Applied Mathematics, 2016, 7, 1868-1881
http://www.scirp.org/journal/am
ISSN Online: 2152-7393
ISSN Print: 2152-7385

A Remark on Polynomial Mappings from $\mathbb{C}^n$ to $\mathbb{C}^{n-1}$ and an Application of the Software Maple in Research

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How to cite this paper: Nguyen, T.B.T. (2016) A Remark on Polynomial Mappings from $\mathbb{C}^n$ to $\mathbb{C}^{n-1}$ and an Application of the Software Maple in Research. *Applied Mathematics*, **7**, 1868-1881.
http://dx.doi.org/10.4236/am.2016.715154

Received: July 16, 2016
Accepted: September 26, 2016
Published: September 29, 2016

Abstract

In [1], we construct singular varieties $\mathcal{V}_G$ associated to a polynomial mapping $G:\mathbb{C}^n \to \mathbb{C}^{n-1}$ where $n \geq 2$ such that if G is a local submersion but is not a fibration, then the 2-dimensional homology and intersection homology (with total perversity) of the variety $\mathcal{V}_G$ are not trivial. In [2], the authors prove that if there exists a so-called *very good projection with respect to the regular value* t^0 of a polynomial mapping $G:\mathbb{C}^n \to \mathbb{C}^{n-1}$, then this value is an atypical value of G if and only if the Euler characteristic of the fibers is not constant. This paper provides relations of the results obtained in the articles [1] and [2]. Moreover, we provide some examples to illustrate these relations, using the software *Maple* to complete the calculations of the examples. We provide some discussions on these relations. This paper is an example for graduate students to apply a software that they study in the graduate program in advanced researches.

Keywords

Singularities, Intersection Homology, Polynomial Mappings, Bifurcation

1. Preliminaries

1.1. Intersection Homology

We briefly recall the definition of intersection homology; for details, we refer to the fundamental work of M. Goresky and R. MacPherson [3] (see also [4]).

Definition 1.1. Let X be an m-dimensional variety. A *stratification of X* is the data of a finite filtration

$$X = X_m \supset X_{m-1} \supset \cdots \supset X_0 \supset X_{-1} = \varnothing,$$

such that for every *i*, the set $S_i = X_i \setminus X_{i-1}$ is either an empty set or a manifold of dimension *i*. A connected component of S_i is called *a stratum* of *X*.

We denote by cL, the open cone on the space *L*, the cone on the empty set being a point. Observe that if *L* is a stratified set then cL is stratified by the cones over the strata of *L* and an additional 0-dimensional stratum (the vertex of the cone).

Definition 1.2. A stratification of *X* is said to be *locally topologically trivial* if for every $x \in X_i \setminus X_{i-1}$, $i \geq 0$, there is an open neighborhood U_x of *x* in *X*, a stratified set *L* and a homeomorphism

$$h : U_x \to (0;1)^i \times cL,$$

such that *h* maps the strata of U_x (induced stratification) onto the strata of $(0;1)^i \times cL$ (product stratification).

The definition of perversities has originally been given by Goresky and MacPherson:

Definition 1.3. A *perversity* is an $(m+1)$-uple of integers $\bar{p} = (p_0, p_1, p_2, p_3, \cdots, p_m)$ such that $p_0 = p_1 = p_2 = 0$ and $p_{k+1} \in \{p_k, p_k + 1\}$, for $k \geq 2$.

Traditionally we denote the zero perversity by $\bar{0} = (0, 0, \cdots, 0)$, the maximal perversity by $\bar{t} = (0, 0, 0, 1, \cdots, m-2)$, and the middle perversities by $\bar{m} = \left(0, 0, 0, 0, 1, 1, \cdots, \left[\frac{m-2}{2}\right]\right)$ (lower middle) and $\bar{n} = \left(0, 0, 0, 1, 1, 2, 2, \cdots, \left[\frac{m-1}{2}\right]\right)$ (upper middle). We say that the perversities $\bar{p}$ and $\bar{q}$ are *complementary* if $\bar{p} + \bar{q} = \bar{t}$.

