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New Result for Strongly Starlike Functions

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Abstract

In this paper, using Salagean differential operator, we define and investigate a new subclass of univalent functions $S_\alpha^n(\beta)$. We also establish a characterization property for functions belonging to the class $S_\alpha^n(\beta)$.

Keywords

Strongly Starlike Functions, Strongly Convex Functions, Salagean Differential Operator

1. Introduction

Let A be the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic in the unit disk $U = \{z \in C : |z| < 1\}$. A function $f(z) \in A$ is said to be starlike of order α if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad 0 \le \alpha < 1 \quad (z \in U) \tag{2}$$

We denote by $S^*(\alpha)$ the subclass of A consisting of functions which are starlike of order α in U.

Also, a function $f(z) \in A$ is said to be convex of order α if and only if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \quad 0 \le \alpha < 1 \quad (z \in U) \tag{3}$$

We denote by $C(\alpha)$ the subclass of A consisting of functions which are convex of order α in U.

If $f(z) \in A$ satisfies

$$\left|\arg\left(\frac{zf'(z)}{f(z)} - \alpha\right)\right| < \frac{\pi\beta}{2}, \quad 0 \le \alpha < 1, \quad 0 < \beta \le 1, \quad (z \in U) \tag{4}$$

then $f(z)$ is said to be strongly starlike of order β and type α in U, denoted by [1].

If $f(z) \in A$ satisfies

$$\left|\arg\left(1+\frac{zf''(z)}{f'(z)}-\alpha\right)\right| < \frac{\pi\beta}{2}, \quad 0 \le \alpha < 1, \quad 0 < \beta \le 1, \quad (z \in U) \tag{5}$$

then $f(z)$ is said to be strongly convex of order β and type α in U, denoted by $C_\alpha(\beta)$ [1].

The following lemma is needed to derive our result for class $S_\alpha^n(\beta)$.

Lemma (1) [2] [3] [4] [5]. Let a function $p(z)$ be analytic in U, $p(0)=1$ and $p(z) \ne 0\,(z \in U)$, if there exists a point $z_0 \in U$ such that $\left|\arg(p(z))\right| < \frac{\pi\beta}{2}$ $(|z| < |z_0|)$ and $\left|\arg(p(z_0))\right| = \frac{\pi\beta}{2}$ with $0 < \beta \le 1$, then

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\beta \tag{6}$$

where

$$k \ge \frac{1}{2}\left(a+\frac{1}{a}\right) \quad \left(\text{when } \arg(p(z_0)) = \frac{\pi\beta}{2}\right)$$

$$k \le -\frac{1}{2}\left(a+\frac{1}{a}\right) \quad \left(\text{when } \arg(p(z_0)) = -\frac{\pi\beta}{2}\right)$$

And $p(z_0)^{\frac{1}{\beta}} = \pm ia \ (a > 0)$.

Definition 1. A function $f(z) \in A$ is said to be in the class $S_\alpha^n(\beta)$ if

$$\left|\arg\left(\frac{D^{n+1}f(z)}{D^n f(z)}-\alpha\right)\right| < \frac{\pi\beta}{2}, \quad (z \in U) \tag{7}$$

For some $\alpha,\ 0 \le \alpha < 1,\ n \in N_0 = N \cup \{0\}\ 0 < \beta \le 1$.

Remark

When $n = 0$ then $S_\alpha^n(\beta)$ is the class studied by [1].

Definition 2. For functions $f(z) \in A$ the Salagean differential operator [6] is $D^n : A \to A$

$$D^0 f(z) = f(z),\ D^1 f(z) = zf'(z), \cdots D^n f(z) = D\left[D^{n-1} f(z)\right],\ n = 0,1,2,3,\cdots$$

The main focus of this work is to provide a characterization property for the class of functions belonging to the class $S_\alpha^n(\beta)$.

2. Main Result

Theorem 1. If $f(z) \in A$ satisfies

$$(i)\ \frac{D^{n+1}f(z)}{D^n f(z)} \ne \frac{1}{2}$$

$$(ii)\ \left|\frac{D^{n+2}f(z)/D^{n+1}f(z)}{D^{n+1}f(z)/D^n f(z)} - 1\right| < \frac{\beta}{2}, (z \in U)$$

for some $\beta,\ 0 < \beta \le 1,\ n \in N_0 = N \cup \{0\}$, then $f(z) \in S_{\frac{1}{2}}^n(\beta)$

Proof. Let

$$p(z) = 2\frac{D^{n+1}f(z)}{D^n f(z)} - 1, \quad n \in N_0 n = 0,1,2,\cdots \tag{8}$$

Taking the logarithmic differentiation in both sides of Equation (8), we have

$$\begin{aligned}
\frac{p'(z)}{p(z)} &= \left[\frac{D^n f(z)2\left(D^{n+1}f(z)\right)' - 2D^{n+1}f(z)\left[D^n f(z)\right]'}{\left[D^n f(z)\right]^2}\right]\left[\frac{D^n f(z)}{2D^{n+1}f(z) - D^n f(z)}\right] \\
&= \left[\frac{D^n f(z)2\left(D^{n+1}f(z)\right)' - 2D^{n+1}f(z)\left[D^n f(z)\right]'}{D^n f(z)}\right]\left[\frac{1}{2D^{n+1}f(z) - D^n f(z)}\right] \qquad (9) \\
&= \frac{2\left(D^{n+1}f(z)\right)'}{D^n f(z)p(z)} - \frac{2D^{n+1}f(z)\left[D^n f(z)\right]'}{\left[D^n f(z)\right]^2 p(z)}
\end{aligned}$$

