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A Common Fixed Point Theorem for Compatible Mappings of Type (C)

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Abstract

We establish a common fixed-point theorem for six self maps under the compatible mappings of type (C) with a contractive condition [1], which is independent of earlier contractive conditions.

Keywords: Fixed Point, Compatible Mappings of Type (C), Complete Metric Space

1. Introduction

The study of common fixed point of mappings satisfying contractive type conditions has been a very active field of research activity during the last two decades. Researchers like R. P. Pant *et al.* [2,3] have shown that how the three types of contractive conditions (Banach, Meir keeler and contractive gauge function/φ contractive condition) hold simultaneously or independent of each other and as a result of this study they have proved a fixed point theorem using Lipschitz type contractive condition [3] and gauge function [2].

In this paper we generalize the result of K. Jha, R. P. Pant, S. L. Singh [1] and prove a fixed point theorem for six self mappings in a complete metric space.

$$d(Ax,By) \le c\lambda(x,y), \quad 0 \le c < 1 \tag{1.1}$$

where,

$$d(y_n, y_{n+1}) d(y, y) \lambda(x,y)$$
$$= \max\left\{k_1\left[d(Sx,Ty)+d(Ax,Sx)\right.\right.$$
$$\left.\left.+d(By,Ty), \frac{1}{2}\left[d(Sx,By)+d(Ax,Ty)\right]\right]\right\}$$

or a Meir-Keeler type (ε,δ)—contractive condition of the form, given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\varepsilon \le \lambda(x,y) < \varepsilon+\delta \ \text{ implies } \ d(Ax,By) < \varepsilon \tag{1.2}$$

or, a φ-contractive condition of the form

$$d(Ax,By) \le \varphi(\lambda(x,y)) \tag{1.3}$$

involving a contractive function φ: $R_+ \to R_+$ is such that $\varphi(t) < t$ for each $t > 0$. Clearly, condition (1.1) is a special case of both conditions (1.2) and (1.3). Pant *et al.* [2] have shown the two type of contractive condition (1.2) and (1.3) are independent. The contractive conditions (1.2) and (1.3) hold simultaneously whenever (1.2) or (1.3) is assumed with additional conditions on δ and φrespectively. It follows, therefore, that the known common fixed point theorems can be extended and generalized if instead of assuming one of the contractive condition (1.2) or (1.3) with additional conditions on δ and φ. we assume contractive condition [2] which is condition (1.2) together with the following condition of the form

$$d(Ax,By) < \max\left\{k_1[d(Sx,Ty)+d(Ax,Sx)\right.$$
$$\left.+d(By,Ty), \frac{k_2}{2}\left[d(Sx,By)+d(Ax,Ty)\right]\right\} \tag{1.4}$$

for $0 \le k_1 < 1$, $1 \le k_2 < 2$,

instead of assuming one of the contractive conditions (1.2) or (1.3) with additional conditions on δ and φ.

Definition: Two self mappings A and S of a metric space (X, d) are said to be compatible (see Jungck [4]) if, $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$ whenever $\langle x_n \rangle$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

Definition: Two self mappings *A* and *S* of a metric space (*X*, *d*) are said to be compatible mappings of type (A) (See [5]) if $\lim_{n\to\infty} d(ASx_n, SSx_n) = 0$ and $\lim_{n\to\infty} d(SAx_n, AAx_n) = 0$ whenever $\langle x_n \rangle$ is a sequence

in X such that
$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

Definition: Two self mappings A and S of a metric space (X,d) are said to be compatible mappings of type (B) (See[6]) if,

$$\lim_{n\to\infty} d\left(ASx_n, SSx_n\right) \le \frac{1}{2}\left[\lim_{n\to\infty} d\left(ASx_n, At\right) + \lim_{n\to\infty} d\left(At, AAx_n\right)\right]$$

and

$$\lim_{n\to\infty} d\left(SAx_n, AAx_n\right) \le \frac{1}{2}\left[\lim_{n\to\infty} d\left(SAx_n, St\right) + \lim_{n\to\infty} d\left(St, SSx_n\right)\right]$$

whenever $\langle x_n \rangle$ is a sequence in X such that
$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

Definition: Two self mappings A and S of a metric space (X,d) are said to be compatible mappings of type (C) (see [7]) if,

$$\lim_{n\to\infty} d\left(ASx_n, SSx_n\right) \le \frac{1}{3}\left[\lim_{n\to\infty} d\left(ASx_n, At\right) + \lim_{n\to\infty} d\left(At, AAx_n\right) + \lim_{n\to\infty} d\left(At, SSx_n\right)\right]$$

and

$$\lim_{n\to\infty} d\left(SAx_n, AAx_n\right) \le \frac{1}{3}\left[\lim_{n\to\infty} d\left(SAx_n, St\right) + \lim_{n\to\infty} d\left(St, SSx_n\right) + \lim_{n\to\infty} d\left(St, AAx_n\right)\right]$$

whenever $\langle x_n \rangle$ is a sequence in X such that
$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

