

# New Common Fixed Point Results for Four Maps on Cone Metric Spaces\*

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## Abstract

In this paper, some new existence and uniqueness of common fixed points for four mappings are obtained, which do not satisfy continuity and commutation on non-normal cone metric spaces. These results improve and generalize several well-known comparable results in the literature.

**Keywords:** Common Fixed Point, Cone Metric Space, Non-Normal Cone

## 1. Introduction and Preliminaries

Since Huang and Zhang [1] introduced the concept of cone metric space, the study of common fixed points of mappings satisfying certain contractive conditions on cone metric spaces has been at the center of strong research activity, because it has not only important theoretical meaning but also wide applications. Recently, some authors obtained a number of meaningful consulting fixed point theorems for one or two mappings on cone metric spaces (see [1-3,5-8]). The aim of this paper is to present coincidence points results for four mappings without satisfying the notion of continuity and commutation on non-normal cone metric spaces. Common fixed point theorems are obtained under weakly compatible maps. Our results generalized and unified these main results in [1-5].

We recall some definitions and properties of cone metric spaces in [1]. Let  $E$  be a real Banach space and  $P$  be a subset of  $E$ ,  $\theta$  denotes the zero element of  $E$  and  $\text{int } P$  denotes the interior of  $P$ . The subset  $P$  is called a cone if and only if

- 1)  $P$  is closed, nonempty and  $P \neq \{\theta\}$ ,
- 2)  $a, b \in P, a, b \geq 0, x, y \in P \Rightarrow ax + by \in P$ ,
- 3)  $x \in P$  and  $-x \in P \Rightarrow x = \theta$ .

Given a cone  $P \subset E$ , we define a partial ordering  $\leq$  with respect to  $P$  by  $x \leq y$  if and only if  $y - x \in P$ . We shall write  $x < y$  if  $x \leq y$  and  $x \neq y$ , while  $x \ll y$  will stand for  $y - x \in \text{int } P$ . A cone  $P$  is called normal if there is a number  $N > 0$  such that for all  $x, y \in P$ ,

$$\theta \leq x \leq y \text{ implies } \|x\| \leq N \|y\|.$$

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The least positive number satisfying the above inequality is called the normal constant of  $P$ .

**Definition 1.1.** ([1]) Let  $X$  be a nonempty set. Suppose that the mapping  $d : X \times X \rightarrow E$  satisfies:

- (d1)  $\theta \leq d(x, y)$  for all  $x, y \in X$  and  $d(x, y) = \theta$  if and only if  $x = y$ ;
- (d2)  $d(x, y) = d(y, x)$  for all  $x, y \in X$ ;
- (d3)  $d(x, y) \leq d(x, z) + d(z, y)$  for all  $x, y, z \in X$ .

Then  $d$  is called a cone metric on  $X$  and  $(X, d)$  is called a cone metric space. It is clear that the cone metric space is more general than metric space.

**Definition 1.2** ([1]) Let  $(X, d)$  be a cone metric space. Then we say that  $\{x_n\}$  is:

- 1) a Cauchy sequence if for every  $c \in E$  with  $c \gg \theta$ , there is  $N$  such that for all  $n, m > N, d(x_n, x_m) \ll c$ ;
- 2) a convergent sequence if for every  $c \in E$  with  $c \gg \theta$ , there is  $N$  such that for all  $m > N, d(x_m, x) \ll c$  for some fixed  $x$  in  $X$ .

A cone metric space  $X$  is said to be complete if every Cauchy sequence in  $X$  is convergent in  $X$ .

**Definition 1.3.** ([2]) Let  $f$  and  $g$  be self maps of a set  $X$ . If  $w = fx = gx$  for some  $x$  in  $X$ , then  $x$  is called a coincidence point of  $f$  and  $g$ , and  $w$  is called a point of coincidence of  $f$  and  $g$ .