Let *X* be a variety such that *X* admits a locally topologically trivial stratification. We say that an *i*-dimensional subset $Y \subset X$ is $(\bar{p}, i)$*-allowable* if

$$\dim(Y \cap X_{m-k}) \leq i - k + p_k \quad \text{for all } k.$$

Define $IC_i^{\bar{p}}(X)$ to be the $\mathbb{R}$-vector subspace of $C_i(X)$ consisting in the chains ξ such that $|\xi|$ is $(\bar{p}, i)$-allowable and $|\partial\xi|$ is $(\bar{p}, i-1)$-allowable.

Definition 1.4. The i^{th} *intersection homology group with perversity* $\bar{p}$, denoted by $IH_i^{\bar{p}}(X)$, is the i^{th} homology group of the chain complex $IC_*^{\bar{p}}(X)$.

The notation $IH_*^{\bar{p},c}(X)$ will refer to the intersection homology with compact supports, and the notation $IH_*^{\bar{p},cl}(X)$ will refer to the intersection homology with closed supports. In the compact case, they coincide and will be denoted by $IH_*^{\bar{p}}(X)$. In general, when we write $H_*(X)$ (*resp.*, $IH_*^{\bar{p}}(X)$), we mean the homology (*resp.*, the intersection homology) with both compact supports and closed supports.

Goresky and MacPherson proved that the intersection homology is independent on the choice of the stratification satisfying the locally topologically trivial conditions.

The Poincaré duality holds for the intersection homology of a (singular) variety:

Theorem 1.5. (Goresky, MacPherson [3]) *For any orientable compact stratified semi-algebraic m-dimensional variety X, the generalized Poincaré duality holds:*

$$IH_k^{\bar{p}}(X) \simeq IH_{m-k}^{\bar{q}}(X),$$

where $\bar{p}$ and $\bar{q}$ are complementary perversities.

For the non-compact case, we have:

$$IH_k^{\overline{p},c}(X) \simeq IH_{m-k}^{\overline{q},cl}(X).$$

1.2. The Bifurcation Set, the Set of Asymptotic Critical Values and the Asymptotic Set

Let $G:\mathbb{C}^n \to \mathbb{C}^m$ where $n \geq m$ be a polynomial mapping.

i) The bifurcation set of *G*, denoted by $B(G)$ is the smallest set in $\mathbb{C}^m$ such that *G* is not C^∞-fibration on this set (see, for example, [5]).

ii) When $n = m$, we denote by S_G the set of points at which the mapping *G* is not proper, *i.e.*

$$S_G := \left\{\alpha \in \mathbb{C}^m : \exists \{z_k\} \subset \mathbb{C}^n, |z_k| \to \infty \text{ such that } G(z_k) \to \alpha\right\},$$

and call it the *asymptotic variety* (see [6]). The following holds: $B(G) = S_G$ ([6]).

2. Varieties $\mathcal{V}_G$ Associated to a Polynomial Mapping $G:\mathbb{C}^n \to \mathbb{C}^{n-1}$

In [1], we construct singular varieties associated to a polynomial mapping $G:\mathbb{C}^n \to \mathbb{C}^{n-1}$ as follows: let $G:\mathbb{C}^n \to \mathbb{C}^{n-1}$ such that $K_0(G) = \varnothing$, where $K_0(G)$ is the set of critical values of *G*. Let $\rho:\mathbb{C}^n \to \mathbb{R}$ be a real function such that

$$\rho = a_1|z_1|^2 + \cdots + a_n|z_n|^2,$$

where $\sum_{i=1}^n a_i^2 \neq 0$, $a_i \geq 0$ and $a_i \in \mathbb{R}$. Let us denote $\varphi = \dfrac{1}{1+\rho}$ and consider (G,φ) as a real mapping from $\mathbb{R}^{2n}$ to $\mathbb{R}^{2n-1}$. Let us define

$$\mathcal{M}_G := Sing(G,\varphi) = \left\{x \in \mathbb{R}^{2n} \text{ such that } \mathrm{Rank}_{\mathbb{R}} D(G,\varphi)(x) \leq 2n-2\right\},$$

where $D(G,\varphi)(x)$ is the (real) Jacobian matrix of $(G,\varphi):\mathbb{R}^{2n} \to \mathbb{R}^{2n-1}$ at *x*. Notice that $Sing(G,\varphi) = Sing(G,\rho)$, so we have $\mathcal{M}_G = Sing(G,\rho)$.