Multiply Equation (9) through by $p(z)$, to get

$$p'(z) = \frac{2\left(D^{n+1}f(z)\right)'}{D^n f(z)} - \frac{2D^{n+1}f(z)\left(D^n f(z)\right)'}{\left(D^n f(z)\right)^2} \tag{10}$$

Multiply Equation (10) by z to obtain

$$\begin{aligned}
zp'(z) &= \frac{2z\left(D^{n+1}f(z)\right)'}{D^n f(z)} - \frac{2D^{n+1}f(z)z\left(D^n f(z)\right)'}{\left(D^n f(z)\right)^2} \\
&= \frac{2\left(D^{n+2}f(z)\right)}{D^n f(z)} - \frac{\left(1+p(z)\right)^2}{2}
\end{aligned} \tag{11}$$

Multiply Equation (11) through by 2 and divide through by $\left(1+p(z)\right)^2$ to give

$$\frac{2zp'(z)}{\left(1+p(z)\right)^2} = \frac{4\left(D^{n+2}f(z)\right)}{D^n f(z)\left(1+p(z)\right)^2} - 1 \tag{12}$$

Multiplying Equation (12) by $\frac{D^{n+1}f(z)}{D^n f(z)} = \frac{1+p(z)}{2}$, and further simplification, we obtain

$$\frac{D^{n+1}f(z)}{D^n f(z)}\left(1 + \frac{2zp'(z)}{\left(1+p(z)\right)^2}\right) = \frac{D^{n+2}f(z)}{D^{n+1}f(z)}, \ z \in U, \ n \in N_0 \tag{13}$$

therefore,

$$\frac{D^{n+2}f(z)/D^{n+1}f(z)}{D^{n+1}f(z)/D^n f(z)} = 1 + \frac{2zp'(z)}{\left(1+p(z)\right)^2} \tag{14}$$

If $\exists$ a point $z_0 \in U$ which satisfies $\left|\arg p(z)\right| < \frac{\pi\beta}{2}$ $\left(|z| < |z_0|\right)$ and $\left|\arg p(z_0)\right| = \frac{\pi\beta}{2}$

then by lemma [2]

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\beta$$

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \quad \text{and} \quad p(z_0) = a^{\beta} e^{\frac{i\pi\beta}{2}} \quad \text{or} \quad p(z_0) = a^{\beta} e^{\frac{-i\beta}{2}} \quad (a > 0)$$

Now,

$$\left| \frac{D^{n+2} f(z_0) / D^{n+1} f(z_0)}{D^{n+1} f(z_0) D^n f(z_0)} - 1 \right| = 2k\beta \left| \frac{p(z_0)}{\left(1 + p(z_0)\right)^2} \right|$$

$$\geq \frac{2\beta \frac{1}{2}\left(a + \frac{1}{a}\right) \left| p(z_0) \right|}{\left| \left(1 + p(z_0)\right)^2 \right|} \tag{15}$$

Since,

$$\frac{1}{\left| \left(1 + p(z_0)\right)^2 \right|} \geq \frac{1}{1 + 2\left| p(z_0) \right| + \left| p(z_0)^2 \right|} \tag{16}$$

$$\left| \frac{D^{n+2} f(z_0) / D^{n+1} f(z_0)}{D^{n+1} f(z_0) / D^n f(z_0)} - 1 \right| \geq \frac{\beta\left(a + \frac{1}{a}\right) \left| p(z_0) \right|}{1 + 2\left| p(z_0) \right| + \left| p(z_0) \right|^2} \tag{17}$$

$$p(z_0) = a^{\beta} e^{\frac{i\pi\beta}{2}}, \quad a > 0 \Rightarrow \left| p(z_0) \right| = a^{\beta}$$

But

$$= \frac{\beta\left(a + \frac{1}{a}\right) a^{\beta}}{1 + 2a^{\beta} + a^{2\beta}}$$

$$= \frac{\left(a + \frac{1}{a}\right) \beta}{a^{-\beta} + 2 + a^{\beta}}$$

Let

$$S(a) = \frac{a + \frac{1}{a}}{a^{-\beta} + 2 + a^{\beta}}$$

then

$$S'(a) = \frac{2\left(a^2 - 1\right) + (1 - \beta) a^{-\beta}\left(a^{2(1+\beta)} - 1\right) + (1 + \beta) a^{\beta}\left(a^{2(1-\beta)} - 1\right)}{a^2\left(a^{\beta} + 2 + a^{-\beta}\right)^2} \tag{18}$$

Hence, $S'(a) = 0 \Rightarrow a = 1$.

It implies that

$S'(a) < 0$ when $0 < a < 1$ and $S'(a) > 0$ when $a > 1$, hence, $a = 1$ is a minimum point of $S(a) \cdot S(1) = \frac{1}{2}$.

Therefore, we have that

$$\left| \frac{D^{n+2} f(z) f(z_0) / D^{n+1} f(z_0)}{D^{n+1} f(z_0) / D^n f(z_0)} - 1 \right| \geq \frac{\beta}{2}, \; n \in N_0, \; z \in U \tag{19}$$

which contradicts the condition of the theorem.

Hence, it is concluded from lemma [2] that

$$|\arg p(z)| = \left|\arg\left(\frac{D^{n+1}f(z)}{D^n f(z)} - \frac{1}{2}\right)\right| < \frac{\pi\beta}{2},\ z \in U,\ n \in N_0 \tag{20}$$

so that

$$f(z) \in S^n_{\frac{1}{2}}(\beta).$$

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