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Hölder Regularity for Abstract Fractional Cauchy Problems with Order in (0,1)

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Abstract

In this paper, we study the regularity of mild solution for the following fractional abstract Cauchy problem $D_t^\alpha u(t) = Au(t) + f(t), \quad t \in (0,T] \quad u(0) = x_0$ on a Banach space X with order $\alpha \in (0,1)$, where the fractional derivative is understood in the sense of Caputo fractional derivatives. We show that if A generates an analytic α-times resolvent family on X and $f \in L^p([0,T]; X)$ for some $p > 1/\alpha$, then the mild solution to the above equation is in $C^{\alpha-1/p}[\epsilon, T]$ for every $\epsilon > 0$. Moreover, if f is Hölder continuous, then so are the $D_t^\alpha u(t)$ and $Au(t)$.

Keywords

Fractional Cauchy Problem, Fractional Resolvent Family, Generator, Regularity, Hölder Continuity

1. Introduction

Recently there are increasing interests on fractional differential equations due to their wide applications in viscoelasticity, dynamics of particles, economic and science *et al.* For more details we refer to [1] [2].

Many evolution equations can be rewritten as an abstract Cauchy problem, and then they can be studied in an unified way. For example, a heat equation with different initial data or boundary conditions can be written as a first order Cauchy problem, in which the governing operator generates a C_0-semigroup, and then the solution is given by the operation of this semigroup on the initial data. See for instance [3] [4]. Prüss [5] developed the theory of solution operators to research some abstract Volterra integral equations and it was Bajlekova [6] who first use solution operators to discuss the fractional abstract Cauchy problems. If the coefficient operator of a fractional abstract Cauchy problem generates a C_0-semigroup, we can invoke an operator described by the C_0-semigroup and a probability density function to solve this problem, for more details we refer to [7] [8] [9]. The vector-valued Laplace transform developed in [3] is an important tool in the theory of fractional differential equations.

There are some papers devoted to the fractional differential equations in many different respects: the connection between solutions of fractional Cauchy problems and Cauchy problems of first order [10]; the existence of solution of several kinds of fractional equations [11] [12]; the Hölder regularity for a class of fractional equations [13] [14]; the maximal L^p regularity for fractional order equations [6]; the boundary regularity for the fractional heat equation [15]; the relation of continuous regularity for fractional order equations with semi-variations [12]. In this paper we are mainly interested in the Hölder regularity for abstract Cauchy problems of fractional order.

Pazy [4] considered the regularity for the abstract Cauchy problem of first order:

$$\begin{aligned} &u'(t) = Au(t) + f(t), \quad t \in [0,T] \\ &u(0) = x_0 \end{aligned} \tag{1.1}$$

where A is the infinitesimal generator of an analytic C_0-semigroup. He showed that if $f \in L^p[0,T]$ for some $1 < p < \infty$, then $u(t)$ is Hölder continuous with exponent $\dfrac{p-1}{p}$ in $[\epsilon, T]$; if moreover $x \in D(A)$, then u is Hölder continuous with the same exponent in $[0,T]$. If in addition f is Hölder continuous, then Pazy showed that there are some further regularity of $Au(t)$ and $\dfrac{\mathrm{d}u}{\mathrm{d}t}$. Li [16] gave similar results for fractional differential equations with order $\alpha \in (1,2)$. In this paper we will extend their results to fractional Cauchy problems with order in $(0,1)$.

Our paper is organized as follows. In Section 2 there are some preliminaries on fractional derivatives, fractional Cauchy problems and fractional resolvent families. In Section 3 we give the regularity of the mild solution under the condition that $f \in L^p([0,T], X)$. And some further continuity results are given in Section 4.