**Definition 1.4.** ([6]) The mappings  $f, g : X \rightarrow X$  are weakly compatible if, for every  $x \in X$ , holds  $fgx = ggx$  whenever  $fx = gx$ .

**Lemma 1.5.** ([2]) Let  $f$  and  $g$  be weakly compatible self maps of a set  $X$ . If  $f$  and  $g$  have a unique point of coincidence  $w = fx = gx$ , then  $w$  is the unique common fixed point of  $f$  and  $g$ .

**Remark 1.6.** Let  $(X, d)$  be a cone metric space with a cone  $P$ . If  $d(x, y) \leq hd(x, y)$  for all

$x, y \in X, h \in (0, 1)$ , then  $d(x, y) = \theta$ , which implies that  $x = y$ .

### 2. Main Results

In this section, we give some common fixed point theorems for four mappings defined on a cone metric space. Normality of the cone is not assumed.

**Theorem 2.1.** *Let  $(X, d)$  be a cone metric space. Suppose mappings  $S, I, T, J: X \rightarrow X$  satisfy*

$$d(Sx, Ty) \leq a_1 d(Ix, Jy) + a_2 d(Ix, Sx) + a_3 d(Jy, Ty) + a_4 d(Ix, Ty) + a_5 d(Jy, Sx)$$

for all  $x, y \in X$ , where  $a_i \geq 0 (i = 1, 2, 3, 4, 5)$  satisfying  $a_1 + a_2 + a_3 + 2 \max\{a_4, a_5\} < 1$

$$\text{or } a_1 + 2 \max\{a_2, a_3\} + a_4 + a_5 < 1.$$

If  $S(X) \subseteq J(X), T(X) \subseteq I(X)$  and one of  $S(X), I(X), J(X)$  and  $T(X)$  is a complete subspace of  $X$ , then the four mappings  $S, I, T$  and  $J$  have a unique point of coincidence in  $X$ . Moreover, if  $\{T, J\}$  and  $\{S, I\}$  are weakly compatible, respectively, all of the mappings  $S, I, T$  and  $J$  have a unique common fixed point.

*Proof.* Suppose  $x_0$  is an arbitrary point in  $X$ . Since  $S(X) \subseteq J(X), T(X) \subseteq I(X)$ , there exist  $x_1, x_2 \in X$  such that  $Sx_0 = Jx_1, Tx_1 = Ix_2$ . Continuing this process, we can define  $\{x_n\}$  by

$$Jx_{2n+1} = Sx_{2n}, Ix_{2n+2} = Tx_{2n+1}, \quad n = 0, 1, 2, \dots \text{ Denote} \\ y_{2n} = Jx_{2n+1} = Sx_{2n}, \quad y_{2n+1} = Ix_{2n+2} = Tx_{2n+1}, \\ n = 0, 1, 2, \dots$$

Now, we shall show that  $\{y_n\}$  is a Cauchy sequence. From (2.1), we know

$$d(y_{2n}, y_{2n+1}) = d(Sx_{2n}, Tx_{2n+1}) \\ \leq a_1 d(Ix_{2n}, Jx_{2n+1}) + a_2 d(Ix_{2n}, Sx_{2n}) + a_3 d(Jx_{2n+1}, Tx_{2n+1}) \\ + a_4 d(Ix_{2n}, Tx_{2n+1}) + a_5 d(Jx_{2n+1}, Sx_{2n}) \\ \leq a_1 d(y_{2n-1}, y_{2n}) + a_2 d(y_{2n-1}, y_{2n}) + a_3 d(y_{2n}, y_{2n+1}) \\ + a_4 [d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})] + a_5 d(y_{2n}, y_{2n}),$$

which implies that

$$d(y_{2n+1}, y_{2n}) \leq \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4} d(y_{2n}, y_{2n-1}). \quad (2.2)$$

