \tag{4.18}$$

where both $p_*:(\mathrm{D},\{\mathrm{J}'_\alpha\})\to(\mathrm{D},\{\mathrm{J}'_\alpha\})$ and $p_*:(\mathrm{D}^0,\{\mathrm{J}_\alpha^0\})\to(\mathrm{D}^0,\{\mathrm{J}_\alpha^0\})$ are isomorphisms such that

$$p_* \circ J'_\alpha = J'_\alpha \circ p_* \text{ and } p_* \circ J^0_\alpha = J^0_\alpha \circ p_* \ (\alpha = 1,2,3). \tag{4.19}$$

Recall from (3.5) that $g^0 = \sigma^0 \odot p^*\omega^0 + dp^*\omega^0_\alpha \circ J^0_\alpha$. (We write p more precisely.) Consider the pull-back metric

$$\begin{aligned}(\mathrm{dev}\times\rho)^* g^0(X,Y) &= \sigma^0 \odot p^*\omega^0\left((\mathrm{dev}\times\rho)_* X, (\mathrm{dev}\times\rho)_* Y\right)\\ &\quad + dp^*\omega^0_\alpha\left(J^0_\alpha(\mathrm{dev}\times\rho)_* X, (\mathrm{dev}\times\rho)_* Y\right).\end{aligned} \tag{4.20}$$

Calculate the first and the second summand of (4.20) respectively.

$$\begin{aligned}(\mathrm{dev}\times\rho)^*\left(\sigma^0 \odot p^*\omega^0\right) &= (\mathrm{dev}\times\rho)^*\sigma^0 \odot (\mathrm{dev}\times\rho)^* p^*\omega^0\\ &= \rho^*\mathrm{dev}^*\sigma^0 \odot p^*\mathrm{dev}^*\omega^0.\end{aligned} \tag{4.21}$$

$$\begin{aligned}& dp^*\omega^0_\alpha\left(J^0_\alpha(\mathrm{dev}\times\rho)_* X, (\mathrm{dev}\times\rho)_* Y\right)\\ &= d\omega^0_\alpha\left(J^0_\alpha p_*(\mathrm{dev}\times\rho)_* X, p_*(\mathrm{dev}\times\rho)_* Y\right)\\ &= d\omega^0_\alpha\left(J^0_\alpha \mathrm{dev}_* p_* X, \mathrm{dev}_* p_* Y\right)\\ &= d\omega^0_\alpha\left(\mathrm{dev}_* J'_\alpha p_* X, \mathrm{dev}_* p_* Y\right) \quad (4.16)\\ &= d\omega^0_\alpha\left(\mathrm{dev}_* p_* J'_\alpha X, \mathrm{dev}_* p_* Y\right) \quad (4.19)\\ &= dp^*\mathrm{dev}^*\omega^0_\alpha\left(J'_\alpha X, Y\right) = d\left(p^*\mathrm{dev}^*\omega^0_\alpha\right)\circ J'_\alpha(X,Y).\end{aligned} \tag{4.22}$$

Thus

$$(\mathrm{dev}\times\rho)^* g^0 = R^*_{\tilde{g}}\mathrm{dev}^*\sigma^0 \odot p^*\mathrm{dev}^*\omega^0 + d\left(p^*\mathrm{dev}^*\omega^0_\alpha\right)\circ J'_\alpha.$$

Then it follows by the construction of (3.5) that $(\mathrm{dev}\times\rho)^* g^0$ is the corresponding pseudo-Riemannian metric for $\mathrm{dev}^*\omega^0 = \omega'$ and so $(\mathrm{dev}\times\rho)^* g^0 = g' = u\tilde{g}$ by (4.17). Therefore $(\tilde{M}\times S^3, \tilde{g})$ is conformally flat and so is $(M\times S^3, g)$.

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