2. Preliminaries

Let A be a closed densely defined linear operator on a Banach space X. In this paper we consider the following equation:

$$\begin{aligned} &D_t^\alpha u(t) = Au(t) + f(t), \quad t \in (0,T] \\ &u(0) = x_0 \end{aligned} \tag{2.1}$$

where u and f are X-valued functions, $0 < \alpha < 1$, and D_t^α is the Caputo fractional derivative defined by

$$D_t^\alpha f(t) := \int_0^t g_{1-\alpha}(t-s) f'(s) ds,$$

in which for $\alpha > 0$,

$$g_\alpha(t) := \begin{cases} \frac{t^{\alpha-1}}{\Gamma(\alpha)}, & t > 0 \\ 0, & t \le 0 \end{cases},$$

and $g_0(t)$ is understood as the Dirac measure δ at 0. The convolution of two functions f and g is defined by

$$(f * g)(t) = \int_0^t f(t-s)g(s)ds = \int_0^t f(s)g(t-s)ds$$

when the above integrals exist.

The classical (or strong) solution to (2.1) is defined as:

Definition 2.1. If $0 < \alpha \le 1$, $u \in C([0,T], X)$ is called a solution of (2.1) if

1) $u \in C([0,T], D(A))$.

2) $(g_{1-\alpha} * (u - x_0))(t) \in C^1([0,T], X)$.

3) u satisfies (2.1) on $[0,T]$.

By integration (2.1) for α-times, we are able to define a kind of weak solutions.

Definition 2.2. If $0 < \alpha \le 1$, $u \in C([0,T], X)$ is called a mild solution of (2.1) if $(g_\alpha * u)(t) \in D(A)$ for every $t \in [0,T]$ and

$$u(t) = x_0 + A(g_\alpha * u)(t) + (g_\alpha * f)(t).$$

And it is therefore natural to give the following definition of α-resolvent family for the operator A.

Definition 2.3. A family $\{S_\alpha(t)\}_{t\ge 0} \subset B(X)$ is called an α-resolvent family for the operator A if the following conditions are satisfied:

1) $S_\alpha(t)x : \mathbb{R}_+ \to X$ is continuous for every $x \in X$ and $S_\alpha(0) = I$;

2) $S_\alpha(t)D(A) \subset D(A)$ and $AS_\alpha(t)x = S_\alpha(t)Ax$ for all $x \in D(A)$ and $t \ge 0$;

3) the resolvent equation

$$S_\alpha(t)x = x + (g_\alpha * S_\alpha)(t)Ax$$

holds for every $x \in D(A)$.

If there is an α-times resolvent family $S_\alpha(t)$ for the operator A, then the mild solution of (2.1) is given by the following lemma.

Lemma 2.4. [10] *Let A generate an α-times resolvent family* S_α *and let* $f \in L^1([0,T]; X)$. *If* (2.1) *has a mild solution, then it is given by*

$$u(t) = S_\alpha(t)x_0 + \frac{d}{dt}(g_\alpha * S_\alpha * f)(t), \quad t \ge 0.$$

For the strong solution of (2.1), we have

Lemma 2.5. [10] *Let A generate an α-times resolvent family* S_α *and let* $x_0 \in D(A)$, $f \in C([0,\tau); X)$. *If* $\alpha \in (0,1]$, *then the following statements are equivalent*:

(a) (2.1) has a strong solution on $[0,T]$.

(b) $S_\alpha * f$ is differentiable on $[0,T]$.

(c) $\frac{d}{dt}(g_\alpha * S_\alpha * f)(t) \in D(A)$ for $t \in [0,T]$ and $A(\frac{d}{dt}(g_\alpha * S_\alpha * f)(t))$ is continuous on $[0,T]$.

If in addition, the α-times resolvent family $S_\alpha(t)$ admits an analytic extension to some sector $\Sigma_{\theta+\pi/2}:=\{\lambda\in\mathbb{C}:|\arg(\lambda)|<\theta+\pi/2\}$, and $\|S_\alpha(t)\|\le Me^{\omega \mathrm{Re} t}$ for all $t\in\Sigma_{\theta+\pi/2}$, we will then denote it by $A\in\mathcal{A}^\alpha$.