Similarly it is not difficult to show that

$$d(y_{2n+2}, y_{2n+1}) = d(Sx_{2n+2}, Tx_{2n+1}) \\ \leq a_1 d(Ix_{2n+2}, Jx_{2n+1}) + a_2 d(Ix_{2n+2}, Sx_{2n+2}) \\ + a_3 d(Jx_{2n+1}, Tx_{2n+1}) \\ + a_4 d(Ix_{2n+2}, Tx_{2n+1}) + a_5 d(Jx_{2n+1}, Sx_{2n+2}) \\ \leq a_1 d(y_{2n}, y_{2n+1}) + a_2 d(y_{2n+1}, y_{2n+2}) + a_3 d(y_{2n}, y_{2n+1}) \\ + a_5 [d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})],$$

which implies that

$$d(y_{2n+2}, y_{2n+1}) \leq \frac{a_1 + a_3 + a_5}{1 - a_2 - a_5} d(y_{2n+1}, y_{2n}). \quad (2.3)$$

Now, from (2.2) and (2.3), we deduce that

$$d(y_{2n+2}, y_{2n+1}) \leq \frac{a_1 + a_3 + a_5}{1 - a_2 - a_5} d(y_{2n+1}, y_{2n}) \\ \leq \frac{a_1 + a_3 + a_5}{1 - a_2 - a_5} \cdot \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4} d(y_{2n}, y_{2n-1}) \\ \leq \frac{a_1 + a_3 + a_5}{1 - a_2 - a_5} \cdot \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4} \cdot \frac{a_1 + a_3 + a_5}{1 - a_2 - a_5} d(y_{2n-1}, y_{2n-2}) \\ \leq \dots \leq \frac{a_1 + a_3 + a_5}{1 - a_2 - a_5} \left( \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4} \cdot \frac{a_1 + a_3 + a_5}{1 - a_2 - a_5} \right)^n d(y_1, y_0)$$

and

$$d(y_{2n+3}, y_{2n+2}) \leq \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4} d(y_{2n+2}, y_{2n+1}) \\ \leq \dots \leq \left( \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4} \cdot \frac{a_1 + a_3 + a_5}{1 - a_2 - a_5} \right)^{n+1} d(y_1, y_0).$$

Let

$$M = \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4}, \quad N = \frac{a_1 + a_3 + a_5}{1 - a_2 - a_5}.$$

If  $a_1 + a_2 + a_3 + 2 \max\{a_4, a_5\} < 1$ , then

$$MN = \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4} \cdot \frac{a_1 + a_3 + a_5}{1 - a_2 - a_5} < 1 \cdot 1 = 1.$$

If  $a_1 + 2 \max\{a_2, a_3\} + a_4 + a_5 < 1$ , then

$$MN = \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4} \cdot \frac{a_1 + a_3 + a_5}{1 - a_2 - a_5} \\ = \frac{a_1 + a_3 + a_5}{1 - a_3 - a_4} \cdot \frac{a_1 + a_2 + a_4}{1 - a_2 - a_5} < 1 \cdot 1 = 1.$$

Now, for any  $n > m$ , we have

$$d(y_{2n+1}, y_{2m+1}) \\ \leq d(y_{2n+1}, y_{2n}) + d(y_{2n}, y_{2n-1}) + \dots + d(y_{2m+2}, y_{2m+1}) \\ \leq \left( \sum_{i=m+1}^n (MN)^i + N \sum_{i=m}^{n-1} (MN)^i \right) d(y_1, y_0) \\ \leq \left( \frac{(MN)^{m+1}}{1 - MN} + \frac{N(MN)^m}{1 - MN} \right) d(y_1, y_0) \\ = (M + 1) \frac{N(MN)^m}{1 - MN} d(y_1, y_0).$$