Proposition 2.1. [1] For an open and dense set of polynomial mappings $G:\mathbb{C}^n \to \mathbb{C}^{n-1}$ such that $K_0(G) = \varnothing$, the variety $\mathcal{M}_G$ is a smooth manifold of dimension $2n-2$.

Now, let us consider:

a) $F := G|_{\mathcal{M}_G}$ the restriction of *G* on $\mathcal{M}_G$,

b) $\mathcal{N}_G = \mathcal{M}_G \setminus F^{-1}(K_0(F))$.

Since the dimension of $\mathcal{M}_G$ is $2n-2$ (Proposition 2.1), then locally, in a neighbourhood of any point x_0 in $\mathcal{M}_G$, we get a mapping $F:\mathbb{R}^{2n-2} \to \mathbb{R}^{2n-2}$. Then there exists a covering $\{U_1,\cdots,U_p\}$ of $\mathcal{N}_G$ by open semi-algebraic subsets (in $\mathbb{R}^{2n}$) such that on every element of this covering, the mapping *F* induces a diffeomorphism onto its image (see Lemma 2.1 of [7]). We can find semi-algebraic closed subsets $V_i \subset U_i$ (in $\mathcal{N}_G$) which cover $\mathcal{N}_G$ as well. Thanks to Mostowski's Separation Lemma (see Separation Lemma in [7], p. 246), for each $i = 1,\cdots,p$, there exists a Nash function $\psi_i:\mathcal{N}_G \to \mathbb{R}$, such that ψ_i is positive on V_i and negative on $\mathcal{N}_G \setminus U_i$. We can choose the Nash functions ψ_i such that $\psi_i(x_k)$ tends to zero when $\{x_k\} \subset \mathcal{N}_G$

tends to infinity. Let the Nash functions ψ_i and ρ be such that $\psi_i(x_k)$ tends to zero and $\rho(x_k)$ tends to infinity when $x_k \subset \mathcal{N}_G$ tends to infinity. Define a variety $\mathcal{V}_G$ associated to (G,ρ) as

$$\mathcal{V}_G := \overline{(F,\psi_1,\cdots,\psi_p)(\mathcal{N}_G)},$$

that means $\mathcal{V}_G$ is the closure of $\mathcal{N}_G$ by $(F,\psi_1,\cdots,\psi_p)$.

In order to understand better the construction of the variety $\mathcal{V}_G$, see the example 4.13 in [1].

Proposition 2.2. [1] Let $G:\mathbb{C}^n \to \mathbb{C}^{n-1}$ be a polynomial mapping such that $K_0(G)=\varnothing$ and let $\rho:\mathbb{C}^n \to \mathbb{R}$ be a real function such that

$$\rho = a_1|z_1|^2 + \cdots + a_n|z_n|^2,$$

where $\sum_{i=1}^n a_i^2 \neq 0$, $a_i \geq 0$ and $a_i \in \mathbb{R}$ for $i=1,\cdots,n$. Then, there exists a real algebraic variety $\mathcal{V}_G$ in $\mathbb{R}^{2n-2+p}$, where $p>0$, such that:

1) The real dimension of $\mathcal{V}_G$ is $2n-2$,

2) The singular set at infinity of the variety $\mathcal{V}_G$ is contained in $\mathcal{S}_G(\rho)\times\{0_{\mathbb{R}^p}\}$, where

$$\mathcal{S}_G(\rho) := \left\{\alpha \in \mathbb{C}^{n-1} \mid \exists\{z_k\} \subset Sing(G,\rho): z_k \text{ tends to infinity}, G(z_k) \text{ tends to } \alpha\right\}.$$

3. The Bifurcation Set $B(G)$ and the Homology, Intersection Homology of Varieties $\mathcal{V}_G$ Associated to a Polynomial Mapping $G:\mathbb{C}^n \to \mathbb{C}^{n-1}$

We have the two following theorems dealing with the homology and intersection homology of the variety $\mathcal{V}_G$.