If $A\in\mathcal{A}^\alpha$, then there exists constants C, ω and θ_0 such that $\lambda^\alpha\in\rho(A)$ and

$$\|\lambda^{\alpha-1}R(\lambda^\alpha,A)\|\le\frac{C}{|\lambda-\omega|}\tag{2.2}$$

for each $\lambda\in\omega+\Sigma_{\theta_0+\pi/2}$. The α-times resolvent family generated by A can be given by

$$S_\alpha(t)=\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\lambda^{\alpha-1}R(\lambda^\alpha,A)\mathrm{d}\lambda,\quad t>0$$

where

$$\Gamma:=\{\omega+re^{-i(2/\pi\pm\delta)}:\rho\le r<\infty\}\bigcup\{\omega+\rho e^{i\phi}:|\phi|\le\pi/2+\delta\}$$

is oriented counter-clockwise. And the corresponding operators $P_\alpha(t)$ are defined by

$$P_\alpha(t)=\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}R(\lambda^\alpha,A)\mathrm{d}\lambda,\quad t>0.$$

Lemma 2.6. *Let* $0<\alpha<1$ *and* $A\in\mathcal{A}^\alpha$. *We have*

(1) $P_\alpha(t)\in B(X)$ for every $t>0$ and $\|P_\alpha(t)\|\le Ce^{\omega t}(1+t^{\alpha-1})$ for $t>0$;

(2) for every $x\in X$, $P_\alpha(t)x\in D(A)$ and $\|AP_\alpha(t)\|\le Ce^{\omega t}(1+t^{-1})$ for $t>0$;

(3) $S'_\alpha(t)=-A(g_{\alpha-1}*S_\alpha)(t)=AP_\alpha(t)$ for $t>0$, $R(P_\alpha^{(l)}(t))\subset D(A)$ for any integer $l\ge0$ and $\|A^kP_\alpha^{(l)}(t)\|\le C_\alpha e^{\omega t}(1+t^{-l-1-\alpha(k-1)})$ for $t>0$, where $k=0,1$.

Proof. (1) By the definition of $P_\alpha(t)$ and (2.2),

$$\begin{aligned}\|P_\alpha(t)\|&\le\frac{1}{2\pi}\int_\Gamma e^{\mathrm{Re}(\lambda t)}\cdot\frac{|\lambda|^{1-\alpha}}{|\lambda-\omega|}|d\lambda|\\&\le\frac{1}{2\pi}\int_\Gamma e^{\mathrm{Re}(\lambda t)}\cdot\frac{c(|\lambda-\omega|^{1-\alpha}+1)}{|\lambda-\omega|}|d\lambda|\\&\le\frac{1}{2\pi}\Big(\int_\Gamma e^{\mathrm{Re}(\lambda t)}\frac{|d\lambda|}{|\lambda-\omega|^\alpha}+\int_\Gamma e^{\mathrm{Re}(\lambda t)}\frac{|d\lambda|}{|\lambda-\omega|}\Big).\end{aligned}$$

Since

$$\int_\Gamma e^{\mathrm{Re}(\lambda t)}\frac{|d\lambda|}{|\lambda-\omega|^\alpha}\le2\int_\rho^\infty e^{\omega t}e^{-rt\sin\delta}\frac{dr}{r^\alpha}+\int_0^\pi e^{\omega t}e^{\rho t\cos\phi}\rho^{1-\alpha}d\phi,$$

taking $\rho=1/t$, we can obtain that the above integral is bounded by

$$2e^{\omega t}\int_1^\infty e^{-r\sin\delta}\frac{t^{\alpha-1}dr}{r^\alpha}+e^{\omega t}t^{\alpha-1}\int_0^\pi e^{\cos\varphi}d\phi\le Ce^{\omega t}t^{\alpha-1}.$$

Analogously one can show the estimate

$$\int_\Gamma e^{\mathrm{Re}(\lambda t)}\frac{|d\lambda|}{|\lambda-\omega|}\le Ce^{\omega t}.$$

It thus follows the estimate for $\|P_\alpha(t)\|$.

