

Determining Efficient Solutions of Multi-Objective Linear Fractional Programming Problems and Application

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Abstract

In this paper, a modified method to find the efficient solutions of multi-objective linear fractional programming (MOLFP) problems is presented. While some of the previously proposed methods provide only one efficient solution to the MOLFP problem, this modified method provides multiple efficient solutions to the problem. As a result, it provides the decision makers flexibility to choose a better option from alternatives according to their financial position and their level of satisfaction of objectives. A numerical example is provided to illustrate the modified method and also a real life oriented production problem is modeled and solved.

Keywords

Linear Programming (LP), Linear Fractional Programming (LFP), Multi-Objective Linear Programming (MOLP), Multi-Objective Linear Fractional Programming (MOLFP)

1. Introduction

Making decisions is part of our daily lives. A major concern is that almost all decision problems have multiple, usually conflicting criteria. Multi-objective programming (also known as multi-objective optimization, vector optimization, multi-criteria optimization, multi-attribute optimization or Pareto optimization) is an area of multiple criteria decision making, that is concerned with mathematical optimization problems involving more than one objective function to be optimized simultaneously.

In multi-objective analysis, linear fractional objectives are sometimes encoun-

tered. Fractional programming concerns with the optimization problem of one or several ratios of functions subject to some linear constraints. These ratios are quantities that measure the efficiency of system. When one or more objectives of a multi-objective programming problem are linear fractional *i.e.* ratio of two linear functions under some technological linear restrictions, then the problem is called “multi-objective linear fractional programming (MOLFP) problem”.

When there is more than one fractional objective function, it is difficult to talk about the optimal solutions of these problems. In the case when several fractional objective functions exist, the optimal solution for an objective function may not be an optimal solution for some other objective functions. Therefore, one needs to find the notion of the “best compromise solution”, also known as “non-dominated solution”, “efficient solution”, “Pareto optimal solution”, “Pareto efficient solution” etc. Thus the concept of optimality in the MOLFP problem is replaced with that of efficiency. A solution is called efficient if none of the objective functions can be improved in value without demeaning the value of any other objective. There may exist a good number of efficient solutions; as vectors cannot be ordered completely, all efficient solutions are equally good.

There exist several methodologies to solve MOLFP problem in the literature. Among them, few approaches have been reported in the early age. Kornbluth and Steuer [1] considered MOLFP problem and presented a simplex-based solution procedure to find all weakly efficient vertices of the augmented feasible region. Benson [2] showed that the procedure suggested by Kornbluth and Steuer for computing the numbers to find break points may not work all the time and he proposed a fail-safe method for computing these numbers.

In the recent years, some other approaches have been reported for solving MOLFP problems. Guzel and Sivri [3] worked together to propose a method for finding an efficient solution of MOLFP problem using goal programming. Later Guzel [4] presented a simplex-based algorithm to find an efficient solution of MOLFP problem based on a theorem studied in a work by Dinkelbach [5], where he converted the main problem into a single linear programming problem. Jain [6] proposed a method using Gauss elimination technique to derive numerical solution of multi-objective linear programming (MOLP) problem. Then Jain [7] in 2014 extended his work for MOLFP problem. Porchelvi *et al.* [8] presented procedures for solving multi-objective linear fractional programming problems for both crisp and fuzzy cases using the complementary development method [9], where the fractional linear programming is transformed into linear programming problem. All of these methods provide only one efficient solution of MOLFP problem. S.F. Tantway [10] proposed a feasible direction method to find all efficient solutions of MOLFP problem. But his proposed method is applicable only for a special class of MOLFP problem, where all denominators of the fractional objectives are equal.

Though some approaches to find an efficient solution of MOLFP problem can be observed in the literature but hardly any method for finding multiple efficient solutions.