In analogous way, we gain

$$d(y_{2n}, y_{2m+1}) \leq (M + 1) \frac{N(MN)^m}{1 - MN} d(y_1, y_0),$$

$$d(y_{2n}, y_{2m}) \leq (N+1) \frac{(MN)^{m+1}}{1-MN} d(y_1, y_0)$$

and

$$d(y_{2n+1}, y_{2m}) \leq (N+1) \frac{(MN)^{m+1}}{1-MN} d(y_1, y_0).$$

Thus, for  $n > m > 0$

$$\begin{aligned} & d(y_n, y_m) \\ & \leq \max \left\{ (N+1) \frac{(MN)^{m+1}}{1-MN}, (M+1) \frac{N(MN)^m}{1-MN} \right\} d(y_1, y_0) \\ & = \lambda_m d(y_1, y_0), \end{aligned}$$

where  $\lambda_m \rightarrow 0$  as  $m \rightarrow \infty$ .

For each  $c \gg \theta$ , choose  $\delta > 0$  such that  $c - x \in \text{int}P$ , where  $\|x\| \ll \delta$ , i.e.,  $x \ll c$ . For this  $\delta$ , we can choose a natural number  $N_1$  such that  $\|\lambda_m d(y_1, y_0)\| < \delta$  for  $m > N_0$ . Thus, we get

$$d(y_n, y_m) \leq \lambda_m d(y_1, y_0) \ll c, \text{ for all } n > m > N_0.$$

Therefore  $\{y_n\}$  is a Cauchy sequence in  $(X, d)$ . Suppose  $J(X)$  is complete, there exists  $q \in J(X)$  such that  $y_{2n} = Sx_{2n} = Jx_{2n+1} \rightarrow q$  as  $n \rightarrow \infty$ . So we can find a  $p \in X$  such that  $Jp = q$ . (If  $S(X)$  is complete, there exists  $q \in S(X) \subseteq J(X)$ , then the conclusions remain the same.) Letting  $n \rightarrow \infty$ , and by  $Sx_{2n} = Jx_{2n+1} \rightarrow q$ , as  $n \rightarrow \infty$ , we can choose a natural

number  $N_1$  such that  $d(y_{2n}, q) \ll \frac{(1-a_3-a_4)c}{4}$  and

$$d(y_{2n}, y_{2n-1}) \ll \frac{(1-a_3-a_4)c}{2} \text{ for } n \geq N_1.$$

Now we show that  $Tp = q$ . By (2.1), we have

$$\begin{aligned} & d(Tp, q) \leq d(Sx_{2n}, Tp) + d(Sx_{2n}, q) \\ & \leq a_1 d(Ix_{2n}, Jp) + a_2 d(Ix_{2n}, Sx_{2n}) + a_3 d(Jp, Tp) \\ & \quad + a_4 d(Ix_{2n}, Tp) + a_5 d(Jp, Sx_{2n}) + d(Sx_{2n}, q) \\ & = a_1 d(y_{2n-1}, q) + a_2 d(y_{2n-1}, y_{2n}) + a_3 d(q, Tp) \\ & \quad + a_4 d(y_{2n-1}, Tp) + a_5 d(q, y_{2n}) + d(y_{2n}, q) \\ & \leq a_1 [d(y_{2n-1}, y_{2n}) + d(y_{2n}, q)] \\ & \quad + a_2 d(y_{2n-1}, y_{2n}) + a_3 d(q, Tp) \\ & \quad + a_4 [d(y_{2n-1}, y_{2n}) + d(y_{2n}, q) + d(q, Tp)] \\ & \quad + (a_5 + 1) d(q, y_{2n}). \end{aligned}$$

Taking  $n \rightarrow \infty$ , we get

$$\begin{aligned} (1-a_3-a_4)d(q, Tp) & \leq (a_1+a_2+a_4)d(y_{2n-1}, y_{2n}) \\ & \quad + (a_1+a_4+a_5+1)d(y_{2n}, q). \end{aligned}$$

Then, we get  $d(q, Tp) \ll \frac{c}{2} + \frac{c}{2} = c$ , i.e.  $Tp = q = Jp$ .