Theorem 3.1. [1] Let $G=(G_1,G_2):\mathbb{C}^3 \to \mathbb{C}^2$ be a polynomial mapping such that $K_0(G)=\varnothing$. If $B(G)\neq\varnothing$ then

1) $H_2(\mathcal{V}_G,\mathbb{R})\neq 0$,

2) $IH_2^{\bar{t}}(\mathcal{V}_G,\mathbb{R})\neq 0$, where $\bar{t}$ is the total perversity.

Theorem 3.2. [1] Let $G=(G_1,\cdots,G_{n-1}):\mathbb{C}^n \to \mathbb{C}^{n-1}$, where $n\geq 4$, be a polynomial mapping such that $K_0(G)=\varnothing$ and $\mathrm{Rank}_{\mathbb{C}}(D\hat{G}_i)_{i=1,\cdots,n-1} \geq n-2$, where $\hat{G}_i$ is the leading form of G_i, that is the homogenous part of highest degree of G_i, for $i=1,\cdots,n-1$. If $B(G)\neq\varnothing$ then

1) $H_2(\mathcal{V}_G,\mathbb{R})\neq 0$,

2) $H_{2n-4}(\mathcal{V}_G,\mathbb{R})\neq 0$,

3) $IH_2^{\bar{t}}(\mathcal{V}_G,\mathbb{R})\neq 0$, where $\bar{t}$ is the total perversity.

Remark 3.3. The singular set at infinity of $\mathcal{V}_G$ depends on the choice of the function ρ, since when ρ changes, the set $\mathcal{S}_G$ also changes. However, we have alway the property $B(G)\subset\mathcal{S}_G(\rho)$ (see [8]).

Remark 3.4. The variety $\mathcal{V}_G$ depends on the choice of the function ρ and the functions ψ_i, but the theorems 3.1 and 3.2 do not depend on the varieties $\mathcal{V}_G$. Form now, we denote by $\mathcal{V}_G(\rho)$ any variety $\mathcal{V}_G$ associated to (G,ρ). If we refer to $\mathcal{V}_G$, that means a variety $\mathcal{V}_G$ associated to (G,ρ) for any ρ.

4. The Bifurcation Set $B(G)$ and the Euler Characteristic of the Fibers of a Polynomial Mapping $G:\mathbb{C}^n \to \mathbb{C}^{n-1}$

Let $G=(G_1,G_2,\cdots,G_{n-1}):\mathbb{C}^n \to \mathbb{C}^{n-1}$ be a non-constant polynomial mapping and $t^0=(t_1^0,t_2^0,\cdots,t_{n-1}^0)\in\mathbb{C}^{n-1}$ be a regular value of G.

Definition 4.1. [2] A linear function $L:\mathbb{C}^n\to\mathbb{C}$ is said to be a very good projection with respect to the value t^0 if there exists a positive number δ such that for all $t\in D_\delta(t^0)=\left\{t=(t_1,t_2,\cdots,t_{n-1})\in\mathbb{C}^{n-1}:\left|t_i-t_i^0\right|<\delta\right\}$:

i) The restriction $L_t:=L|_{G^{-1}(t)}\to\mathbb{C}$ is proper,

ii) The cardinal of $L^{-1}(\lambda)$ does not depend on λ, where λ is a regular value of L.

Theorem 4.2. [2] Let t^0 be a regular value of G. Assume that there exists a very good projection with respect to the value t^0. Then, t^0 is an atypical value of G if and only if the Euler characteristic of $G^{-1}(t^0)$ is bigger than that of the generic fiber.

Theorem 4.3. [2] Assume that the zero set $\left\{z\in\mathbb{C}^n:\hat{G}_i(z)=0, i=1,\cdots,n-1\right\}$, where $\hat{G}_i$ is the leading form of G_i, has complex dimension one. Then any generic linear mapping L is a very good projection with respect to any regular value t^0 of G.