Here we concentrate on finding more than one (depending on the number of objectives) efficient solution of MOLFP problem by using the methods proposed by Dheyab [9] and Porchelvi *et al.* [8]. We modify the method for solving MOLFP problem provided by Porchelvi *et al.* [8] which is based on the concept of Dheyab [9]. Our modified method gives improved result in a sense that it provides multiple efficient solutions including also the one obtained by Porchelvi's method. We provide a numerical example to show the comparison. We also provide an application to show the advantage of our method.

2. Mathematical Definitions

Linear Fractional Programming (LFP) Problem: An LFP problem is defined as follows:

$$\begin{aligned}
 \text{(LFP)} \quad & \text{Maximize } z = \frac{N(\mathbf{x})}{D(\mathbf{x})} = \frac{\mathbf{c}^T \mathbf{x} + \alpha}{\mathbf{d}^T \mathbf{x} + \beta} \\
 & \text{subject to } \mathbf{Ax} \leq \mathbf{b} \\
 & \mathbf{x} \geq \mathbf{0}
 \end{aligned}$$

Here,

$S = \{ \mathbf{x} \in \mathbb{R}^n \mid \mathbf{Ax} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}, \mathbf{b} \in \mathbb{R}^m \}$ is the feasible set in decision space,

A is an $m \times n$ matrix,

$\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^m; (\mathbf{b} \geq \mathbf{0})$,

$\mathbf{c}^T, \mathbf{d}^T \in \mathbb{R}^n$ and

$\alpha, \beta \in \mathbb{R}$.

The denominator $D(\mathbf{x}) = \mathbf{d}^T \mathbf{x} + \beta \neq 0$ for all $\mathbf{x} = (x_1, x_2, \dots, x_n) \in S$.

Multi-objective Linear Fractional Programming (MOLFP) Problem: An MOLFP problem is defined as follows:

$$\begin{aligned}
 \text{(MOLFP)} \quad & \text{Maximize } \{ \mathbf{Z}(\mathbf{x}) = (z_1(\mathbf{x}), z_2(\mathbf{x}), \dots, z_k(\mathbf{x})) \} \\
 & \text{subject to } \mathbf{Ax} \leq \mathbf{b} \\
 & \mathbf{x} \geq \mathbf{0}
 \end{aligned}$$

Here,

$S = \{ \mathbf{x} \in \mathbb{R}^n \mid \mathbf{Ax} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}, \mathbf{b} \in \mathbb{R}^m \}$ is the feasible set in decision space,

A is an $m \times n$ matrix,

$\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^m; (\mathbf{b} \geq \mathbf{0})$,

$k \geq 2$,

$$z_i(\mathbf{x}) = \frac{\mathbf{c}_i^T \mathbf{x} + \alpha_i}{\mathbf{d}_i^T \mathbf{x} + \beta_i} = \frac{N_i(\mathbf{x})}{D_i(\mathbf{x})}; \mathbf{c}_i^T, \mathbf{d}_i^T \in \mathbb{R}^n; \alpha_i, \beta_i \in \mathbb{R}; \text{ for all } i = 1, 2, \dots, k$$

and $D_i(\mathbf{x}) = \mathbf{d}_i^T \mathbf{x} + \beta_i > 0$, for all $i = 1, 2, \dots, k$, for all $\mathbf{x} \in S$.

Efficient solution of MOLFP problem: A solution $\bar{\mathbf{x}} \in S$ is an efficient solution of the problem (MOLFP) if and only if there is no $\mathbf{x} \in S$ such that $z_i(\mathbf{x}) \geq z_i(\bar{\mathbf{x}})$ for all $i = 1, 2, \dots, k$ and $z_i(\mathbf{x}) > z_i(\bar{\mathbf{x}})$ for at least one i .

Note that, for vectors \mathbf{x} and \mathbf{y} ;

$\mathbf{x} \geq \mathbf{y}$ implies $x_i \geq y_i$ for each i
 $\mathbf{x} \geq \mathbf{y}$ implies $x_i \geq y_i$ for i and $x_r > y_r$ for at least one $i = r$,
 and $\mathbf{x} > \mathbf{y}$ implies $x_i > y_i$ for each i .