Definition: Two self mappings A and S of a metric space (X,d) are said to be compatible mappings of type (P) (see [8]), if $\lim_{n\to\infty} d\left(SSx_n, AAx_n\right) = 0$ whenever $\langle x_n \rangle$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$. From the propositions given in [4-8] all compatibility conditions are equivalent when A and S are continuous. We observe that they are independent if the functions are discontinuous.

We give an example which is compatible mapping of type (C) but is neither compatible nor compatible mapping of type (A), compatible mapping of type (B) and compatible mapping of type (P).

Example: Let X = [1,10] with $d\left(x,y\right) = |x-y|$ Define self maps S and A of X by

$$Sx = \begin{cases} 1 & if\ x = 1 \\ 3 & if\ 1 < x \le 5 \\ x-4 & if\ 5 < x \le 10 \end{cases} \text{ and } Ax = \begin{cases} 1 & if\ x \in \{1\} \cup (5,10] \\ 1 & if\ 1 < x \le 5 \end{cases}$$

Let $x_n = 5 + \frac{1}{n}$ for $n \ge 1$ be a sequence in X. Hence for such a sequence $\langle x_n \rangle$ both Sx_n, Ax_n converge to 1 as $n \to \infty$.

Let t = 1. Now, $SAx_n \to 1$, $ASx_n \to 2$, $SSx_n \to 3$, $AAx_n \to 1$ as $n \to \infty$. The pair (S, A) is not compatible, compatible of type (A), compatible of type (B), compatible of type (P) but is only compatible of type (C).

2. K. Jha, R. P. Pant and S. L. Singh [1] Proved the Following Common Fixed Point.

2.1. Theorem

Let (A, S) and (B, T) be compatible pairs of self mappings of a complete metric space (X, d) such that

$$AX \subset TX \quad \text{and} \quad BX \subset SX \qquad (2.1.1)$$

given $\varepsilon > 0$ there exist $\delta > 0$ such that for all $x, y \in X$ $\varepsilon \le \lambda\left(x,y\right) < \varepsilon + \delta$ implies

$$d\left(Ax, By\right) < \varepsilon \qquad (2.1.2)$$

$$d\left(Ax, By\right) < \max\left\{k_1[d\left(Sx, Ty\right) + d\left(Ax, Sx\right) + d\left(By, Ty\right), \frac{k_2}{2}\left[d\left(Sx, By\right) + d\left(Ax, Ty\right)\right]\right\} \qquad (2.1.3)$$

for $0 \le k_1 < 1$, $1 \le k_2 < 2$.

If one of the mappings A, B, S and T is continuous then A, B, S and T have a unique common fixed point.

We generalise this theorem by extending four self maps to six self maps and replacing the condition of compatibility of self maps by the compatible mapping of type (C).

To prove our theorem we shall use the following lemma.

2.2. Lemma

Let A, B, S, T, L and M be self mappings of (X,d) such that

$$L\left(X\right) \subset ST\left(X\right), M\left(X\right) \subset AB\left(X\right). \qquad (2.2.1)$$

Assume further that given $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x, y \in X$

$$\varepsilon \le M\left(x,y\right) < \varepsilon + \delta \text{ implies } d\left(Lx, My\right) < \varepsilon \qquad (2.2.2)$$

where

$$M\left(x,y\right) = \max\left\{d\left(ABx, STy\right) + d\left(Lx, ABx\right) + d\left(My, STy\right), \frac{1}{2}\left[d\left(ABx, My\right) + d\left(Lx, STy\right)\right]\right\} \qquad (2.2.3)$$

If $x_0 \in X$ and the sequence $\{y_n\}$ in X defined by the rule

$$y_{2n-1} = STx_{2n-1} = Lx_{2n-2} \quad (2.2.4)$$

and

$$y_{2n} = ABx_{2n} = Mx_{2n-1}$$

for $n = 1,2,3\cdots$

Then we have the following

for every $\varepsilon > 0$, $\varepsilon \le d(y_p, y_q) < \delta + \varepsilon$ implies $d(y_{p+1}, y_{q+1}) < \varepsilon$ (2.2.5)

where p and q are of opposite parity.

