Advances in Linear Algebra & Matrix Theory, 2017, 7, 67-71
http://www.scirp.org/journal/alamt
ISSN Online: 2165-3348
ISSN Print: 2165-333X

A Note on the Inclusion Sets for Tensors

Jun He*, **Yanmin Liu, Junkang Tian, Xianghu Liu**

School of Mathematics, Zunyi Normal College, Zunyi, China
Email: *hejunfan1@163.com

How to cite this paper: He, J., Liu, Y.M., Tian, J.K. and Liu, X.H. (2017) A Note on the Inclusion Sets for Tensors. *Advances in Linear Algebra & Matrix Theory*, **7**, 67-71.
https://doi.org/10.4236/alamt.2017.73006

Received: August 30, 2017
Accepted: September 24, 2017
Published: September 27, 2017

Abstract

In this paper, we give a note on the eigenvalue localization sets for tensors. We show that these sets are tighter than those provided by Li *et al.* (2014) [1].

Keywords

Tensor Eigenvalue, Localization Set, Tensor

1. Introduction

Eigenvalue problems of higher order tensors have become an important topic of study in a new applied mathematics branch, numerical multilinear algebra, and they have a wide range of practical applications [2]-[9].

First, we recall some definitions on tensors. Let $\mathbb{R}$ be the real field. An m-th order n dimensional square tensor $\mathcal{A}$ consists of nm entries in $\mathbb{R}$, which is defined as follows:

$$\mathcal{A}=\left(a_{i_1 i_2\cdots i_m}\right),\ a_{i_1 i_2\cdots i_m}\in\mathbb{R},\quad 1\le i_1,i_2,\cdots i_m\le n.$$

To an n-vector x, real or complex, we define the n-vector:

$$\mathcal{A}x^{m-1}=\left(\sum_{i_2,\cdots,i_m=1}^{n} a_{i i_2\cdots i_m}x_{i_2}\cdots x_{i_m}\right)_{1\le i\le n}.$$

and

$$x^{[m-1]}=\left(x_i^{m-1}\right)_{1\le i\le n}.$$

If $\mathcal{A}x^{m-1}=\lambda x^{[m-1]}$, x and λ are all real, then λ is called an H-eigenvalue of $\mathcal{A}$ and x an H-eigenvector of $\mathcal{A}$ associated with λ [10] [11].

Qi [10] generalized Geršgorin eigenvalue inclusion theorem from matrices to real supersymmetric tensors, which can be easily extended to generic tensors; see [1].

Theorem 1. *Let* $\mathcal{A}=\left(a_{i_1 i_2\cdots i_m}\right)$ *be a complex tensor of order* m *dimension*

n. *Then*

$$\sigma(\mathcal{A}) \subseteq \Gamma(\mathcal{A}) = \bigcup_{i \in N} \Gamma_i(\mathcal{A})$$

where $\tau(\mathcal{A})$ is the set of all the eigenvalues of $\mathcal{A}$ and

$$\Gamma_i(\mathcal{A}) = \left\{ z \in \mathbb{C} : \left| z - a_{i \cdots i} \right| \le r_i(\mathcal{A}) \right\},$$

where

$$\delta_{i_1 \cdots i_m} = \begin{cases} 1, & \text{if } i_1 = \cdots = i_m \\ 0, & \text{otherwise,} \end{cases}$$

and

$$r_i(\mathcal{A}) = \sum_{\delta_{i i_2 \cdots i_m} = 0} \left| a_{i i_2 \cdots i_m} \right|.$$

Recently, Li *et al.* [1] obtained the following result, which is also used to identify the positive definiteness of an even-order real supersymmetric tensor.

Theorem 2. *Let* $\mathcal{A} = \left(a_{i_1 i_2 \cdots i_m} \right)$ *be a complex tensor of order* m *dimension* n. *Then*

$$\sigma(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}) = \bigcup_{i, j \in N, i \ne j} \mathcal{K}_{i,j}(\mathcal{A})$$

where $\sigma(\mathcal{A})$ is the set of all the eigenvalues of $\mathcal{A}$ and

$$\mathcal{K}_{i,j}(\mathcal{A}) = \left\{ z \in \mathbb{C} : \left(\left| z - a_{i \cdots i} \right| - r_i^j(\mathcal{A}) \right) \left| z - a_{j \cdots j} \right| \le \left| a_{ij \dots j} \right| r_j(\mathcal{A}) \right\},$$

where

$$r_i^j(\mathcal{A}) = \sum_{\substack{\delta_{i i_2 \cdots i_m} = 0, \\ \delta_{j i_2 \cdots i_m} = 0}} \left| a_{i i_2 \cdots i_m} \right| = r_i(\mathcal{A}) - \left| a_{ij \cdots j} \right|.$$

In this paper, we give some new eigenvalue localization sets for tensors, which are tighter than those provided by Li *et al.* [1].