(2) By the identity $AR(\lambda^\alpha, A) = \lambda^\alpha R(\lambda^\alpha, A) - I$, we have

$$\int_\Gamma e^{\lambda t} AR(\lambda^\alpha, A) d\lambda = \int_\Gamma e^{\lambda t} \lambda^\alpha R(\lambda^\alpha, A) d\lambda - \int_\Gamma e^{\lambda t} d\lambda = \int_\Gamma e^{\lambda t} \lambda^\alpha R(\lambda^\alpha, A) d\lambda,$$

since $\int_\Gamma e^{\lambda t} d\lambda = 0$. Moreover,

$$\begin{aligned} \left\| \int_\Gamma e^{\lambda t} \lambda^\alpha R(\lambda^\alpha, A) d\lambda \right\| &\le \int_\Gamma e^{\mathrm{Re}(\lambda t)} \frac{|\lambda|^\alpha |d\lambda|}{|\lambda|^{\alpha-1} |\lambda - \omega|} \\ &= \int_\Gamma e^{\mathrm{Re}(\lambda t)} \frac{|\lambda| |d\lambda|}{|\lambda - \omega|} \\ &\le \int_\Gamma e^{\mathrm{Re}(\lambda t)} |d\lambda| + \int_\Gamma e^{\mathrm{Re}(\lambda t)} \frac{|\omega| |d\lambda|}{|\lambda - \omega|} \\ &\le Ce^{\omega t}(t^{-1} + 1). \end{aligned}$$

By the closedness of the operator A, the assertion of (2) follows.

(3) By the proof of (2) and the closedness of A,

$$S'_\alpha(t) = \int_\Gamma e^{\lambda t} \lambda^\alpha R(\lambda^\alpha, A) d\lambda = \int_\Gamma e^{\lambda t} AR(\lambda^\alpha, A) d\lambda = A \int_\Gamma e^{\lambda t} R(\lambda^\alpha, A) d\lambda = AP_\alpha(t).$$

And the second part of (3) can be proved similarly. □

Remark 2.7. Similar results for $\alpha \in (1,2)$ were given in [16]. It is obvious that

$$P_\alpha(t) = (g_{\alpha-1} * S_\alpha)(t)$$

if $1 < \alpha < 2$ and

$$S_\alpha(t) = (g_{1-\alpha} * P_\alpha)(t)$$

if $0 < \alpha < 1$.

3. Regularity of the Mild Solutions

In this section we consider the mild solution of (2.1) with $0 < \alpha < 1$. Suppose that the operator A generates an analytic α-resolvent family, then by Lemma 2.4 and Remark 2.7 the mild solution of (2.1) is given by

$$u(t) = S_\alpha(t) x_0 + (P_\alpha * f)(t). \tag{3.1}$$

Theorem 3.1. *Let* $0 < \alpha < 1$, $A \in \mathcal{A}^\alpha$, *and* $f \in L^p([0,T], X)$ *with* $p > 1/\alpha$. *Then for every* $x_0 \in X$ *and* $\epsilon > 0$, $u \in C^{\alpha - \frac{1}{p}}([\epsilon, T], X)$, *where* $u(t)$ *is given by* (3.1). *If moreover* $x_0 \in D(A^n)$ *such that* $n\alpha \ge 1$, *then* $u \in C^{\alpha - \frac{1}{p}}([0,T], X)$.

Proof. Since $S_\alpha(t)$ is analytic, we only need to show that $(P_\alpha * f)(t) \in C^{\alpha - \frac{1}{p}}$. Let $h > 0$ and $t \in [0, T-h]$, then

$$\begin{aligned} &(P_\alpha * f)(t+h) - (P_\alpha * f)(t) \\ &= \int_0^{t+h} P_\alpha(t+h-s) f(s) \mathrm{d}s - \int_0^t P_\alpha(t-s) f(s) \mathrm{d}s \\ &= \int_t^{t+h} P_\alpha(t+h-s) f(s) \mathrm{d}s + \int_0^t (P_\alpha(t+h-s) - P_\alpha(t-s)) f(s) \mathrm{d}s \\ &= I_1 + I_2. \end{aligned}$$