3. Complementary Development Method to Find Efficient Solution of MOLFP Problem

The complementary development method studied by Dheyab [9] is used to transform linear fractional programming problem into linear programming problem. The concept is that, to maximize a ratio function, the numerator of the function should be maximized and the denominator of the function should be minimized. To maximize a fractional objective, that fractional objective can be linearized by subtracting its denominator from the numerator and this linearized objective is then maximized subject to the original restrictions of the problem. Thus LFP problem converts into LP problem.

Later, Porchelvi *et al.* [8] extended this concept for solving MOLFP problem. They proposed an algorithm for solving MOLFP problem which gives an efficient solution to the problem. The algorithm is based on the concept that, if a linear programming problem is formed, where any one of the objective functions of MOLP problem should be optimized subject to the original constraints in addition to optimization of the remaining objective functions used as constraints, then the optimal solution of that problem becomes an efficient solution of the MOLP problem, as all the objectives are satisfied simultaneously.

3.1. Algorithm of Complementary Development Method to Find Efficient Solution of MOLFP Problem

Step I: At first consider the first objective function $z_1(\mathbf{x}) = \frac{N_1(\mathbf{x})}{D_1(\mathbf{x})}$, in which

$N_1(\mathbf{x})$ is the numerator function and $D_1(\mathbf{x})$ is the denominator function. The value of the objective function is taken as maximum ($\max N_1(\mathbf{x})$) for the numerator and minimum ($\min D_1(\mathbf{x})$) for the denominator function.

Step II: An LP problem is formulated as: $\max \bar{z}_1(\mathbf{x})$ subject to the constraints of the original problem, where the objective function $\bar{z}_1(\mathbf{x})$ is formed by subtracting the denominator function ($D_1(\mathbf{x})$) from the numerator function ($N_1(\mathbf{x})$) *i.e.* $\bar{z}_1(\mathbf{x}) = N_1(\mathbf{x}) - D_1(\mathbf{x})$. This LP problem is solved by regular simplex method.

Step III: The same LP formulation procedure is repeated for the second objective function $z_2(\mathbf{x}) = \frac{N_2(\mathbf{x})}{D_2(\mathbf{x})}$. This time, the LP problem is solved including the maximization of prior objective function $\bar{z}_1(\mathbf{x})$ as one of the constraints.

Step IV: Again the same LP formulation procedure is repeated for the third objective function $z_3(\mathbf{x}) = \frac{N_3(\mathbf{x})}{D_3(\mathbf{x})}$. And then, the LP problem is solved

including the maximization of prior objective function $\bar{z}_2(\mathbf{x})$ as one of the constraints.

Step V: The same procedure is repeated until all the objective functions are optimized. Calculations up to this step provide one efficient solution of the problem (MOLFP). Reclamation of the values of $\max z_i(\mathbf{x})$ is done by substituting the efficient solution into the original objective functions $z_i(\mathbf{x})$.

3.2. Numerical Example (NE 1)

Consider an MOLFP problem

$$\max \left\{ z_1(\mathbf{x}) = \frac{-3x_1 + 2x_2}{x_1 + x_2 + 3}, z_2(\mathbf{x}) = \frac{7x_1 + x_2}{5x_1 + 2x_2 + 1}, z_3(\mathbf{x}) = \frac{x_1 + 4x_2}{2x_1 + 3x_2 + 2} \right\}$$

subject to

$$\begin{aligned} x_1 - x_2 &\geq 1 \\ 2x_1 + 3x_2 &\leq 15 \\ x_1 + 9x_2 &\geq 9 \\ x_1 &\geq 3 \\ x_1, x_2 &\geq 0 \end{aligned}$$