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0 \quad (2.2.6)$$

$\{y_n\}$ is a cauchy sequence in *X*. (2.2.7)

Proof: Since from (2.2.2) for every $\varepsilon > 0$

$$\varepsilon \le \max\left\{d(ABx, STy) + d(Lx, ABx) + d(My, STy), \frac{1}{2}\left[d(ABx, My) + d(Lx, STy)\right]\right\} < \delta + \varepsilon$$

implies $d(Lx, My) < \varepsilon$ for all $x, y \in X$ suppose that $\varepsilon \le d(y_p, y_q) < \delta + \varepsilon$.

Putting $p = 2n$ and $q = 2m-1$ in the above inequality, we have

$$d(y_{p+1}, y_{q+1}) = d(y_{2n+1}, y_{2m}) = d(Lx_{2n}, Mx_{2m-1})$$

and

$$\varepsilon \le d(y_p, y_q) = d(y_{2n}, y_{2m-1}) = d(ABx_{2n}, STx_{2m-1})$$

$$\varepsilon \le \max\Big\{d(ABx_{2n}, STx_{2n-1}) + d(Lx_{2n}, ABx_{2n}) + d(Mx_{2n-1}, STx_{2n-1}), \frac{1}{2}\Big[d(ABx_{2n}, Mx_{2n-1}) + d(Lx_{2n}, STx_{2n-1})\Big]\Big\} < \delta + \varepsilon$$

which implies that

$$d(y_{p+1}, y_{q+1}) = d(Lx_{2n}, Mx_{2m-1}) < \varepsilon$$

Now, for $x_0 \in X$, by (2.2.3), we have

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &= d(y_{2n+1}, y_{2n}) = d(Lx_{2n}, Mx_{2n-1}) \\ &\le \max\Big\{d(ABx_{2n}, STx_{2n-1}) + d(Lx_{2n}, ABx_{2n}) \\ &\quad + d(Mx_{2n-1}, STx_{2n-1}), \frac{1}{2}\Big[d(ABx_{2n}, Mx_{2n-1}) \\ &\quad + d(Lx_{2n}, STx_{2n-1})\Big]\Big\} \qquad (2.2.8) \\ &= \max\left\{d(y_{2n}, y_{2n-1}) + d(y_{2n}, y_{2n+1}) + d(y_{2n-1}, y_{2n}), \frac{1}{2}\left[d(y_{2n}, y_{2n}) + d(y_{2n-1}, y_{2n+1})\right]\right\} \\ &= d(y_{2n-1}, y_{2n}) \end{aligned}$$

Similarly, we have $d(y_{2n+1}, y_{2n+2}) < d(y_{2n}, y_{2n+1})$.

Thus the sequence $\{d(y_n, y_{n+1})\}$ is non increasing and converges to the greatest lower bound of its range $t \ge 0$. Now we prove that $t = 0$

If $t \ne 0$, (2.2.2) implies that $d(y_{m+1}, y_{m+2}) < t$ whenever $t \le d(y_m, y_{m+1}) < \delta(t)$.

But since $\{d(y_m, y_{m+1})\}$ converges to *t*, there exists a *k* such that $\{d(y_m, y_{m+1})\} < \delta(t)$ so that $t \le d(y_k, y_{k+1}) < \delta(t)$ which by (2.2.5) implies $d(y_{k+1}, y_{k+2}) < t$, which contradicts the infrimum nature of *t*.

Therefore, we have $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$.

We shall prove that $\{y_n\}$ is a cauchy sequence in X. In virtue of (2.2.6), it is sufficient to show that $\{y_{2n}\}$ is a cauchy sequence.

Suppose that $\{y_{2n}\}$ is not a cauchy sequence. Then there is an $\varepsilon > 0$ such that for each integer 2*k*, there exists even integers 2*m*(*k*) and 2*n*(*k*) with 2*m*(*k*) > 2*n*(*k*) ≥ 2*k* such that

$$d\left(y_{2m(k)}, y_{2n(k)}\right) > \varepsilon \quad (2.2.9)$$

For each even integer 2*k*, let 2*m*(*k*) be the least even integer exceeding 2*n*(*k*) satisfying (2.2.9), that is

$$d\left(y_{2n(k)}, y_{2n(k)-2}\right) \le \varepsilon \quad (2.2.10)$$

and

$$d\left(y_{2n(k)}, y_{2m(k)}\right) > \varepsilon.$$

Then for each even integer 2*k*, we have

$$\begin{aligned} \varepsilon \le d\left(y_{2n(k)}, y_{2m(k)}\right) &\le d\left(y_{2n(k)}, y_{2n(k)-2}\right) \\ &+ d\left(y_{2n(k)-2}, y_{2n(k)-1}\right) + d\left(y_{2n(k)-1}, y_{2n(k)}\right) \end{aligned}$$