5. Relations between [1] and [2]

Let $G=(G_1,\cdots,G_{n-1}):\mathbb{C}^n\to\mathbb{C}^{n-1}$ $(n\geq 3)$ be a polynomial mapping such that $K_0(G)=\varnothing$. Then any $t^0\in\mathbb{C}^{n-1}$ is a regular value of G. Let $\rho:\mathbb{C}^n\to\mathbb{R}$ be a real function such that $\rho=a_1|z_1|^2+\cdots+a_n|z_n|^2$, where $\sum_{i=1}^n a_i^2\neq 0$, $a_i\geq 0$ and $a_i\in\mathbb{R}$ for $i=1,\cdots,n$. From theorems 3.1 and 4.2, we have the following corollary.

Corollary 5.1. Let $G=(G_1,G_2):\mathbb{C}^3\to\mathbb{C}^2$ be a polynomial mapping such that $K_0(G)=\varnothing$. Assume that there exists a very good projection with respect to $t^0\in\mathbb{C}^2$. If the Euler characteristic of $G^{-1}(t^0)$ is bigger than that of the generic fiber, then

1) $H_2(\mathcal{V}_G(\rho),\mathbb{R})\neq 0$, for any ρ,

2) $IH_2^{\bar{t}}(\mathcal{V}_G(\rho),\mathbb{R})\neq 0$, for any ρ, where $\bar{t}$ is the total perversity.

Proof. Let $G=(G_1,G_2):\mathbb{C}^3\to\mathbb{C}^2$ be a polynomial mapping such that $K_0(G)=\varnothing$. Then every point $t^0\in\mathbb{C}^2$ is a regular point of G. Assume that there exists a very good projection with respect to $t^0\in\mathbb{C}^2$. If the Euler characteristic of $G^{-1}(t^0)$ is bigger than that of the generic fiber, then by the theorem 4.2, the bifurcation set $B(G)$ is not empty. Then by the theorem 3.1, we have $H_2(\mathcal{V}_G(\rho),\mathbb{R})\neq 0$, for any ρ and $IH_2^{\bar{t}}(\mathcal{V}_G(\rho),\mathbb{R})\neq 0$, for any ρ, where $\bar{t}$ is the total perversity. ■

From theorems 3.2 and 4.2, we have the following corollary.

Corollary 5.2. Let $G=(G_1,\cdots,G_{n-1}):\mathbb{C}^n\to\mathbb{C}^{n-1}$, where $n\geq 4$, be a polynomial mapping such that $K_0(G)=\varnothing$ and $\mathrm{Rank}_{\mathbb{C}}\left(D\hat{G}_i\right)_{i=1,\cdots,n-1}\geq n-2$, where $\hat{G}_i$ is the leading form of G_i. Assume that there exists a very good projection with respect to $t^0\in\mathbb{C}^{n-1}$. If the Euler characteristic of $G^{-1}(t^0)$ is bigger than that of the generic fiber, then

1) $H_2(\mathcal{V}_G(\rho),\mathbb{R})\neq 0$, for any ρ,

2) $H_{2n-4}\left(\mathcal{V}_G(\rho),\mathbb{R}\right)\neq 0$, for any ρ,
3) $IH_2^{\overline{t}}\left(\mathcal{V}_G,\mathbb{R}\right)\neq 0$, for any ρ, where $\overline{t}$ is the total perversity.

Proof. Let $G=\left(G_1,\cdots,G_{n-1}\right):\mathbb{C}^n\to\mathbb{C}^{n-1}$, where $n\geq 4$, be a polynomial mapping such that $K_0(G)=\varnothing$. Then every point $t^0\in\mathbb{C}^{n-1}$ is a regular point of *G*. Assume that there exists a very good projection with respect to $t^0\in\mathbb{C}^{n-1}$. By the theorem 4.2, the bifurcation set $B(G)$ is not empty. If $\operatorname{Rank}_{\mathbb{C}}\left(D\hat{G}_i\right)_{i=1,\cdots,n-1}\geq n-2$, then by the theorem 3.2, we have

1) $H_2\left(\mathcal{V}_G(\rho),\mathbb{R}\right)\neq 0$, for any ρ,
2) $H_{2n-4}\left(\mathcal{V}_G(\rho),\mathbb{R}\right)\neq 0$, for any ρ,
3) $IH_2^{\overline{t}}\left(\mathcal{V}_G,\mathbb{R}\right)\neq 0$, for any ρ, where $\overline{t}$ is the total perversity. ■

We have also the following corollary.