3.3. Solution

First consider the first objective function $z_1(\mathbf{x}) = \frac{-3x_1 + 2x_2}{x_1 + x_2 + 3}$ and separate it into sub-functions (numerator and denominator). Now as per the algorithm we have to construct an LP problem subject to the original constraints as follows:

$$\max \bar{z}_1(\mathbf{x}) = (-3x_1 + 2x_2) - (x_1 + x_2 + 3) = -4x_1 + x_2 - 3$$

subject to

$$\begin{aligned} x_1 - x_2 &\geq 1 \\ 2x_1 + 3x_2 &\leq 15 \\ x_1 + 9x_2 &\geq 9 \\ x_1 &\geq 3 \\ x_1, x_2 &\geq 0. \end{aligned}$$

Solving this LP problem by regular simplex method (for convenience we use Mathematica), we get the following optimal solution:

$$x_1 = 3, x_2 = 2 \text{ with } \max \bar{z}_1 = -13.$$

Next, consider the second objective function $z_2(\mathbf{x}) = \frac{7x_1 + x_2}{5x_1 + 2x_2 + 1}$.

According to the algorithm, we construct a new LP problem as follows:

$$\max \bar{z}_2(\mathbf{x}) = (7x_1 + x_2) - (5x_1 + 2x_2 + 1) = 2x_1 - x_2 - 1$$

subject to

$$\begin{aligned} x_1 - x_2 &\geq 1 \\ 2x_1 + 3x_2 &\leq 15 \\ x_1 + 9x_2 &\geq 9 \\ x_1 &\geq 3 \\ 4x_1 - x_2 &\leq 10 \end{aligned}$$

$$x_1, x_2 \geq 0.$$

Here the set of constraints include the maximization of prior objective \bar{z}_1 as a constraint, that is, the constraint $4x_1 - x_2 \leq 10$ is obtained by simplifying $-4x_1 + x_2 - 3 \geq -13$.

Solving this LP problem we can get the following optimal solution:

$$x_1 = 3, x_2 = 2 \quad \text{with} \quad \max \bar{z}_2 = 3.$$

At last, we consider the third objective function $z_3(\mathbf{x}) = \frac{x_1 + 4x_2}{2x_1 + 3x_2 + 2}$ and solve the following LP problem:

$$\max \bar{z}_3(\mathbf{x}) = (x_1 + 4x_2) - (2x_1 + 3x_2 + 2) = -x_1 + x_2 - 2$$

$$\begin{aligned} \text{subject to } & x_1 - x_2 \geq 1 \\ & 2x_1 + 3x_2 \leq 15 \\ & x_1 + 9x_2 \geq 9 \\ & x_1 \geq 3 \\ & 2x_1 - x_2 \geq 4 \\ & x_1, x_2 \geq 0. \end{aligned}$$

Here the set of constraints include the maximization of prior objective \bar{z}_2 as a constraint, that is, the constraint $2x_1 - x_2 \geq 4$ is obtained by simplifying $2x_1 - x_2 - 1 \geq 3$.

Solving this LP problem we can get the following optimal solution:

$$x_1 = 3, x_2 = 2 \quad \text{with} \quad \max \bar{z}_3 = -3.$$

This gives an efficient solution for the given MOLFP problem which is,

$$x_1 = 3, x_2 = 2 \quad \text{with} \quad \max z_1 = -\frac{5}{8}, \max z_2 = \frac{23}{20}, \max z_3 = \frac{11}{14}.$$

4. Proposed Modified Method to Find Efficient Solutions of MOLFP Problem

For any MOLFP problem, there may exist a good number of efficient solutions. As vectors cannot be ordered completely, all efficient solutions are equally acceptable. It depends on the situation that which of the efficient solutions is preferable to the decision makers. Decision makers' preference may depend on their financial position, time limit etc. So it is better to find more than one efficient solution for any MOLFP problem and provide decision maker facility to choose a better option from alternatives according to their level of satisfaction of objectives.