At the same time, as  $q = Tp \in T(X) \subseteq I(X)$ , there exists  $u$  in  $X$  such that  $Iu = q$ . From (2.1),

$$\begin{aligned} & d(Su, q) = d(Su, Tp) \\ & \leq a_1 d(Iu, Jp) + a_2 d(Iu, Su) + a_3 d(Jp, Tp) \\ & \quad + a_4 d(Iu, Tp) + a_5 d(Jp, Su) \\ & = (a_2 + a_5) d(Su, q). \end{aligned}$$

Hence, from Remark 1.6, we know  $Su = q$ . Therefore,  $Tp = Jp = q, Su = Iu = q$ .

Next if we assume  $I(X)$  is complete, there exists  $q \in I(X)$  such that

$y_{2n+1} = Ix_{2n+2} = Tx_{2n+1} \rightarrow q$  as  $n \rightarrow \infty$ . So we can find a  $u \in X$  such that  $Iu = q$ . (If  $T(X)$  is complete, there exists  $q \in T(X) \subseteq I(X)$ , then the conclusions remain the same.) Then, we can choose a natural number  $N_2$

$$\text{such that } d(y_{2n+1}, q) \ll \frac{(1-a_2-a_5)c}{4} \text{ and}$$

$$d(y_{2n+1}, y_{2n}) \ll \frac{(1-a_2-a_5)c}{2} \text{ for } n \geq N_2.$$

Now we show that  $Su = q$ . By (2.1), we get

$$\begin{aligned} & d(Su, q) \leq d(Su, Tx_{2n+1}) + d(Tx_{2n+1}, q) \\ & \leq a_1 d(Iu, Jx_{2n+1}) + a_2 d(Iu, Su) \\ & \quad + a_3 d(Jx_{2n+1}, Tx_{2n+1}) + a_4 d(Iu, Tx_{2n+1}) \\ & \quad + a_5 d(Jx_{2n+1}, Su) + d(Tx_{2n+1}, q) \\ & = a_1 d(y_{2n}, q) + a_2 d(q, Su) + a_3 d(y_{2n}, y_{2n+1}) \\ & \quad + a_4 d(q, y_{2n+1}) + a_5 d(y_{2n}, Su) + d(y_{2n+1}, q) \\ & \leq a_1 [d(y_{2n+1}, q) + d(y_{2n+1}, y_{2n})] \\ & \quad + a_2 d(q, Su) + a_3 d(y_{2n+1}, y_{2n}) \\ & \quad + (a_4 + 1) d(q, y_{2n+1}) \\ & \quad + a_5 [d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, q) + d(q, Su)]. \end{aligned}$$

Letting  $n \rightarrow \infty$ , by Lemma 1.6 we get

$$\begin{aligned} (1-a_2-a_5)d(Su, q) & \leq (a_1+a_3+a_5)d(y_{2n+1}, y_{2n}) \\ & \quad + (a_1+a_4+a_5+1)d(y_{2n+1}, q). \end{aligned}$$

Thus, we have  $d(q, Su) \ll \frac{c}{2} + \frac{c}{2} = c$ , i.e.

$Su = q \in S(X) \subseteq J(X)$ , then there exists  $p \in X$  such that  $Jp = q$ . In the same method we can also obtain  $Tp = q$ .