By Hölder's inequality and Lemma 2.6,

$$
\begin{aligned}
\| I_1 \| &\le \| f \|_{L^p} \left(\int_t^{t+h} \| P_\alpha(t+h-s) \|^{\frac{p}{p-1}} \mathrm{d}s \right)^{\frac{p-1}{p}} \\
&\le C \| f \|_{L^p} \| \left(\int_0^h s^{\frac{p(\alpha-1)}{p-1}} \mathrm{d}s \right)^{\frac{p-1}{p}} \\
&= C \| f \|_{L^p} h^{\frac{p\alpha-1}{p}}.
\end{aligned}
$$

We remark that the constant C here and in the sequel may be vary line by line, but not depending on t and h. Next, we estimate I_2. For $h>0$, $t \in [0,T]$, first assume that $t>h$,

$$
\begin{aligned}
\| I_2 \| &= \| \int_0^t (P_\alpha(t+h-s) - P_\alpha(t-s)) f(s) ds \| \\
&\le \int_0^t \| P_\alpha(s+h) - P_\alpha(s) \| \cdot \| f(t-s) \| \mathrm{d}s \\
&\le \left[\int_0^h (\| P_\alpha(s+h) \| + \| P_\alpha(s) \|)^{\frac{p}{p-1}} \mathrm{d}s \right]^{\frac{p-1}{p}} \cdot \| f \|_{L^p} \\
&\quad + \left[\int_h^t (\| P_\alpha(s+h) - P_\alpha(s) \|)^{\frac{p}{p-1}} \mathrm{d}s \right]^{\frac{p-1}{p}} \cdot \| f \|_{L^p} \\
&\le C \| f \|_{L^p} \left\{ \left[\int_0^h (2s^{\alpha-1})^{\frac{p}{p-1}} \mathrm{d}s \right]^{\frac{p-1}{p}} + \left[\int_h^t (hs^{\alpha-2})^{\frac{p}{p-1}} \mathrm{d}s \right]^{\frac{p-1}{p}} \right\} \\
&\le C \| f \|_{L^p} \left[h^{\frac{p\alpha-1}{p}} + \left(\int_1^{\frac{t}{h}} h^{\frac{\alpha p-1}{p-1}} \tau^{\frac{(\alpha-2)p}{p-1}} d\tau \right)^{\frac{p-1}{p}} \right] \\
&\le C \| f \|_{L^p} h^{\frac{\alpha p-1}{p}} \left[1 + \left(\int_1^\infty \tau^{\frac{(\alpha-2)p}{p-1}} d\tau \right)^{\frac{p-1}{p}} \right] \\
&\le C \| f \|_{L^p} h^{\frac{\alpha p-1}{p}},
\end{aligned}
$$

since $\dfrac{(\alpha-2)p}{p-1} < -1$; if $0<t<h$, then

$$
\begin{aligned}
\| I_2 \| &= \| \int_0^t (P_\alpha(s+h) - P_\alpha(s)) f(t-s) ds \| \\
&\le \int_0^h (\| P_\alpha(s+h) \| + \| P_\alpha(s) \|) \| f(t-s) \| ds
\end{aligned}
$$

from which it follows also that $\| I_2 \| \le C \| f \|_{L^p} h^{\frac{\alpha p-1}{p}}$.

If $x_0 \in D(A^n)$ with $n\alpha \ge 1$, then by [[10], Lemma 4.5] we have that $S_\alpha(t)x_0$ is differentiable and thus Lipschitz continuous. □

If we put more conditions on $f(t)$, the regularity of $u(t)$ can be raised.

Proposition 3.2. *Let* $0<\alpha<1$, $A \in \mathcal{A}^\alpha$ *and* $f \in L^p([0,T],X)$ *with* $1 \le p < \infty$. *For every* $\epsilon > 0$, *we define the function* $\psi_\epsilon(t)$ *by*

$$
\psi_\epsilon(t) = \int_0^{t-\epsilon} \frac{f(t)-f(s)}{(t-s)^{2-\alpha}} \mathrm{d}s.
$$

If $\lim_{\epsilon\to 0} \psi_\epsilon(t) = \psi(t)$ exists, then for every $x_0 \in X$, $u \in C^{1-\frac{1}{p}}([\epsilon,T]:X)$. If moreover $x_0 = 0$, then $u \in C^{1-\frac{1}{p}}([0,T]:X)$.