Keeping all these in mind we tried to find more than one efficient solution for an MOLFP problem. We modified the complementary method (developed by Porchelvi *et al.* [8] for MOLFP problem) by constructing LP problems in more ways. It is done by repeating the whole procedure (step I to V) of complementary development algorithm by altering the order of the objective functions in all

possible way (for n objectives there will be $n!$ ways).

The same problem (NE 1) using this modified method is solved here which gives three efficient solutions.

In this modified method, the objective functions are considered in different orders (in six possible ways).

So the following cases arise:

| Order of consideration Case | First | Second | Third |
|--------------------------------|-------------------|-------------------|-------------------|
| 1 | $z_1(\mathbf{x})$ | $z_2(\mathbf{x})$ | $z_3(\mathbf{x})$ |
| 2 | $z_2(\mathbf{x})$ | $z_1(\mathbf{x})$ | $z_3(\mathbf{x})$ |
| 3 | $z_1(\mathbf{x})$ | $z_3(\mathbf{x})$ | $z_2(\mathbf{x})$ |
| 4 | $z_2(\mathbf{x})$ | $z_3(\mathbf{x})$ | $z_1(\mathbf{x})$ |
| 5 | $z_3(\mathbf{x})$ | $z_1(\mathbf{x})$ | $z_2(\mathbf{x})$ |
| 6 | $z_3(\mathbf{x})$ | $z_2(\mathbf{x})$ | $z_1(\mathbf{x})$ |

We proceed as follows:

Case 1: This case is same as the complementary development algorithm, which gives the following efficient solution for the given MOLFP problem:

$$x_1 = 3, x_2 = 2 \text{ with } \max z_1 = -\frac{5}{8}, \max z_2 = \frac{23}{20}, \max z_3 = \frac{11}{14}.$$

To get more efficient solution we have to consider the following cases:

Case 2: Here in this case, first considering the second objective $z_2(\mathbf{x})$ we formulate and solve the LP problem: $\max \bar{z}_2(\mathbf{x})$ subject to the constraints of the original problem. Next consider the first objective function $z_1(\mathbf{x})$ and formulate the LP problem: $\max \bar{z}_1(\mathbf{x})$ subject to the original constraints in addition to the maximization of prior objective $\bar{z}_2(\mathbf{x})$ as a constraint. After solving this LP problem, finally consider the third objective $z_3(\mathbf{x})$. Then formulate and solve the LP problem: $\max \bar{z}_3(\mathbf{x})$ subject to the original constraints in addition to the maximization of prior objective $\bar{z}_1(\mathbf{x})$ as a constraint. This will give the following efficient solution:

$$x_1 = \frac{36}{5}, x_2 = \frac{1}{5} \text{ with } \max z_1 = -\frac{53}{26}, \max z_2 = \frac{23}{17}, \max z_3 = \frac{8}{17}.$$

Proceeding in this manner, the remaining cases will give the following efficient solutions:

Case 3: $x_1 = \frac{18}{5}, x_2 = \frac{13}{5}$ with $\max z_1 = -\frac{14}{23}, \max z_2 = \frac{139}{121}, \max z_3 = \frac{14}{17}$;

Case 4: $x_1 = 3, x_2 = 2$ with $\max z_1 = -\frac{5}{8}, \max z_2 = \frac{23}{20}, \max z_3 = \frac{11}{14}$;

Case 5: $x_1 = 3, x_2 = 2$ with $\max z_1 = -\frac{5}{8}, \max z_2 = \frac{23}{20}, \max z_3 = \frac{11}{14}$;

Case 6: $x_1 = \frac{36}{5}, x_2 = \frac{1}{5}$ with $\max z_1 = -\frac{53}{26}, \max z_2 = \frac{23}{17}, \max z_3 = \frac{8}{17}$.

