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On a Subordination Result of a Subclass of Analytic Functions

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Abstract

In this paper, we investigate a subordination property and the coefficient inequality for the class $M(1,b)$, The lower bound is also provided for the real part of functions belonging to the class $M(1,b)$.

Keywords

Analytic Function, Univalent Function, Hadamard Product, Subordination

1. Introduction

Let A denote the class of function $f(z)$ analytic in the open unit disk $U=\{z\in\mathbb{C}:|z|<1\}$ and let S be the subclass of A consisting of functions univalent in U and have the form

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k, \tag{1.1}$$

The class of convex functions of order α in *U*, denoted as $K(\alpha)$ is given by

$$K(\alpha)=\left\{f\in S: Re\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha, 0\le\alpha<1, z\in U\right\}$$

Definition 1.1. *The Hadamard product or convolution* $f*g$ *of the function* $f(z)$ *and* $g(z)$, *where* $f(z)$ *is as defined in* (1.1) *and the function* $g(z)$ *is given by*

$$g(z)=z+\sum_{k=2}^{\infty}b_k z^k,$$

is defined as:

$$(f*g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g*f)(z), \tag{1.2}$$

Definition 1.2. *Let* $f(z)$ *and* $g(z)$ *be analytic in the unit disk* U *. Then* $f(z)$ *is said to be subordination to* $g(z)$ *in* U *and written as:*

$$f(z) \prec g(z),\ z \in U$$

if there exist a Schwarz function $\omega(z)$, analytic in U with $\omega(0) = 0$, $|\omega(z)| < 1$ such that

$$f(z) = g(\omega(z)),\ z \in U \tag{1.3}$$

In particular, if the function $g(z)$ is univalent in *U*, then $f(z)$ is said to be subordinate to $g(z)$ if

$$f(0) = g(0),\quad f(u) \subset g(u) \tag{1.4}$$

Definition 1.3. *The sequence* $\{c_k\}_{k=1}^{\infty}$ *of complex numbers is said to be a subordinating factor sequence of the function* $f(z)$ *if whenever* $f(z)$ *in the form* (1.1) *is analytic, univalent and convex in the unit disk* U , *the subordination is given by*

$$\sum_{k=1}^{\infty} a_k c_k z^k \prec f(z),\ z \in U, a_1 = 1$$

We have the following theorem:

Theorem 1.1. (Wilf [1]) *The sequence* $\{c_k\}_{k=1}^{\infty}$ *is a subordinating factor sequence if and only if*

$$Re\left\{1 + 2\sum_{k=1}^{\infty} c_k z^k\right\} > 0,\ z \in U \tag{1.5}$$

Definition 1.4. *A function* $P \in A$ *which is normalized by* $P(0) = 1$ *is said to be in* $P(1,b)$ *if*

$$|P(z) - 1| < b,\ b > 0, z \in U.$$

The class $P(1,b)$ was studied by Janwoski [2]. The family $P(1,b)$ contains many interesting classes of functions. For example, for $f(z) \in A$, if

$$\left(\frac{zf'(z)}{f(z)}\right) \in P(1,1-\alpha), 0 \le \alpha < 1$$

Then $f(z)$ is starlike of order α in *U* and if

$$\left(1 + \frac{zf''(z)}{f'(z)}\right) \in P(1,1-\alpha), 0 \le \alpha < 1$$

Then $f(z)$ is convex of order α in *U*.

Let $F(1,b)$ be the subclass of $P(1,1-\alpha)$ consisting of functions $P(f)$ such that

$$P(f) = \frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)}\right) \tag{1.6}$$

we have the following theorem

Theorem 1.2. [3] *Let* $P(f)$ *be given by* Equation (1.6) *with* $f(z)=z+\sum a_k z^k$. *If*

$$\sum_{k=2}^{\infty}\left(k^2+b-1\right)\left|a_k\right|<b, b>0$$

then $P(f)\in F(1,b)$, $0<b<0.935449$.

It is natural to consider the class

$$M(1,b)=\left\{f\in A:\sum_{k=2}^{\infty}\left(k^2+b-1\right)\left|a_k\right|<b, b>0\right\}$$

$$0<b<0.935449$$

Remark 1.1. [4] *If* $b=1-\alpha$, *then* $M(1,1-\alpha)$ *consists of starlike functions of order* α, $0\le\alpha<1$ *since*

$$\sum_{k=2}^{\infty}(k-\alpha)\left|a_k\right|<\sum_{k=2}^{\infty}\left(k^2-\alpha\right)\left|a_k\right|$$

Our main focus in this work is to provide a subordination results for functions belonging to the class $M(1,b)$

2. Main Results

2.1. Theorem

Let $f(z)\in M(1,b)$, then

$$\frac{3+b}{2(3+2b)}(f*g)(z)\prec g(z) \tag{2.1}$$

where $0<b<0.935449$ and $g(z)$ is convex function.

Proof:

Let

$$f(z)\in M(1,b)$$

and suppose that

$$g(z)=z+\sum b_k z^k\in C(\alpha)$$

that is $g(z)$ is a convex function of order α.