Finally, we show that  $T$  and  $J$ ,  $S$  and  $I$  have a unique point of coincidence in  $X$ . Assume there exists another point  $z \in X$  such that  $Tx = Jx = z$ , then

$$\begin{aligned} & d(q, z) = d(Su, Tx) \leq a_1 d(Iu, Jx) + a_2 d(Iu, Su) \\ & \quad + a_3 d(Jx, Tx) + a_4 d(Iu, Tx) + a_5 d(Jx, Su), \end{aligned}$$

which gives

$$d(q, z) \leq (a_1 + a_4 + a_5)d(q, z),$$

thus  $d(q, z) = \theta$ , i.e.,  $Tx = Tp = Jx = Jp = q = z$ . That is to say,  $q$  is a unique point of coincidence in  $X$  of  $T$  and  $J$ . Similarly, we also have  $q$  is a unique point of coincidence of  $S$  and  $I$  by induction. So, according to Lemma 1.5,  $q$  is the unique common fixed point of  $\{T, J\}$  and  $\{S, I\}$ . Therefore,  $q$  is the unique common fixed point of the four mappings  $T, J, S$  and  $I$ . The proof of the theorem is completed.

As a variant of Theorem 2.1, we get the following Corollary.

**Corollary 2.2.** *Let  $(X, d)$  be a cone metric space. Suppose mappings  $S, I, T, J : X \rightarrow X$  satisfy*

$$\begin{aligned} d(S^m x, T^n y) &\leq a_1 d(I^m x, J^n y) + a_2 d(I^m x, S^m x) \\ &+ a_3 d(J^n y, T^n y) + a_4 d(I^m x, T^n y) + a_5 d(J^n y, S^m x) \end{aligned} \quad (2.4)$$

for all  $x, y \in X$ , where  $a_i \geq 0 (i = 1, 2, 3, 4, 5)$  satisfying

$$\begin{aligned} a_1 + a_2 + a_3 + 2\max\{a_4, a_5\} &< 1 \\ \text{or } a_1 + 2\max\{a_2, a_3\} + a_4 + a_5 &< 1. \end{aligned}$$

If  $S(X) \subseteq J(X), T(X) \subseteq I(X)$  and one of  $S(X), I(X), J(X)$  and  $T(X)$  is a complete subspace of  $X$ , then the four mappings  $S, I, T$  and  $J$  have a unique point of coincidence in  $X$ . Moreover, if  $\{T, J\}$  and  $\{S, I\}$  are weakly compatible, respectively, all of the mappings  $S, I, T$  and  $J$  have a unique common fixed point.

*Proof.* It follows from Theorem 2.1, that the four mappings  $S^m, I^m, T^n$  and  $J^n$  have a unique common fixed point  $q$ . Now,  $Sq = S(S^m q) = S^{m+1} q = S^m(Sq)$  and  $Iq = I(I^m q) = I^{m+1} q = I^m(Iq)$  implies that  $Sq$  and  $Iq$  are also fixed points for  $S^m$  and  $I^m$ . Hence,  $Sq = Iq = q$ . By using the same method, we have  $Tq = Jq = q$ .

**Remark 2.3.** *Compared with corresponding results in the literature [4, Theorem 2.8 and Corollary 2.9], Theorem 2.1 show that our hypotheses are greatly weaker than those conditions. On the one hand, we ensure the existence and uniqueness of a common fixed point of four mappings without continuity and commutation; on the other hand, they are special cases of Theorem 2.1 and Corollary 2.2 with  $a_4 = a_5$ . Hence, Theorem 2.1 present a more general format of common fixed point for four mappings without continuity and commutation on non-normal cone metric spaces, which extends the main results in [4].*

Now we use Theorem 2.1 to obtain a series of new common fixed point theorems for four mappings in non-normal cones metric spaces, which improve and

generalize several known results in [1-3].

**Corollary 2.4.** *Let  $(X, d)$  be a cone metric space. Suppose mappings  $S, I, T, J : X \rightarrow X$  satisfy*

$$\begin{aligned} d(Sx, Ty) &\leq \alpha d(Ix, Jy) + \beta [d(Ix, Sx) + d(Jy, Ty)] \\ &+ \gamma [d(Ix, Ty) + d(Jy, Sx)], \end{aligned} \quad (2.5)$$

for all  $x, y \in X$ , where  $\alpha, \beta, \gamma \geq 0$  and  $\alpha + 2\beta + 2\gamma < 1$ . If  $S(X) \subseteq J(X), T(X) \subseteq I(X)$  and one of  $S(X), I(X), J(X)$  and  $T(X)$  is a complete subspace of  $X$ , then the four mappings  $S, I, T$  and  $J$  have a unique point of coincidence in  $X$ . Moreover, if  $\{T, J\}$  and  $\{S, I\}$  are weakly compatible, respectively, all of the mappings  $S, I, T$  and  $J$  have a unique common fixed point.