Concluding the results we get the following three efficient solutions:

| |
|---|
| 1. $x_1 = 3, x_2 = 2$ with $\max z_1 = -\frac{5}{8}, \max z_2 = \frac{23}{20}, \max z_3 = \frac{11}{14};$ |
| 2. $x_1 = \frac{36}{5}, x_2 = \frac{1}{5}$ with $\max z_1 = -\frac{53}{26}, \max z_2 = \frac{23}{17}, \max z_3 = \frac{8}{17};$ |
| 3. $x_1 = \frac{18}{5}, x_2 = \frac{13}{5}$ with $\max z_1 = -\frac{14}{23}, \max z_2 = \frac{139}{121}, \max z_3 = \frac{14}{17}.$ |

So our modified method provides multiple efficient solutions to the problem (NE 1) including the efficient solution (no. 1) obtained by the previous method.

Remark: Using this modified method, for an MOLFP problem with n objectives we can get at best n efficient solutions.

5. Application

We have partially taken data for the following problem from S.K. Saha *et al.* [11].

5.1. Production Problem of a Certain Industry

Suppose an industry has Tk. 30,000,000/= by which it can produce six different products Dalda, Coconut oil, Mustard oil, Sunflower oil, Soybean oil and Palm oil. The net refined oil from per metric ton of dalda, coconut, mustard seeds, sunflower seeds, soybean crude oil and palm crude oil are respectively 300 kg, 400 kg, 400 kg, 980 kg, 970 kg and 980 kg; moreover the time needed are 4 days, 5 days, 6 days, 6 days, 3.5 days and 5 days respectively. The industry has a fixed establishment cost and time of Tk. 500,000/= and 20 days respectively. The management of industry wishes to produce maximum 600 metric tons of different types of oil. The cost for different raw materials to produce per metric ton crude oil/ seed in taka is given in **Table 1**.

In addition, the industry has the following limitations on expenditures:

Maximum investment for crude oil/seeds is Tk. 20,000,000/=;

Maximum investment for transportation is Tk. 500,000/=;

Maximum investment for storage is Tk. 15,000/=;

Maximum investment for customs duties and vat is Tk. 6,000,000/=;

Maximum investment for chemicals is Tk. 50,000/=;

Maximum investment for electricity and gas is Tk. 120,000/=;

Maximum investment for maintenance is Tk. 30,000/=;

Maximum investment for labor is Tk. 200,000/=;

Maximum investment for management is Tk. 25,000/=;

Maximum investment for delivery is Tk. 10,000/=.

The management of industry wants to maximize the ratio of return on investment and maximize the ratio of return on time.

This leads to a multi-objective linear fractional programming problem.

5.2. Mathematical Formulation of the Problem

▪ Selection of unknown variables

Let x_1, x_2, x_3, x_4, x_5 and x_6 be the metric tons of crude oil/seeds of dalda, coconut oil, mustard oil, sunflower oil, soybean oil and palm oil has to be refined respectively.

Table 1. Cost for different raw materials (in taka).

| Name of products | Cost of crude oil/seeds | Transportation cost | Storage cost | Customs duties and vat | Chemical cost | Cost of electricity and gas | Maintenance cost | Labor cost | Management cost | Delivery cost | Return (per metric ton) |
|-----------------------|-------------------------|---------------------|--------------|------------------------|---------------|-----------------------------|------------------|------------|-----------------|---------------|-------------------------|
| 1) Dalda | 22,800 | 650 | 20 | 11,400 | 148 | 180 | 60 | 30 | 42 | 15 | 59,890 |
| 2) Coconut oil | 9200 | 630 | 22 | 3220 | - | 220 | 40 | 32 | 38 | 18 | 23,390 |
| 3) Mustard oil (seed) | 16,000 | 320 | 20 | 1800 | - | 200 | 35 | 28 | 36 | 16 | 30,750 |
| 4) Sunflower oil | 25,500 | 660 | 18 | 12,750 | 238 | 150 | 50 | 35 | 40 | 14 | 59,750 |
| 5) Soybean oil | 20,000 | 360 | 20 | 3250 | - | 100 | 30 | 26 | 37 | 17 | 40,700 |
| 6) Palm oil | 20,000 | 640 | 17 | 3000 | 135 | 160 | 45 | 20 | 35 | 18 | 59,435 |