Proof. If $f(t)$ satisfies the assumption, by [[17], Theorem 13.2] there exists a function $F \in L^p([0,T],X)$ such that $f(t) = (g_{1-\alpha} * F)(t)$. Thus

$$(P_\alpha * f)(t) = (P_\alpha * g_{1-\alpha} * F)(t) = (S_\alpha * F)(t).$$

Since $S_\alpha(t)$ is analytic and bounded, $F(t) \in L^p([0,T], X)$. It is easy to see that $(S_\alpha * F)(t)$ is Hölder continuous with index $1-\frac{1}{p}$. This completes the proof. □

4. Regularity of the Classical Solutions

Motivated by the results in [18] for the C_0-semigroups, we first give the following proposition.

Proposition 4.1. *Let* $0<\alpha<1$ *and* $0<\beta<1$. *Assume* $A \in \mathcal{A}^\alpha$, $x_0 \in D(A)$ *and* $f \in C^\beta([0,T], X)$. *Then the mild solution of* (2.1) *is the classical solution.*

Proof. By Lemma 2.5 we only need to show that $(P_\alpha * f)(t) \in D(A)$ for every $t \in [0,T)$ and $A(P_\alpha * f)(t)$ is continuous on $[0,T)$. We decompose $(P_\alpha * f)(t)$ into two parts:

$$\begin{aligned}(P_\alpha * f)(t) &= \int_0^t P_\alpha(t-s) f(s)\mathrm{d}s \\ &= \int_0^t P_\alpha(t-s) f(t)\mathrm{d}s + \int_0^t P_\alpha(t-s)(f(s)-f(t))\mathrm{d}s \\ &=: I_1 + I_2,\end{aligned}$$

where

$$\begin{aligned}I_1 &= \int_0^t P_\alpha(t-s)\mathrm{d}s f(t) \\ &= (g_1 * P_\alpha)(t) f(t) \\ &= (g_\alpha * S_\alpha)(t) f(t)\end{aligned}$$

belongs to $D(A)$ and is continuous. To prove that $I_2 \in D(A)$, we define the following functions:

$$v(t) = \int_0^t P_\alpha(t-s)(f(s)-f(t))\mathrm{d}s$$

and

$$v_\epsilon(t) = \int_0^{t-\epsilon} P_\alpha(t-s)(f(s)-f(t))\mathrm{d}s$$

for $\epsilon > 0$ small enough. It is clear that $v_\epsilon(t) \to v(t)$ as $\epsilon \to 0^+$. Moreover.

$$P_\alpha(t-s)[f(s)-f(t)] \in D(A)$$

for all $0<s<t-\epsilon$, it follows from the fact that for a fix $t \in [0,T]$ the map $s \mapsto AP_\alpha(t-s)[f(s)-f(t)]$ is a continuous mapping, we conclude that

$$A\int_0^{t-\epsilon} P_\alpha(t-s)[f(s)-f(t)]\mathrm{d}s = \int_0^{t-\epsilon} AP_\alpha(t-s)[f(s)-f(t)]\mathrm{d}s.$$

By our assumption and Lemma 2.6 there exists a constant $C>0$ such that

$$\| AP_\alpha(t-s)[f(s)-f(t)] \| \le C|t-s|^{\beta-1};$$

consequently, the function $s \mapsto AP_\alpha(t-s)[f(s)-f(t)]$ is integrable. Hence by the closedness of A we obtain that

$$v(t) = \int_0^t P_\alpha(t-s)(f(s)-f(t))\mathrm{d}s \in D(A).$$

The continuity of the function $t \mapsto A(P_\alpha * f)(t)$ follows directly from the fact

$$A(P_\alpha * f)(t) = \int_0^t AP_\alpha(t-s)(f(s)-f(t))\mathrm{d}s + S_\alpha(t)f(t) - f(t).$$

This completes the proof. □

We will then give the regularity of such classical solutions.