Corollary 5.3. Let $G=\left(G_1,\cdots,G_{n-1}\right):\mathbb{C}^n\to\mathbb{C}^{n-1}$, where $n\geq 4$, be a polynomial mapping such that $K_0(G)=\varnothing$. Assume that the zero set $\left\{z\in\mathbb{C}^n:\hat{G}_i(z)=0,i=1,\cdots,n-1\right\}$ has complex dimension one, where $\hat{G}_i$ is the leading form of G_i. If the Euler characteristic of $G^{-1}\left(t^0\right)$ is bigger than that of the generic fiber, where $t^0\in\mathbb{C}^{n-1}$, then

1) $H_2\left(\mathcal{V}_G(\rho),\mathbb{R}\right)\neq 0$, for any ρ,
2) $H_{2n-4}\left(\mathcal{V}_G(\rho),\mathbb{R}\right)\neq 0$, for any ρ,
3) $IH_2^{\overline{t}}\left(\mathcal{V}_G(\rho),\mathbb{R}\right)\neq 0$, for any ρ, where $\overline{t}$ is the total perversity.

Proof. At first, since the zero set $\left\{z\in\mathbb{C}^n:\hat{G}_i(z)=0,i=1,\cdots,n-1\right\}$ has complex dimension one, then by the theorem 4.3, any generic linear mapping *L* is a very good projection with respect to any regular value t^0 of *G*. Moreover, the complex dimension of the set $\left\{z\in\mathbb{C}^n:\hat{G}_i(z)=0,i=1,\cdots,n-1\right\}$ is the complex *corank* of $\left(D\hat{G}_i\right)_{i=1,\cdots,n-1}$. Then $\operatorname{Rank}_{\mathbb{C}}\left(D\hat{G}_i\right)_{i=1,\cdots,n-1}=n-2$. By the corollary 5.2, we get the proof of the corollary 5.3. ■

Remark 5.4. We can construct the variety $\mathcal{V}_G(L)$, where *L* is a very good projection defined in 4.2 as the following: Let $G=\left(G_1,\cdots,G_{n-1}\right):\mathbb{C}^n\to\mathbb{C}^{n-1}$, where $n\geq 2$, be a polynomial mapping such that $K_0(G)=\varnothing$. Assume that there exists a very good projection $L:\mathbb{C}^n\to\mathbb{C}$ with respect to $t^0\in\mathbb{C}^{n-1}$. Then *L* is a linear function. Assume that $L=\sum_{i=1}^n a_i z_i$. Then the variety $\mathcal{V}_G(L)$ is defined as the variety $\mathcal{V}_G(\rho)$, where

$$\rho=\sum_{i=1}^n\left|a_i\right|\left|z_i\right|,$$

with $\left|a_i\right|$, $\left|z_i\right|$ are the modules of the complex numbers a_i and z_i, respectively. With this variety $\mathcal{V}_G(L)$, all the results in the corollaries 5.1, 5.2 and 5.3 hold. Moreover, the varieties $\mathcal{V}_G(L)$ make the corollaries 5.1, 5.2 and 5.3 simpler.

Remark 5.5. In the construction of the variety $\mathcal{V}_G$ [1] (see section 2), if we replace *F* by the restriction of (G,φ) to $\mathcal{M}_G$, that means