*Proof.* Let  $a_1 = \alpha, a_2 = a_3 = \beta, a_4 = a_5 = \gamma$  in Theorem 2.1.

**Corollary 2.5.** *Let  $(X, d)$  be a cone metric space. Suppose mappings  $S, I, T, J : X \rightarrow X$  satisfy*

$$d(Sx, Ty) \leq \alpha d(Ix, Jy) + \beta [d(Ix, Sx) + d(Jy, Ty)], \quad (2.6)$$

for all  $x, y \in X$ , where  $\alpha, \beta \geq 0$  and  $\alpha + 2\beta < 1$ . If  $S(X) \subseteq J(X), T(X) \subseteq I(X)$  and one of  $S(X), I(X), J(X)$  and  $T(X)$  is a complete subspace of  $X$ , then the four mappings  $S, I, T$  and  $J$  have a unique point of coincidence in  $X$ . Moreover, if  $\{T, J\}$  and  $\{S, I\}$  are weakly compatible, respectively, all of the mappings  $S, I, T$  and  $J$  have a unique common fixed point.

**Corollary 2.6.** *Let  $(X, d)$  be a cone metric space. Suppose mappings  $S, I, T, J : X \rightarrow X$  satisfy*

$$d(Sx, Ty) \leq kd(Ix, Sx) + ld(Jy, Ty), \quad (2.7)$$

for all  $x, y \in X$ , where  $k, l \in [0, 1)$  and  $k + l < 1$ . If  $S(X) \subseteq J(X), T(X) \subseteq I(X)$  and one of  $S(X), I(X), J(X)$  and  $T(X)$  is a complete subspace of  $X$ , then the four mappings  $S, I, T$  and  $J$  have a unique point of coincidence in  $X$ . Moreover, if  $\{T, J\}$  and  $\{S, I\}$  are weakly compatible, respectively, all of the mappings  $S, I, T$  and  $J$  have a unique common fixed point.

*Proof.* Let  $a_1 = a_4 = a_5 = 0, a_2 = k, a_3 = l$  in Theorem 2.1, the conclusions are true.

**Corollary 2.7.** *Let  $(X, d)$  be a cone metric space. Suppose mappings  $S, I, T, J : X \rightarrow X$  satisfy*

$$d(Sx, Ty) \leq kd(Ix, Ty) + ld(Jy, Sx), \quad (2.8)$$

for all  $x, y \in X$ , where  $k, l \in [0, 1)$  and  $k + l < 1$ . If  $S(X) \subseteq J(X), T(X) \subseteq I(X)$  and one of  $S(X), I(X), J(X)$  and  $T(X)$  is a complete subspace of  $X$ , then the four mappings  $S, I, T$  and  $J$  have a unique point of coincidence in  $X$ . Moreover, if  $\{T, J\}$  and  $\{S, I\}$  are weakly compatible, respectively,

all of the mappings  $S, I, T$  and  $J$  have a unique common fixed point.

**Corollary 2.8.** ([4]) Let  $(X, d)$  be a cone metric space. Suppose mappings  $S, I, T, J : X \rightarrow X$  satisfy

$$d(Sx, Ty) \leq kd(Ix, Jy),$$

for all  $x, y \in X$ , where  $k \in [0, 1)$ . If

$S(X) \subseteq J(X), T(X) \subseteq I(X)$  and one of  $S(X), I(X), J(X)$  and  $T(X)$  is a complete subspace of  $X$ , then  $\{T, J\}$  and  $\{S, I\}$  have a unique point of coincidence in  $X$ . Moreover, if  $\{T, J\}$  and  $\{S, I\}$  are weakly compatible, respectively, all of the mappings  $S, I, T$  and  $J$  have a unique common fixed point.