Lemma 4.2. *Let* $A \in \mathcal{A}^\alpha$ *with* $0<\alpha<1$, *and* $f \in C^\beta([0,T],X)$ *with* $\beta \in (0,1)$. *Define*

$$I_1 = \int_0^t P_\alpha(t-s)(f(s)-f(t))\mathrm{d}s.$$

Then for any $t>0$, $I_1 \in D(A)$ *and* $AI_1 \in C^\beta([0,T],X)$.

Proof. For fixed $t \in (0,T]$, since $f \in C^\beta([0,T],X)$ we have

$$\| AP_\alpha(t-s)(f(s)-f(t)) \| \le C(t-s)^{\beta-1} \in L^1(0,t).$$

By the closedness of *A*, we obtain $I_1 \in D(A)$. Thus it remains to show that $AI_1 \in C^\beta([0,T],X)$. For $h>0$ and $t \in [0,T-h]$ we have the following decomposition:

$$\begin{aligned} AI_1(t+h) - AI_1(t) &= \int_0^t (AP_\alpha(t+h-s) - AP_\alpha(t-s))(f(s)-f(t))\mathrm{d}s \\ &\quad + \int_0^t AP_\alpha(t+h-s)(f(t)-f(t+h))\mathrm{d}s \\ &\quad + \int_t^{t+h} AP_\alpha(t+h-s)(f(s)-f(t+h))\mathrm{d}s \\ &=: h_1 + h_2 + h_3. \end{aligned}$$

Since $f \in C^\beta([0,T],X)$ we have

$$\begin{aligned} \| h_1 \| &\le \int_0^t \| AP_\alpha(\tau+h) - AP_\alpha(\tau) \| \cdot \| f(t-\tau) - f(t) \| \mathrm{d}\tau \\ &\le \int_0^t \| \int_\tau^{\tau+h} AP'_\alpha(s)ds \| \tau^\beta \mathrm{d}\tau \le C\int_0^t (\int_\tau^{\tau+h} s^{-2}ds)\tau^\beta \mathrm{d}\tau \\ &\le C\int_0^t (\frac{1}{\tau} - \frac{1}{\tau+h})\tau^\beta d\tau = C\int_0^t \frac{\tau^{\beta-1}h}{\tau+h} d\tau \\ &= C\int_0^h \frac{h}{\tau+h}\tau^{\beta-1} d\tau + C\int_h^\infty \frac{\tau^{\beta-1}}{\tau+h} h d\tau \le Ch^\beta. \end{aligned}$$

We can estimate h_2 as follows:

$$\begin{aligned} \| h_2 \| &\le \| \int_0^t AP_\alpha(t+h-s)(f(t)-f(t+h))\mathrm{d}s \| \\ &\le \| \int_h^{t+h} AP_\alpha(\tau)(f(t)-f(t+h))\mathrm{d}\tau \| \\ &= \| [(g_1 * AP_\alpha)(t+h) - (g_1 * AP_\alpha)(h)](f(t)-f(t+h)) \| \\ &= \| [(g_\alpha * AS_\alpha)(t+h) - (g_\alpha * AS_\alpha)(h)](f(t)-f(t+h)) \| \\ &\le \| S_\alpha(t+h)(f(t)-f(t+h)) - S_\alpha(h)(f(t)-f(t+h)) \| \\ &\le M \| f(t) - f(t+h) \| \le Mh^\beta. \end{aligned}$$

And it is easy to show that $\| h_3 \| \le Mh^\beta$. Combining all above we have $AI_1 \in C^\beta([0,T];X)$. □

The following theorem extends [[16], Theorem 4.4] to the case that $0<\alpha<1$.

Theorem 4.3. *Let* $A \in \mathcal{A}^\alpha$ *with* $0<\alpha<1$, $f \in C^\beta([0,T],X)$ *with*

$0<\beta<1$, $x_0 \in D(A)$, *and* u *is the classical solution of* (2.1). *The following assertions hold.*

(1) For every $\epsilon>0$, Au, $D_t^\alpha u \in C^\beta([\epsilon,T],X)$.

(2) If moreover $f(0)=0$, then Au and $D_t^\alpha u$ are continuous on $[0,T]$.