2. New Eigenvalue Inclusion Sets

Theorem 3. *Let* $\mathcal{A} = \left(a_{i_1 i_2 \cdots i_m} \right)$ *be a complex tensor of order* m *dimension* n. *Then*

$$\sigma(\mathcal{A}) \subseteq \Delta(\mathcal{A}) = \bigcap_{i \in N} \bigcup_{j \in N, j \ne i} \Delta_{i,j}(\mathcal{A})$$

where $\sigma(\mathcal{A})$ is the set of all the eigenvalues of $\mathcal{A}$ and

$$\Delta_{i,j}(\mathcal{A}) = \left\{ z \in \mathbb{C} : \left| z - a_{i \cdots i} \right| \left(\left| z - a_{j \cdots j} \right| - r_j^i(\mathcal{A}) \right) \le \left| a_{ji \cdots i} \right| r_i(\mathcal{A}) \right\},$$

where

$$r_j^i(\mathcal{A}) = \sum_{\substack{\delta_{j i_2 \cdots i_m} = 0, \\ \delta_{i i_2 \cdots i_m} = 0}} \left| a_{j i_2 \cdots i_m} \right| = r_j(\mathcal{A}) - \left| a_{ji \cdots i} \right|.$$

Proof. Let $x = (x_1, \cdots, x_n)^{\mathrm{T}}$ be an eigenvector of $\mathcal{A}$ corresponding to $\lambda(\mathcal{A})$, that is,

$$\mathcal{A} x^{m-1} = \lambda x^{[m-1]}. \tag{1}$$

Let

$$\left|x_p\right| = \max\left\{\left|x_i\right|, i \in N\right\}.$$

Obviously, $\left|x_p\right| > 0$. For any $q \neq p$, from equality (1), we have

$$\begin{aligned}\left|\lambda - a_{p\cdots p}\right|\left|x_p\right|^{m-1} &\le \sum_{\delta_{pi_2\cdots i_m}=0} \left|a_{pi_2\cdots i_m}\right|\left|x_{i_2}\right|\cdots\left|x_{i_m}\right| \\ &\le \sum_{\substack{\delta_{qi_2\cdots i_m}=0,\\ \delta_{pi_2\cdots i_m}=0}} \left|a_{pi_2\cdots i_m}\right|\left|x_{i_2}\right|\cdots\left|x_{i_m}\right| + \left|a_{pq\cdots q}\right|\left|x_q\right|^{m-1} \\ &\le \sum_{\substack{\delta_{qi_2\cdots i_m}=0,\\ \delta_{pi_2\cdots i_m}=0}} \left|a_{pi_2\cdots i_m}\right|\left|x_p\right|^{m-1} + \left|a_{pq\cdots q}\right|\left|x_q\right|^{m-1} \\ &\le r_p^q\left(\mathcal{A}\right)\left|x_p\right|^{m-1} + \left|a_{pq\cdots q}\right|\left|x_q\right|^{m-1}.\end{aligned} \tag{2}$$

That is,

$$\left(\left|\lambda - a_{p\cdots p}\right| - r_p^q\left(\mathcal{A}\right)\right)\left|x_p\right|^{m-1} \le \left|a_{pq\cdots q}\right|\left|x_q\right|^{m-1}. \tag{3}$$

If $\left|x_q\right| = 0$ for all $q \neq p$, then $\left|\lambda - a_{p\cdots p}\right| - r_p^q\left(\mathcal{A}\right) \le 0$, and $\lambda \in \Delta\left(\mathcal{A}\right)$. If $\left|x_q\right| > 0$, from equality (1), we have

$$\left|\lambda - a_{q\cdots q}\right|\left|x_q\right|^{m-1} \le r_q\left(\mathcal{A}\right)\left|x_p\right|^{m-1}. \tag{4}$$

Multiplying inequalities (3) with (4), we have

$$\left|\lambda - a_{q\cdots q}\right|\left(\left|\lambda - a_{p\ldots p}\right| - r_p^q\left(\mathcal{A}\right)\right) \le r_q\left(\mathcal{A}\right)\left|a_{pq\cdots q}\right|, \tag{5}$$

which implies that $\lambda \in \Delta_{p,q}\left(\mathcal{A}\right)$. From the arbitrariness of *q*, we have $\lambda \in \Delta\left(\mathcal{A}\right)$. □

Remark 1. *Obviously, we can get* $\mathcal{K}\left(\mathcal{A}\right) \subseteq \Delta\left(\mathcal{A}\right)$. *That is to say, our new eigenvalue inclusion sets are always tighter than the inclusion sets in Theorem* 2.