From (2.2.6) and (2.2.10), it follows that

$\varepsilon < d\left(y_{2m(k)}, y_{2n(k)}\right) \le \varepsilon$ from which , we have

$$d\left(y_{2m(k)}, y_{2n(k)}\right) \to\to \varepsilon \text{ as } k \to \infty. \quad (2.2.11)$$

From the triangle inequality, we have

$$\begin{aligned} &\left|d\left(y_{2n(k)}, y_{2m(k)-1}\right) - d\left(y_{2n(k)}, y_{2m(k)}\right)\right| \\ &\le d\left(y_{2m(k)-1}, y_{2m(k)}\right) \\ &\le d\left(y_{2m(k)-1}, y_{2m(k)}\right) + d\left(y_{2n(k)}, y_{2n(k)-1}\right) \end{aligned}$$

From (2.2.6) and (2.2.10), as $k \to \infty$

$$d\left(y_{2n(k)}, y_{2m(k)-1}\right) \to \varepsilon \quad (2.2.12)$$

and $d\left(y_{2m(k)-1}, y_{2m(k)-1}\right) \to \varepsilon$

Therefore by (2.2.2) and (2.2.4), we have

$$d\left(y_{2m(k)}, y_{2n(k)}\right) \le d\left(y_{2n(k)}, y_{2m(k)-1}\right) + d\left(y_{2n(k)-1}, y_{2m(k)}\right) \le d\left(y_{2n(k)}, y_{2n(k)-1}\right) + d\left(Lx_{2n(k)}, Mx_{2m(k)-1}\right) \quad (2.2.13)$$

(Since by (2.2.5) and $d\left(y_{p+1}, y_{q+1}\right) = d\left(Lx_{2n}, Mx_{2m-1}\right) < \varepsilon$ we have

$$d\left(y_{2m(k)}, y_{2n(k)}\right) \le d\left(y_{2n(k)}, y_{2n(k)-1}\right) + \varepsilon$$

From (2.2.5), (2.2.6) and (2.2.12) as $k \to \infty$, we get $\varepsilon < \varepsilon$, which is a contradiction. Therefore, $\{y_{2n}\}$ is a cauchy sequence in *X* and so is $\{y_n\}$.

2.3. Main Theorem

Let *A*, *B*, *S*, *T*, *L* and *M* be self mappings of a complete metric space (*X*, *d*) satisfying (2.3.1)

$$L(X) \subset ST(X), M(X) \subset AB(X) \quad (2.3.2)$$

given $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x, y \in X$ (2.3.3)

$$\varepsilon \le M(x,y) < \varepsilon + \delta \text{ implies } d(Lx, Ly) \le \varepsilon$$

where $M(x,y)$ is defined as in (5.2.3)

$$d(Lx, My) < \max\left\{k_1\left[d(ABx, STy) + d(Lx, ABx) + d(My, STy)\right], \frac{k_2}{2}\left[d(ABx, My) + d(Lx, STy)\right]\right\} \quad (2.3.4)$$

for $0 \le k_1, k_2 < 1$.

The pair (*L*, *AB*) and (*M*, *ST*) be compatible mappings of type (C) (2.3.5)

AB(*X*) is complete one of the mappings *AB*,*ST*,*L* and *M* is continuous. (2.3.6)

Then *AB*, *ST*, *L* and *M* have a unique common fixed point.

Further if the pairs (*A*, *B*), (*A*, *L*), (*B*, *L*), (*S*, *T*), (*S*, *M*) and (*T*, *M*) are commuting mappings then *A*, *B*, *S*, *T*, *L* and *M* have a unique common fixed point.

Proof: Let x_0 be any point in *X*. Define sequences x_n and y_n in *X* given by the rule

(2.3.7) $y_{2n} = Lx_{2n} = STx_{2n+1}$ and $y_{2n+1} = Mx_{2n+1} = ABx_{2n+2}$ for $n = 0,1,2,\cdots$

This can be done by virtue of (2.3.2). since the contractive condition (2.3.3) of the theorem implies the contractive condition (2.2.2) and (2.2.3) of the lemma 2.2.1 so by using the lemma 2.2.1 we conclude that $\{y_n\}$ is a Cauchy sequence in *X*, but by (2.3.6) AB(X) is complete, it converges to a point $z = AB$u for some u in *X*.

Hence $\{y_n\} \to z \in X$.