$$F:=\left.(G,\varphi)\right|_{\mathcal{M}_G},$$

then we have the same results than in [1]. In fact, in this case, since the dimension of $\mathcal{M}_G$ is $2n-2$, then locally, in a neighbourhood of any point x_0 in $\mathcal{M}_G$, we get a mapping $F:\mathbb{R}^{2n-2}\to\mathbb{R}^{2n-1}$. There exists also a covering $\left\{U_1,\cdots,U_p\right\}$ of $\mathcal{N}_G$ by open semi-algebraic subsets (in $\mathbb{R}^{2n}$) such that on every element of this covering, the mapping F induces a diffeomorphism onto its image. We can find semi-algebraic closed subsets $V_i \subset U_i$ (in $\mathcal{N}_G \setminus U_i$) which cover $\mathcal{N}_G$ as well. Thanks to Mostowski's Separation Lemma, for each $i = 1, \cdots, p$, there exists a Nash function $\psi_i : \mathcal{N}_G \to \mathbb{R}$, such that ψ_i is positive on V_i and negative on $\mathcal{N}_G \setminus U_i$. Let the Nash functions ψ_i and ρ be such that $\psi_i(z_k)$ and $\varphi(z_k) = \frac{1}{1+\rho(z_k)}$ tend to zero where $\{z_k\}$ is a sequence in $\mathcal{N}_G$ tending to infinity. Define a variety $\mathcal{V}_G$ associated to (G, ρ) as

$$\mathcal{V}_G := \overline{(F, \psi_1, \cdots, \psi_p)(\mathcal{N}_G)} = \overline{(G, \varphi, \psi_1, \cdots, \psi_p)(\mathcal{N}_G)}.$$

We get the $(2n-2)$-dimensional singular variety $\mathcal{V}_G$ in $\mathbb{R}^{2n-1+p}$, the singular set at infinity of which is $\mathcal{S}_G \times \{0_{\mathbb{R}^{p+1}}\}$.

With this construction of the set $\mathcal{V}_G$, the corollaries 5.1, 5.2 and 5.3 also hold.

6. Some Discussions

A natural question is to know if the converses of the corollaries 5.1 and 5.2 hold. That means, let $G = (G_1, \cdots, G_{n-1}) : \mathbb{C}^n \to \mathbb{C}^{n-1}$ $(n \geq 3)$ be a polynomial mapping such that $K_0(G) = \varnothing$ then

Question 6.1. If there exists a very good projection with respect to $t^0 \in \mathbb{C}^{n-1}$ and if either $IH_2^{\bar{t}}(\mathcal{V}_G, \mathbb{R}) \neq 0$ or $H_2(\mathcal{V}_G, \mathbb{R}) \neq 0$, then is the Euler characteristic of $G^{-1}(t^0)$ bigger than the one of the generic fiber?

By the theorem 4.2, the above question is equivalent to the following question:

Question 6.2. If $B(G) = \varnothing$ then are $IH_2^{\bar{t}}(\mathcal{V}_G, \mathbb{R}) = 0$ and $IH_2^{\bar{t}}(\mathcal{V}_G, \mathbb{R}) = 0$?

This question is equivalent to the converse of the theorems 3.1 and 3.2. Note that by the proposition 2.2, the singular set at infinity of the variety $\mathcal{V}_G$ is contained in $\mathcal{S}_G(\rho) \times \{0_{\mathbb{R}^{1+p}}\}$. Moreover, in the proofs of the theorems 3.1 and 3.2, we see that the characteristics of the homology and intersection homology of the variety $\mathcal{V}_G(\rho)$ depend on the set $\mathcal{S}_G(\rho)$. In [1], we provided an example to show that the answer to the question 6.2 is negative. In fact, let

$$G : \mathbb{C}^3 \to \mathbb{C}^2, \quad G(z, w, \zeta) = (z, z\zeta^2 + w),$$

then $K_0(G) = \varnothing$ and $B(G) = \varnothing$. if we choose the function $\rho = |\zeta|^2$, then $\mathcal{S}_G(\rho) = \varnothing$ and $IH_2^{\bar{t}}(\mathcal{V}_G(\rho), \mathbb{R}) = 0$; if we choose the function $\rho' = |w|^2$, then $\mathcal{S}_G(\rho') = \varnothing$ and $IH_2^{\bar{t}}(\mathcal{V}_G(\rho'), \mathbb{R}) \neq 0$. Then, we suggest the two following conjectures.

Conjecture 6.3. Does there exist a function ρ such that if $B(G) = \varnothing$ then $\mathcal{S}_G(\rho) = \varnothing$?