▪ **Identification of constraints**

1) The management of industry wishes to produce maximum 600 metric tons different types of oil, which yields

$$0.3x_1 + 0.4x_2 + 0.4x_3 + 0.98x_4 + 0.97x_5 + 0.98x_6 \leq 600 ;$$

2) The industry has maximum investment for crude oil/seeds Tk. 20,000,000/=, which results

$$22800x_1 + 9200x_2 + 16000x_3 + 25500x_4 + 20000x_5 + 20000x_6 \leq 20000000 ;$$

Similarly,

3) For transportation,

$$650x_1 + 630x_2 + 320x_3 + 660x_4 + 360x_5 + 640x_6 \leq 500000 ;$$

4) For storage,

$$20x_1 + 22x_2 + 20x_3 + 18x_4 + 20x_5 + 17x_6 \leq 15000 ;$$

5) For customs duties and vat,

$$11400x_1 + 3220x_2 + 1800x_3 + 12750x_4 + 3250x_5 + 3000x_6 \leq 6000000 ;$$

6) For chemicals,

$$148x_1 + 238x_4 + 135x_6 \leq 50000 ;$$

7) For electricity and gas,

$$180x_1 + 220x_2 + 200x_3 + 150x_4 + 100x_5 + 160x_6 \leq 120000 ;$$

8) For maintenance,

$$60x_1 + 40x_2 + 35x_3 + 50x_4 + 30x_5 + 45x_6 \leq 30000 ;$$

9) For labor,

$$30x_1 + 32x_2 + 28x_3 + 35x_4 + 26x_5 + 20x_6 \leq 200000 ;$$

10) For delivery,

$$15x_1 + 18x_2 + 16x_3 + 14x_4 + 17x_5 + 18x_6 \leq 10000 ;$$

11) For management,

$$42x_1 + 38x_2 + 36x_3 + 40x_4 + 37x_5 + 35x_6 \leq 25000 .$$

We must assume that the variables $x_i; i=1,2,\dots,6$ are not allowed to be negative. That means negative quantities of product cannot be produced.

▪ **Identification of objectives**

Using the given information we have,

$$\begin{aligned} \text{Total Return (in Tk.)} &= 59890x_1 + 23390x_2 + 30750x_3 \\ &\quad + 59750x_4 + 40700x_5 + 59435x_6 \end{aligned}$$

$$\begin{aligned} \text{Total Cost (in Tk.)} &= 35345x_1 + 13420x_2 + 18455x_3 + 39455x_4 \\ &\quad + 23840x_5 + 24070x_6 + 500000 \end{aligned}$$

$$\text{Total Time (in hour)} = 96x_1 + 120x_2 + 144x_3 + 144x_4 + 84x_5 + 120x_6 + 480.$$

So the objectives become

$$\text{Maximize } \frac{59890x_1 + 23390x_2 + 30750x_3 + 59750x_4 + 40700x_5 + 59435x_6}{35345x_1 + 13420x_2 + 18455x_3 + 39455x_4 + 23840x_5 + 24070x_6 + 500000}$$

and

$$\text{Maximize } \frac{59890x_1 + 23390x_2 + 30750x_3 + 59750x_4 + 40700x_5 + 59435x_6}{96x_1 + 120x_2 + 144x_3 + 144x_4 + 84x_5 + 120x_6 + 480}.$$