Also its subsequences converge as follows $\{Mx_{2n+1}\} \to z$ and $\{STx_{2n+1}\} \to z$; $\{Lx_{2n}\} \to z$ and $\{ABx_{2n+2}\} \to z$ as $n \to \infty$. (2.3.8)

Now we will prove the theorem by different cases .

Case (i): AB is continuous then from (2.3.8) we have $ABABx_{2n+2}$ and $ABLx_{2n}$ converges ABz as $n \to \infty$. (2.3.9)

Since (*AB*, *L*) are compatible mappings of type (C), we have from (2.3.9),

$$\begin{aligned}\lim_{n\to\infty} d(LLx_{2n}, ABz) &= \lim_{n\to\infty} d(LLx_{2n}, ABLx_{2n}) \\ &\le \frac{1}{3}\lim_{n\to\infty} d(ABLx_{2n}, ABz) + \lim_{n\to\infty} d(ABz, ABABx_{2n}) + \lim_{n\to\infty} d(ABz, LLx_{2n}) \quad (2.3.10) \\ &\le \frac{1}{3}\Big[d(ABz, ABz) + d(ABz, ABz) + \lim_{n\to\infty} d(ABz, LLx_{2n})\Big] \le \frac{1}{3}\lim_{n\to\infty} d(ABz, LLx_{2n})\end{aligned}$$

which shows LLx_{2n} converges to *ABz* as $n \to \infty$.

Now, we show that z is the fixed point of α *AB*.

In view of (2.3.10), (2.3.8), (2.3.4) and (2.3.9)

$$\begin{aligned}d(ABz, z) &= \lim d(LLx_{2n}, Mx_{2n+1}) \\ &< \lim_{n\to\infty} \max\Big\{k_1\big[d(ABLx_{2n}, STx_{2n+1}) + d(LLx_{2n}, ABLx_{2n}) + d(Mx_{2n+1}, STx_{2n+1})\big], \\ &\qquad \frac{k_2}{2}\big[d(ABLx_{2n}, Mx_{2n+1}) + d(LLx_{2n}, STx_{2n+1})\big]\Big\} \quad (2.3.11) \\ &< \max\Big\{k_1\big[d(ABz, z) + d(ABz, z) + d(z, z)\big], \frac{k_2}{2}\big[d(ABz, z) + d(AB, z)\big]\Big\}\end{aligned}$$

$< d(ABz, z)$ a contradiction if $ABz \ne z$ yielding therefore $(\alpha AB)z = z$.

Now, we show that z is also a fixed point of αL.

In view of (2.3.8), (2.3.4) and (2.3.11)

$$\begin{aligned}d(Lz, z) &= \lim_{n\to\infty} d(Lz, Mx_{2n+1}) \\ &< \lim_{n\to\infty} \max\Big\{k_1\big[d(ABz, STx_{2n+1}) + d(Lz, ABz) + d(Mx_{2n+1}, STx_{2n+1})\big], \frac{k_2}{2}\big[d(ABz, Mx_{2n+1}) + d(Lz, STx_{2n+1})\big]\Big\} \quad (2.3.12) \\ &< \max\Big\{k_1\big[d(z,z) + d(Lz, z) + d(z,z)\big], \frac{k_1}{2}\big[d(z,z) + d(L, z)\big]\Big\}\end{aligned}$$

$< d(Lz,z)$, a contradiction if $Lz \neq z$

Implying there by $Lz = z$.

Thus $ABz = Lz = z$.

Since $L(X) \subseteq ST(X)$ there exist $y \in X$ such that $Z = LZ = STy$.

we prove $STy = My$.

In view of (2.3.12) and (2.3.4)

$$\begin{aligned} d(STy,My) &= d(Lz,My) \\ &< \max\Big\{k_1\big[d(ABz,STy)+d(Lz,ABz) \\ &\quad +d(My,STy)\big], \\ &\quad \frac{k_2}{2}\big[d(ABz,My)+d(Lz,STy)\big]\Big\} \qquad (2.3.13) \\ &< \max\Big\{k_1\big[d(STy,STy)+d(z,z) \\ &\quad +d(My,STy)\big] \\ &\quad \frac{k_2}{2}\big[d(STy,My)+d(z,z)\big]\Big\} \end{aligned}$$

$< d(STy,My)$ a contradiction if $STy \neq My$

Therefore $My = STy$.