By definition (1.1) we have

$$\begin{aligned}&\frac{3+b}{2(3+2b)}(f*g)(z)\\&=\frac{3+b}{2(3+2b)}\left(z+\sum_{k=2}^{\infty}a_k b_k z^k\right)\\&=\sum_{k=1}^{\infty}\frac{3+b}{2(3+2b)}a_k b_k z^k, a_1=1, b_1=1\end{aligned} \tag{2.2}$$

Hence, by Definition 1.3…to show subordination (2.1) is by establishing that

$$\left\{\frac{3+b}{2(3+2b)}a_k\right\}_{k=1}^{\infty} \tag{2.3}$$

is a subordinating factor sequence with $a_1 = 1$. By Theorem 1.1, it is sufficient to show that

$$Re\left\{1+2\sum_{k=1}^{\infty}\frac{3+b}{2(3+2b)}a_k z^k\right\}>0,\ z\in U \tag{2.4}$$

Now,

$$\begin{aligned}
&Re\left\{1+2\sum_{k=1}^{\infty}\frac{3+b}{2(3+2b)}a_k z^k\right\}\\
&= Re\left\{1+\frac{3+b}{3+2b}z+\sum_{k=2}^{\infty}\frac{3+b}{3+2b}a_k z^k\right\}\\
&> Re\left\{1-\frac{3+b}{3+2b}r-\frac{3+b}{3+2b}\sum_{k=2}^{\infty}|a_k|r^k\right\}\\
&> Re\left\{1-\frac{3+b}{3+2b}r-\frac{1}{3+2b}\sum_{k=2}^{\infty}\left(k^2-b+1\right)|a_k|r^k\right\}\\
&> Re\left\{1-\frac{3+b}{3+2b}r-\frac{br}{3+2b}\right\}=1-r>0
\end{aligned}$$

Since ($|z| = r < 1$), therefore we obtain

$$Re\left\{1+2\sum_{k=1}^{\infty}\frac{3+b}{2(3+2b)}a_k z^k\right\}>0,\ z\in U$$

which by Theorem 1.1 shows that $\frac{3+b}{2(3+2b)}a_k$ is a subordinating factor, hence, we have established Equation (2.5).

2.2. Theorem

Given $f(z)\in M(1,b)$, then

$$Ref(z) > -\frac{3+2b}{3+b} \tag{2.6}$$

The constant factor $\frac{3+2b}{3+b}$ cannot be replaced by a larger one.

Proof:

Let

$$g(z)=\frac{z}{1-z}$$

which is a convex function, Equation (2.1) becomes

$$\frac{3+b}{2(3+2b)}f(z)*\frac{z}{1-z}\prec\frac{z}{1-z}$$

Since

$$Re\left(\frac{z}{1-z}\right)>-\frac{1}{2},\ |z|=r \tag{2.7}$$

This implies

$$Re\left\{\frac{3+b}{2(3+2b)}f(z)*\frac{z}{1-z}\right\}>-\frac{1}{2} \tag{2.8}$$

Therefore, we have

$$Re\left(f(z)\right)>-\frac{3+2b}{3+b}$$

which is Equation (2.6).

Now to show that sharpness of the constant factor

$$\frac{3+b}{3+2b}$$

We consider the function

$$f_1(z)=\frac{z(3+b)+bz^2}{3+b} \tag{2.9}$$

Applying Equation (2.1) with $g(z)=\frac{z}{1-z}$ and $f(z)=f_1(z)$, we have

$$\frac{z(3+b)+bz^2}{2(3+b)}\prec\frac{z}{1-z} \tag{2.10}$$

Using the fact that

$$\left|Re(z)\right|\le|z| \tag{2.11}$$

We now show that the

$$\min_{z\in U}\left\{Re\left(\frac{z(3+b)+bz^2}{2(3+b)}\right)\right\}=-\frac{1}{2} \tag{2.12}$$

we have

$$\left|Re\left(\frac{z(3+b)+bz^2}{2(3+b)}\right)\right|\le\left|\frac{z(3+b)+bz^2}{2(3+b)}\right|\le|z|\frac{\left|(3+b)+bz\right|}{\left|2(3+b)\right|}$$
$$\le\frac{\left|(3+b)+bz\right|}{2(3+b)}\le\frac{(3+b)+b}{2(3+2b)}\le\frac{3+2b}{2(3+2b)}=\frac{1}{2},\quad(|z|=1)$$

This implies that

$$\left|Re\left(\frac{z(3+b)+bz^2}{2(3+b)}\right)\right|\le\frac{1}{2}$$

and therefore

$$-\frac{1}{2}\le Re\left(\frac{z(3+b)+bz^2}{2(3+b)}\right)\le\frac{1}{2}$$

Hence, we have that

$$\min_{z\in U}\left\{Re\left(\frac{z(3+b)+bz^2}{2(3+b)}\right)\right\}=-\frac{1}{2}$$

That is

$$\min_{z\in U}\left\{Re\frac{3+b}{2(3+2b)}\left(f_1 * g(z)\right)\right\} = -\frac{1}{2}$$

which shows the Equation (2.12).

2.3. Theorem

Let

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in M(1,b), \quad 0 < b < 0.935449$$

then $|a_k| \le \frac{1}{2}$.

Proof:

Let

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in M(1,b)$$

then by definition of the class $M(I,b)$,

$$\sum_{k=2}^{\infty}\left(k^2 + b - 1\right)|a_k| \le b, \quad 0 < b < 0.935449$$

we have that

$$\frac{k^2 + b - 1}{b} - k > 0$$

which gives that

$$\sum_{k=2}^{\infty} k|a_k| \le \frac{k^2 + b - 1}{b}|a_k| \le 1$$

$$i.e \ \sum_{k=2}^{\infty} k|a_k| \le 1$$

hence

$$2\sum|a_k| \le 1$$

$$|a_k| \le \frac{1}{2}$$

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