Here each objective function is a ratio of two linear functions and all of the constraints are linear, so the problem can be modeled as the following MOLFP problem:

$$\max \left\{ \begin{aligned} z_1(x) &= \frac{59890x_1 + 23390x_2 + 30750x_3 + 59750x_4 + 40700x_5 + 59435x_6}{35345x_1 + 13420x_2 + 18455x_3 + 39455x_4 + 23840x_5 + 24070x_6 + 500000}, \\ z_2(x) &= \frac{59890x_1 + 23390x_2 + 30750x_3 + 59750x_4 + 40700x_5 + 59435x_6}{96x_1 + 120x_2 + 144x_3 + 144x_4 + 84x_5 + 120x_6 + 480} \end{aligned} \right\}$$

where, $z_1(x)$ represents return on investment and $z_2(x)$ represents return on time.

Subject to

$$0.3x_1 + 0.4x_2 + 0.4x_3 + 0.98x_4 + 0.97x_5 + 0.98x_6 \leq 600$$

$$22800x_1 + 9200x_2 + 16000x_3 + 25500x_4 + 20000x_5 + 20000x_6 \leq 20000000$$

$$650x_1 + 630x_2 + 320x_3 + 660x_4 + 360x_5 + 640x_6 \leq 500000$$

$$20x_1 + 22x_2 + 20x_3 + 18x_4 + 20x_5 + 17x_6 \leq 15000$$

$$11400x_1 + 3220x_2 + 1800x_3 + 12750x_4 + 3250x_5 + 3000x_6 \leq 6000000$$

$$148x_1 + 238x_4 + 135x_6 \leq 50000$$

$$180x_1 + 220x_2 + 200x_3 + 150x_4 + 100x_5 + 160x_6 \leq 120000$$

$$60x_1 + 40x_2 + 35x_3 + 50x_4 + 30x_5 + 45x_6 \leq 30000$$

$$30x_1 + 32x_2 + 28x_3 + 35x_4 + 26x_5 + 20x_6 \leq 200000$$

$$15x_1 + 18x_2 + 16x_3 + 14x_4 + 17x_5 + 18x_6 \leq 10000$$

$$42x_1 + 38x_2 + 36x_3 + 40x_4 + 37x_5 + 35x_6 \leq 25000$$

$$x_i \geq 0; i = 1, 2, \dots, 6.$$

5.3. Solution

To find efficient solutions of the problem we are going to consider the following two cases:

| Order of consideration | First | Second |
|------------------------|-------------------|-------------------|
| Case | | |
| 1 | $z_1(\mathbf{x})$ | $z_2(\mathbf{x})$ |
| 2 | $z_2(\mathbf{x})$ | $z_1(\mathbf{x})$ |

z_1 : represents return on investment, and z_2 : represents return on time.

Proceeding in according to the proposed modified method, we get the following two efficient solutions:

Solution 1: $x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0, x_5 = 196.078$ and $x_6 = 370.37$
 with $\max z_1 = 2.1288$ and $\max z_2 = 488.531$
 and

Solution 2: $x_1 = 337.838, x_2 = 0, x_3 = 0, x_4 = 0, x_5 = 290.143$ and $x_6 = 0$
 with $\max z_1 = 1.65524$ and $\max z_2 = 559.348$.

5.4. Observation

- ✓ $\max z_1$ for Solution 1 > $\max z_1$ for Solution 2.
- ✓ $\max z_2$ for Solution 1 < $\max z_2$ for Solution 2.
- ✓ The increment in the maximum value of z_2 between two solutions is comparatively poor than that of z_1 .

5.5. Decision

- ✓ If the management of industry is more concerned about cost than that of time, then they will choose Solution 1.
- ✓ If there is shortage of time and the management has to fulfill their target within limited time then they will choose Solution 2.
- ✓ If there is no time shortage and the management concentrate on the overall situation (keeping last observation in mind), then they can choose Solution 1.

6. Conclusion

In this work, we tried to find multiple solutions of MOLFP problem. We have elaborately explained the procedure with numerical example. Further, one application is also shown by discussing a real life-oriented problem.

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