Hence we have $My = Lz = ABz = z = STy$

Now, taking a sequence $\{z_n\}$ in *X* such that $z_n = y$ $\forall n \geq 1$, it follows that

$$Mz_n \to My = z \quad \text{and} \quad STz_n \to STy = z \quad \text{as} \quad n \to \infty.$$

since the pair (M, ST) is compatible of type (C)

$$\begin{aligned} \lim_{n\to\infty} d(STMz_n,MMz_n) &\leq \frac{1}{3}\Big[\lim_{n\to\infty} d(STMz_n,STz) \\ &\quad +\lim_{n\to\infty} d(STz,MMz_n)+\lim_{n\to\infty} d(STz,STSTz_n)\Big] \end{aligned} \qquad (2.3.14)$$

That is

$$\begin{aligned} d(STMy,MMy) &\leq \frac{1}{3}\big[d(STMy,STz) \\ &\quad +d(STz,MMy)+d(STz,STSTy)\big] \end{aligned}$$

which implies in view of the fact that My = z = ST y

$$\begin{aligned} d(STz,Mz) &\leq \frac{1}{3}\big[d(STz,STz)+d(STz,Mz) \\ &\quad +d(STz,STz)\big] \\ d(STz,Mz) &\leq \frac{1}{3}d(STz,Mz) \end{aligned}$$

Therefore $STz = Mz$.

Hence we have $STz = Mz$; $ABz = Lz = z$. (2.3.15)

Now,we show that z is a fixed point of α *ST*.

In view of (2.3.15) and (2.3.4)

$$\begin{aligned} d(z,STz) &= d(z,Mz) \\ &< \max\Big\{k_1\big[d(ABz,STz)+d(Lz,ABz) \\ &\quad +d(Mz,STz)\big]\frac{k_2}{2}\big[d(ABz,Mz)+d(Lz,STz)\big]\Big\} \qquad (2.3.16) \\ &< \max\Big\{k_1\big[d(z,STz)+d(z,z)+d(STz,STz)\big] \\ &\quad \frac{k_2}{2}\big[d(z,STz)+d(z,STz)\big]\Big\} \end{aligned}$$

$< d(z,STz)$ a contradiction if $STz \neq z$

Therefore z = STz.

Hence $z = STz = ABz = Lz = Mz$

which shows that *z* is a α-common fixed point of *AB*, *ST*, *L* and *M*.

Case(ii): L is continuous

From (2.3.8) we have

Since $\{LLx_{2n}\}$ and $\{LABx_{2n+2}\}$ converges to *Lz* as $n \to \infty$. (2.3.17)

since (*L*,*AB*) are compatible mappings of type (C),we have from (2.3.17)

$$\begin{aligned} \lim_{n\to\infty} d(ABABx_{2n},Lz) &= \lim_{n\to\infty} d(ABABx_{2n},LABx_{2n}) \\ &\leq \frac{1}{3}\lim_{n\to\infty} d(LABx_{2n},Lz) \\ &\quad +\lim_{n\to\infty} d(Lz,LLx_{2n})+\lim_{n\to\infty} d(Lz,ABABx_{2n}) \\ &\leq \frac{1}{3}\lim_{n\to\infty} d(Lz,Lz)+\lim_{n\to\infty} d(Lz,Lz) \\ &\quad +\lim_{n\to\infty} d(Lz,ABABx_{2n}) \\ &\leq \frac{1}{3}\lim_{n\to\infty} d(Lz,ABABx_{2n}) \end{aligned} \qquad (2.3.18)$$

which shows $\{ABABx_{2n}\} \to Lz$ as $n \to \infty$.

Now, we show that *z* is a fixed point of *L*.

In view of (2.3.17), (2.3.8), (2.3.4) and (2.3.18)

$$\begin{aligned} d(Lz,z) &= \lim_{n\to\infty} d(LABx_{2n},Mx_{2n+1}) \\ &< \lim_{n\to\infty} \max\Big\{k_1\big[d(ABABx_{2n},STx_{2n+1}) \\ &\quad +d(LABx_{2n},ABABx_{2n})+d(Mx_{2n+1},STx_{2n+1})\big], \\ &\quad \frac{k_2}{2}\big[d(ABABx_{2n},Mx_{2n+1})+d(LABx_{2n},STx_{2n+1}) \qquad (2.3.19) \\ &< \max\Big\{k_1\big[d(Lz,z)+d(Lz,Lz)+d(z,z)\big], \\ &\quad \frac{k_2}{2}\big[d(Lz,z)+d(Lz,z)\big]\Big\} \end{aligned}$$

$< d(Lz,z)$ a contradiction if $L \neq z$ yielding therefore $L = z$.

Since $L(X) \subseteq ST(X)$ there exist $u \in X$ such that $z = Lz = STu$.

We prove that $STu = Mu$.