(3) If $x_0=f(0)=0$, then Au and $D_t^\alpha u \in C^\beta([0,T],X)$.

Proof. (1) If u is the classical solution of (2.1) on $[0,T]$, then $u(t)=S_\alpha(t)x_0+(P_\alpha * f)(t)$. It is only need to prove $A(P_\alpha * f)(t)\in C^\beta([\epsilon,T],X)$. We decompose

$$(P_\alpha * f)(t)=\int_0^t P_\alpha(t-s)(f(s)-f(t))\mathrm{d}s+\int_0^t P_\alpha(t-s)f(t)\mathrm{d}s=I_1+I_2.$$

By Lemma 4.2, $AI_1\in C^\beta([0,T],X)$. Let $h>0$. If $t\in[\epsilon,T-h]$, then

$$\begin{aligned}
&AI_2(t+h)-AI_2(t)\\
&=\int_0^{t+h}AP_\alpha(t+h-s)f(t+h)\mathrm{d}s-\int_0^t AP_\alpha(t-s)f(t)\mathrm{d}s\\
&=\int_0^{t+h}AP_\alpha(s)f(t+h)\mathrm{d}s-\int_0^t AP_\alpha(s)f(t)\mathrm{d}s\\
&=\int_0^{t+h}AP_\alpha(s)(f(t+h)-f(t))\mathrm{d}s+\int_t^{t+h}AP_\alpha(s)f(t)\mathrm{d}s\\
&=A(g_1 * P_\alpha)(t)(f(t+h)-f(t))+\int_t^{t+h}AP_\alpha(s)f(t)\mathrm{d}s\\
&=A(g_\alpha * S_\alpha)(t)(f(t+h)-f(t))+\int_t^{t+h}AP_\alpha(s)f(t)\mathrm{d}s\\
&=(S_\alpha(t)-I)(f(t+h)-f(t))+\int_t^{t+h}AP_\alpha(s)f(t)\mathrm{d}s,
\end{aligned}$$

thus we have

$$\|AI_2(t+h)-AI_2(t)\|\le C\|S_\alpha(t)-I\|h^\beta+C\int_t^{t+h}s^{-1}\mathrm{d}s\|f\|_\infty\le Ch^\beta+\frac{C}{\varepsilon}h.$$

(2) We only need to show that AI_2 is continuous at $t=0$. Since $f(0)=0$ and $f(t)$ is continuous,

$$\|AI_2(t)\|=\|A(g_1 * P_\alpha)(t)f(t)\|\le C\|S_\alpha(t)-I\|\|f(t)\|\le C\|f(t)\|\to 0$$

as $t\to 0$.

(3) We show that $AI_2\in C^\beta([0,T],X)$. Indeed, this follows from

$$\begin{aligned}
&\|AI_2(t+h)-AI_2(t)\|\\
&\le\left\|\int_0^{t+h}AP_\alpha(t+h-s)(f(t+h)-f(t))\mathrm{d}s+\int_0^t AP_\alpha(t-s)f(t)\mathrm{d}s\right\|\\
&\le C\|A(g_1 * P_\alpha)(t)\|h^\beta+\left\|\int_t^{t+h}AP_\alpha(s)(f(t)-f(0))\mathrm{d}s\right\|\\
&\le Ch^\beta+C\int_t^{t+h}s^{-1}t^\beta\mathrm{d}s\le Ch^\beta+C\int_t^{t+h}s^{\beta-1}\mathrm{d}s\\
&=Ch^\beta+C\int_0^h(t+s)^{\beta-1}\mathrm{d}s\le Ch^\beta+C\int_0^h s^{\beta-1}\mathrm{d}s\le Ch^\beta.
\end{aligned}$$

□

5. Conclusion

In this paper, we proved the Hölder regularity of the mild and strong solutions to the α-order abstract Cauchy problem (2.1) with $\alpha\in(0,1)$. Our results are complemental to the existing results of Pazy [18] for the case $\alpha=1$ and Li [16] for the case that $\alpha \in (1,2)$.

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