**Remark 2.9.** Compared to Theorem 2.1 and Corollary 2.2-2.8 in [3], our Corollary 2.2, 2.4-2.8 do not require that condition “ $P$  is a normal cone”. Moreover, when we further restrict  $I = J = I_x$  in Corollary 2.4, which  $I_x$  is the identity map on  $X$ , we get Theorem 2.1 in [3], and we get Corollary 2.2-2.8 in [3] when we even define  $S = T$  in Corollary 2.2, 2.5-2.8. Hence, Corollary 2.2, 2.4-2.8 improve and generalize Theorem 2.1 and Corollary 2.2-2.8 in [3].

**Corollary 2.10.** Let  $(X, d)$  be a cone metric space. Suppose mappings  $f, g : X \rightarrow X$  satisfy

$$d(fx, fy) \leq \alpha d(gx, gy) + \beta [d(gx, fx) + d(gy, fy)] + \gamma [d(gx, fy) + d(gy, fx)], \tag{2.9}$$

for all  $x, y \in X$ , where  $\alpha, \beta, \gamma \geq 0$  and  $\alpha + 2\beta + 2\gamma < 1$ . If the range of  $g$  contains the range of  $f$  and  $g(X)$  or  $f(X)$  is a complete subspace of  $X$ , then  $f$  and  $g$  have a unique point of coincidence in  $X$ . Moreover, if  $f$  and  $g$  are weakly compatible,  $f$  and  $g$  have a unique common fixed point.

*Proof.* Let  $S = T = f$  and  $I = J = g$  in Corollary 2.4.

**Corollary 2.11.** Let  $(X, d)$  be a cone metric space. Suppose mappings  $f, g : X \rightarrow X$  satisfy

$$d(fx, fy) \leq a_1 d(gx, gy) + a_2 d(fx, gx) + a_3 d(fy, gy) + a_4 d(gx, fy) + a_5 d(gy, fx), \tag{2.10}$$

for all  $x, y \in X$ , where  $a_i \geq 0 (i = 1, 2, 3, 4, 5)$  and  $\sum_{i=1}^5 a_i < 1$ . If  $f(X) \subseteq g(X)$  and  $g(X)$  or  $f(X)$  is a complete subspace of  $X$ , then  $f$  and  $g$  have a unique point of coincidence in  $X$ . Moreover, if  $f$  and  $g$  are weakly compatible,  $f$  and  $g$  have a unique common fixed point.

*Proof.* In (2.10) interchanging the roles of  $x$  and  $y$ , and adding the new inequality to (2.10), yield (2.9) with

$$\alpha = a_1, \beta = \frac{a_2 + a_3}{2}, \gamma = \frac{a_4 + a_5}{2}.$$

**Remark 2.12.** We note that in our Corollary 2.10 and

Corollary 2.11  $g(X)$  or  $f(X)$  is complete, and the results remain the same, while Theorem 2.1 in [5] requires that  $g(X)$  is complete. In addition, Theorem 2.1 in [5] and Theorem 2.1 in [3] generalize the corresponding results in [1-2]. Therefore, above all, our results improve and unify all of these main results in [1-5].

**Remark 2.13.** In Theorem 2.1 we do not require mappings  $S, T, I$  and  $J$  to be compact or continuous. Moreover, we delete the condition “ $P$  is a normal cone” in this papers. In addition, when we choose  $E = R, P = [0, \infty)$  in the above theorems and corollaries, similar conclusions will be gained in simple metric spaces.

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### 4. References

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