Now, In view of (2.3.8) and (2.3.4)

$$
\begin{aligned}
d(z,Mu) &= \lim_{n\to\infty} d(Lx_{2n}, Mu) \\
&< \max\{k_1[d(ABx_{2n}, STu) + d(Lx_{2n}, ABx_{2n}) \\
&+ d(Mu, STu)], \\
&\frac{k_2}{2}[d(ABx_{2n}, Mu) + d(Lx_{2n}, STu)]\} \\
&< \max\{k_1[d(z,Mu) + d(z,z) + d(Mu,STu)], \\
&\frac{k_2}{2}[d(z,Mu) + d(z,Mu)]\}
\end{aligned} \tag{2.3.20}
$$

$< d(z,Mu)$ a contradiction if $z \neq Mu$

Thus $Mu = z$.

Therefore ,we have $z = Lz = STu = Mu$.

Now, taking a sequence $\{z_n\}$ in X such that $z_n = u$ $\forall n \geq 1$, it follows that

$$Mz_n \to Mu = z \text{ and } STz_n \to STu = z \text{ as } n \to \infty$$

since (M, ST) are compatible mappings of type (C), we get

$$
\begin{aligned}
\lim_{n\to\infty} d(STMz_n, MMz_n) &\leq \frac{1}{3}\lim_{n\to\infty} d(STMz_n, STz) \\
&+ \lim_{n\to\infty} d(STz, MMz_n) + \lim_{n\to\infty} d(STz, STSTz_n)
\end{aligned} \tag{2.3.21}
$$

That is

$$
\begin{aligned}
d(STMu, MMu) &\leq \frac{1}{3}[d(STMu, STz) \\
&+ d(STz, MMu) + d(STz, STSTu)]
\end{aligned}
$$

which implies in view of the fact that $Mu = z = STu$

$$
\begin{aligned}
d(STz, Mz) &\leq \frac{1}{3}[d(STz, STz) \\
&+ d(STz, Mz) + d(STz, STz)] \\
d(STz, Mz) &\leq \frac{1}{3} d(STz, Mz)
\end{aligned}
$$

which shows that $STz = Mz$.

Now, we show that z is also a fixed point of M

In view of (2.3.8) and (2.3.4)

$$
\begin{aligned}
d(z,Mz) &= \lim_{n\to\infty} d(Lx_{2n}, Mz) \\
&< \max\{k_1[d(ABx_{2n}, STz) + d(Lx_{2n}, ABx_{2n}) \\
&+ d(Mz, STz)], \frac{k_2}{2}[d(ABx_{2n}, Mz) + d(Lx_{2n}, STz) \\
&< \max\{k_1[d(z,Mz) + d(z,z) + d(Mz,Mz)], \\
&\frac{k_2}{2}[d(z,Mz) + d(z,Mz)]\}
\end{aligned} \tag{2.3.22}
$$

$< d(z,Mz)$ a contradiction if $z \neq Mz$

which shows that $z = Mz$.

Since $(\alpha M)(x) \subseteq (\alpha AB)(x)$ there exist a $v \in X$ such that

$$z = Mz = ABv.$$

We prove $z = Lv$, from (2.3.4) we have

$$
\begin{aligned}
d(Lv, z) &= d(Lv, Mz) \\
&< \max\{k_1[d(ABv, STz) \\
&+ d(Lv, ABv)] \\
&+ d(Mz, STzv)], \frac{k_2}{2}[d(ABv, M)z \\
&+ d(Lv, STz)]\} \\
&< \max\{k_1[d(z,z) \\
&+ d(Lv,z) + d(z,z)], \\
&\frac{k_2}{2}[d(z,z) + d(Lv,z)]\}
\end{aligned} \tag{2.3.23}
$$

$< d(Lv,z)$ a contradiction if $Lv \neq z$

Thus $Lv = z$

Now, taking a sequence $\{v_n\}$ in X such that $v_n = v$ $\forall n \geq 1$, it follows that $Lv_n \to Lv = z$ and $ABv_n \to ABv = z$ as $n \to \infty$, since (L, AB) are compatible mappings of type (C), we have

$$
\begin{aligned}
\lim_{n\to\infty} d(ABLv_n, LLv_n) &\leq \frac{1}{3}\lim_{n\to\infty}[d(ABLv_n, ABz) \\
&+ \lim_{n\to\infty} d(ABz, LLv_n) + \lim_{n\to\infty} d(ABz, ABABv_n)]
\end{aligned} \tag{2.3.24}
$$

That is

$$
\begin{aligned}
d(ABLv, LLv_n) &\leq \frac{1}{3}[d(ABLv, ABz) \\
&+ d(ABz, LLv) + d(ABz, ABABv)]
\end{aligned}
$$

which implies in view of the fact that (α L)v = z = (α AB)v $d(ABz, Lz) \leq \frac{1}{3}[d(ABz, ABz) + d(ABz, Lz) + d(ABz, ABz)] \leq \frac{1}{3} d(ABz, Lz)$

which shows $ABz = Lz$

Since $ABz = Lz = z$ also $z = Mz = STz$.