Remark 2. *If the tensor* $\mathcal{A}$ *is nonnegative, from* (5), *we can get*

$$\left(\lambda - a_{q\cdots q}\right)\left(\lambda - a_{p\cdots p} - r_p^q\left(\mathcal{A}\right)\right) \le r_q\left(\mathcal{A}\right)a_{pq\cdots q}.$$

Then, we can get,

$$\lambda \le \frac{1}{2}\left\{a_{p\cdots p} + a_{q\cdots q} + r_p^q\left(\mathcal{A}\right) + \Theta_{p,q}^{\frac{1}{2}}\left(\mathcal{A}\right)\right\}$$

where

$$\Theta_{p,q}\left(\mathcal{A}\right) = \left(a_{p\cdots p} - a_{q\cdots q} + r_p^q\left(\mathcal{A}\right)\right)^2 + 4a_{pq\cdots q}r_q\left(\mathcal{A}\right).$$

From the arbitrariness of *q*, we have

$$\lambda \le \max_{i\in N}\min_{j\in N, j\neq i}\frac{1}{2}\left\{a_{j\cdots j} + a_{i\cdots i} + r_j^i\left(\mathcal{A}\right) + \Theta_{j,i}^{\frac{1}{2}}\left(\mathcal{A}\right)\right\}.$$

That is to say, from Theorem 3, we can get another proof of the result in Theorem 13 in [12].

Funds

Jun He is supported by Science and technology Foundation of Guizhou province

(Qian ke he Ji Chu [2016]1161); Guizhou province natural science foundation in China (Qian Jiao He KY [2016]255); The doctoral scientific research foundation of Zunyi Normal College (BS [2015]09); High-level innovative talents of Guizhou Province (Zun Ke He Ren Cai [2017]8). Yan-Min Liu is supported by National Natural Science Foundations of China (71461027); Science and technology talent training object of Guizhou province outstanding youth (Qian ke he ren zi [2015]06); Guizhou province natural science foundation in China (Qian Jiao He KY [2014]295); 2013, 2014 and 2015 Zunyi 15,851 talents elite project funding; Zhunyi innovative talent team(Zunyi KH (2015)38). Tian is supported by Guizhou province natural science foundation in China (Qian Jiao He KY [2015]451); Scienceand technology Foundation of Guizhou province (Qian ke he J zi [2015]2147). Xiang-Hu Liu is supported by Guizhou Province Department of Education Fund KY [2015]391, [2016]046; Guizhou Province Department of Education teaching reform project [2015]337; Guizhou Province Science and technology fund (qian ke he ji chu) [2016]1160.

References

[1] Li, C., Li, Y. and Kong, X. (2014) New Eigenvalue Inclusion Sets for Tensors. *Numerical Linear Algebra with Applications*, **21**, 39-50. https://doi.org/10.1002/nla.1858

[2] Chang, K.C., Pearson, K. and Zhang, T. (2008) Perron-Frobenius Theorem for Nonnegative Tensors. *Communications in Mathematical Sciences*, **6**, 507-520. https://doi.org/10.4310/CMS.2008.v6.n2.a12

[3] Chang, K.C., Pearson, K. and Zhang, T. (2009) On Eigenvalue Problems of Real Symmetric Tensors. *Journal of Mathematical Analysis and Applications*, **350**, 416-422. https://doi.org/10.1016/j.jmaa.2008.09.067

[4] Qi, L. (2007) Eigenvalues and Invariants of Tensor. *Journal of Mathematical Analysis and Applications*, **325**, 1363-1377. https://doi.org/10.1016/j.jmaa.2006.02.071

[5] Qi, L. (2013) Symmetric Nonnegative Tensors and Copositive Tensors. *Linear Algebra and Its Applications*, **439**, 228-238. https://doi.org/10.1016/j.laa.2013.03.015

[6] Ng, M.K., Qi, L. and Zhou, G. (2009) Finding the Largest Eigenvalue of a Non-Negative Tensor. *SIAM Journal on Matrix Analysis and Applications*, **31**, 1090-1099.

[7] Zhang, L., Qi, L. and Zhou, G. (2012) M-Tensors and the Positive Definiteness of a Multivariate Form. Preprint, arXiv:1202.6431.

[8] Ding, W., Qi, L. and Wei, Y. (2013) M-Tensors and Nonsingular M-Tensors. *Linear Algebra and Its Applications*, **439**, 3264-3278. https://doi.org/10.1016/j.laa.2013.08.038

[9] Li, C., Wang, F., Zhao, J.X., Zhu, Y. and Li, Y.T. (2014) Criterions for the Positive Definiteness of Real Supersymmetric Tensors. *Journal of Computational and Applied Mathematics*, **255**, 1-14. https://doi.org/10.1016/j.cam.2013.04.022

[10] Qi, L. (2005) Eigenvalues of a Real Supersymmetric Tensor. *Journal of Symbolic Computation*, **40**, 1302-1324. https://doi.org/10.1016/j.jsc.2005.05.007

[11] Lim, L.H. (2005) Singular Values and Eigenvalues of Tensors: A Variational Approach. *Proceedings of the IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing* (*CAMSAP* 05), Vol. 1, IEEE Computer Society Press, Piscataway, NJ, 129-132.

[12] Li, C., Chen, Z. and Li, Y. (2015) A New Eigenvalue Inclusion Set for Tensors and Its Applications. *Linear Algebra and Its Applications*, **481**, 36-53. https://doi.org/10.1016/j.laa.2015.04.023