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Besicovitch-Eggleston Function

Manav Das
Department of Mathematics, University of Louisville, Louisville, USA
E-mail: manav@louisville.edu
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Abstract

In this work we introduce a function based on the well-known Besicovitch-Eggleston sets, and prove that the Hausdorff dimension of its graph is 2.

Keywords: Hausdorff Dimension, Multifractal, Binomial Measure, Dyadic Intervals

1. Introduction

Let $x \in [0,1]$, and let $x = .x_1 x_2, \cdots, x_i = 0 \text{ or } 1$ denote its binary expansion. For any $0 < p < 1/2$ we may consider the set

$$K_p = \left\{ x \in [0,1] : \limsup_{n\to\infty} \frac{x_1 + x_2 + \cdots + x_n}{n} \le p \right\}$$

Besicovitch [2] proved that

$$\dim_H K_p = \frac{-p \log p - (1-p)\log(1-p)}{\log 2}$$

where $\dim_H A$ denotes the Hausdorff dimension of the set A. This result was generalized to the N-ary case by Eggleston [8]. Billingsley proved a more general version of this result in the context of probability spaces [3]. Billingsley's result was related to densities in [5], and a similar result involving packing dimensions was proved in [6]. Sets such as K_p are studied in the context of multifractal theory (see [1,7,9,11,14-16]) and Billingsley-type results have been proved by several authors in this context. Recently, such a result has been proved for a countable symbol space in [13].

In this paper, we are interested in a natural function that may be defined using Besicovitch's result. We call this the Besicovitch-Eggleston function: let $N[1,n,x] = \#$ of 1's in the first n digits of the dyadic expansion for x. Define

$$f(x) = \begin{cases} \lim_{n\to\infty} \dfrac{N[1,n,x]}{n} & \text{if it exists} \\ 1 & \text{otherwise} \end{cases}$$

This function allows us to visualize the multifractal components of $[0,1]$ as level sets. If we let μ_p denote the invariant multifractal measure on K_p then it is clear that $\int f(y)\,\mathrm{d}\mu_p(y) = p.$

In the next section, we state and prove our main result, and finally we close with some open problems.

2. Main Result

We will need the following result by Besicovitch and Moran. This is not the form in which it was originally stated and proved. However, this modern version may be found in [12].

Theorem 1. For any $s \in (0,1)$ there exists a constant $b_s > 0$ such that for all Borel sets $E \subseteq \mathbf{R}^2$ we must have

$$\mathcal{H}^{1+s}(E) \ge b_s \int \mathcal{H}^s(E_y)\,\mathrm{d}y$$

where $E_y = \{x : (x,y) \in E\}$.

We are now ready to state the main result:

Theorem 2. *Let* $B = Graph(f) \subseteq [0,1]\times[0,1]$. *Then* $\dim_H B = 2$ *and* $\mathcal{H}^2(B) = 0$.

Proof. The upper bound is obvious while the lower bound follows from Theorem 1. Fix $s \in (0,1)$ and choose p such that $\dfrac{p\log p + q\log q}{-\log 2} > s.$ Therefore $\mathcal{H}^s(f^{-1}(\{p\})) = \infty$. We can choose an interval I_s containing p such that $\mathcal{H}^s(B_y) = \infty$ for every $y \in I_s$. It follows from Theorem 1 that $\mathcal{H}^{1+s}(B) = \infty$. Observing that $s < 1$ was arbitrary gives us the lower bound. Moreover, since every vertical line meets B exactly once, Fubini's theorem tells us that in fact $\mathcal{H}^2(B) = 0$.

3. Concluding Remarks

Here we pose some problems related to the Besicovitch-Eggleston function.

1) Can one find the precise scaling function ϕ such that $0 < \mathcal{H}^{\phi}(B) < \infty$? Is the set $[0,1]\times[0,1]\setminus B$ immesaurable, that is, either null or non-σ-finite for every translation invariant Borel measure on $\mathbf{R}^2$? See [10], where it is shown that the set $\mathbf{R}\setminus\bigcup_{0<p<1/2}K_p$ is immeasurable.

2) What is the relationship between $f(x)$ and $f^2(x)$?

3) How large is the set of points x such that $x = f(x)$? Can we characterize this set of fixed points?

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