If the mappings M or ST is continuous instead of L or AB then the proof that z is a Common fixed point of L, M, AB, and ST is similar.

Uniqueness:

Let w be another common fixed point of L, M, AB, and ST then $Lw = Mw = ABw = STw = w$.

From (2.3.4) we have

$$
\begin{aligned}
d(z,w) &= d(Lz,Mw) \\
&< \max\Big\{k_1\big[d(ABz,STw)+d(Lz,ABz) \\
&\quad +d(Mw,STw)\big],\frac{k_2}{2}\big[d(ABz,Mw) \\
&\quad +d(Lz,STw)\big]\Big\} \qquad (2.3.25) \\
&< \max\Big\{k_1\big[d(z,w)+d(z,z)+d(w,w)\big], \\
&\quad \frac{k_2}{2}\big[d(z,w)+d(z,w)\big]\Big\}
\end{aligned}
$$

$< d(z,w)$ a contradiction if $z \neq w$
yielding there by $z = w$.

Finally we need to show that *z* is a common fixed point of *L*, *M*, *A*, *B*, *S* and *T*.

For this let *z* is the unique common fixed point of (*AB*, *L*) and (*ST*, *M*).

Since (*A*, *B*), (*A*, *L*), (*B*, *L*) are commutative

$$Az = A(ABz) = A(BAz) = (AB)(Az); Az = ALz = LAz$$

$$Bz = B(ABz) = (BA)(Bz) = (AB)(Az); Bz = BLz = LBz.$$

which shows that *Az*, *Bz* are common fixed points of (*AB*, *L*) yielding there by $Az = Z = Bz = Lz = ABz$ in the view of uniqueness of common fixed point of the pairs (*AB*, *L*) .

Similarly using the, commutativity of (*S*,*T*), (*S*,*M*) and (*T*,*M*) it can be shown that *Sz* = *z* = *Tz* = *Mz* = *STz*.

Now, we need to show that *Az* = *Sz* (*Bz* = *Tz*) also remains a common fixed point of both the pairs (*AB*, *L*) and (*ST*, *M*).

From (2.3.4) we have

$$
\begin{aligned}
d(Az,Sz) &= d(Lz,Mz) \\
&\le \max\Big\{k_1\big[d(ABz,STz)+d(Lz,ABz) \\
&\quad +d(Mz,STz)\big],\frac{k_2}{2}\big[d(ABz,Mz)+d(Lz,STz)\big]\Big\} \\
&< \max\Big\{k_1\big[d(z,z)+d(z,z)+d(z,z)\big], \\
&\quad \frac{k_2}{2}\big[d(z,z)+d(z,z)\big]\Big\} \le 0
\end{aligned}
$$

implies that *Az* = *Sz*.

similarly it can be shown that *Bz* = *Tz*. Thus *z* is the unique common fixed point of *A*, *B*, *S*, *T*, *L* and *M*.

This establishes the theorem.

Now we give an example to claim our result.

Example: Let $X = [1,\infty]$ with $d(x,y) = |x-y|$. Define self maps *A*, *B*, *S*, *T*, *L* and $M : X \to X$ by $Lx = x$, $Mx = x^2$, $AB = 2x^2 - 1$, $ST = 2x^4 - 1$, $x \ge 1$

Let $x_n \to 1 + \frac{1}{n}$ for $n \ge 1$

Then $x_n \to 1$ as $n \to \infty$.

$$d(ABLx_n, LLx_n) \to 0 \text{ iff } x_n \to 1$$

$$d(ABLx_n, ABt) \to 0 \text{ iff } x_n \to 1$$

$$d(ABt, ABABx_n) \to 0 \text{ iff } x_n \to 1$$

$$d(ABt, LLx_n) \to 0 \text{ iff } x_n \to 1$$

ABx_n, STx_n, Lx_n, Mx_n converges to $1 = t \in X$ as $n \to \infty$.

The pairs (*L*, *AB*) and (*M*, *ST*) are compatible mappings of type (c) and also satisfies the conditions (2.3.2), (2.3.3), (2.3.4), (2.3.5) and (2.3.6)

Remarks: Main theorem remains true if we replace condition compatible mappings of type (C) by

1) compatible mappings of type (A) or
2) compatible mappings of type (B) or
3) compatible mappings of type (P)

3. References

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