Conjecture 6.4. Let $G = (G_1, \cdots, G_{n-1}) : \mathbb{C}^n \to \mathbb{C}^{n-1}$ $(n \geq 2)$ be a polynomial mapping such that $K_0(G) = \varnothing$. Assume that there exists a very good projection with respect to $t^0 \in \mathbb{C}^{n-1}$. If the Euler characteristic of $G^{-1}(t)$ is constant, for any $t \in \mathbb{C}^{n-1}$, then there exists a real positive function $\rho : \mathbb{C}^n \to \mathbb{R}$ such that $H_2(\mathcal{V}_G(\rho), \mathbb{R}) = 0$ and $IH_2^{\bar{t}}(\mathcal{V}_G(\rho), \mathbb{R}) = 0$.

Remark 6.5. The construction of the variety $\mathcal{V}_G$ in [1] (see section 2) can be applied for any polynomial mapping $G:\mathbb{C}^n \to \mathbb{C}^m$, where $1 \le m \le n-2$, such that $K_0(G) = \varnothing$. In fact, if G is generic then similarly to the proposition 2.1, the variety

$$\mathcal{M}_G := Sing(G,\varphi) = \left\{ x \in \mathbb{R}^{2n} \text{ such that } \mathrm{Rank}_{\mathbb{R}} D(G,\varphi)(x) \le 2m \right\},$$

has the real dimension 2 m. Hence, if we consider $F := G|_{\mathcal{M}_G}$, that means F is the restriction of G to $\mathcal{M}_G$, then locally we get a real mapping $F:\mathbb{R}^{2m} \to \mathbb{R}^{2m}$. Moreover, in this case, we also have $B(G) \subset \mathcal{S}_G(\rho)$ for any ρ (see [8]), where

$$\mathcal{S}_G(\rho) := \left\{ \alpha \in \mathbb{C}^m \mid \exists \{z_k\} \subset Sing(G,\rho) : z_k \text{ tends to infinity}, G(z_k) \text{ tends to } \alpha \right\}.$$

So, we can use the same arguments in [1], and we have the following results.

Proposition 6.6. Let $G:\mathbb{C}^n \to \mathbb{C}^m$ be a polynomial mapping, where $1 \le m \le n-2$, such that $K_0(G) = \varnothing$. Let $\rho:\mathbb{C}^n \to \mathbb{R}$ be a real function such that

$$\rho = a_1|z_1|^2 + \cdots + a_n|z_n|^2,$$

where $\sum_{i=1}^n a_i^2 \ne 0$, $a_i \ge 0$ and $a_i \in \mathbb{R}$ for $i = 1,\cdots,n$. Then, there exists a real variety $\mathcal{V}_G$ in $\mathbb{R}^{2m+p}$, where $p > 0$, such that:

1) The real dimension of $\mathcal{V}_G$ is 2 m,

2) The singular set at infinity of the variety $\mathcal{V}_G$ is contained in $\mathcal{S}_G(\rho) \times \left\{0_{\mathbb{R}^p}\right\}$.

Similarly to [1], we have the two following theorems (see theorems 3.1 and 3.2).

Theorem 6.7. Let $G = (G_1, G_2):\mathbb{R}^n \to \mathbb{R}^2$, where $n \ge 4$, be a polynomial mapping such that $K_0(G) = \varnothing$. If $B(G) \ne \varnothing$ then

1) $H_2(\mathcal{V}_G, \mathbb{R}) \ne 0$,

2) $IH_2^{\bar{t}}(\mathcal{V}_G, \mathbb{R}) \ne 0$, where $\bar{t}$ is the total perversity.

Theorem 6.8. Let $G = (G_1,\cdots,G_m):\mathbb{C}^n \to \mathbb{C}^m$, where $3 \le m \le n-2$, be a polynomial mapping such that $K_0(G) = \varnothing$. Assume that $\mathrm{Rank}_{\mathbb{C}}\left(D\hat{G}_i\right)_{i=1,\cdots,m} \ge m-1$, where $\hat{G}_i$ is the leading form of G_i. If $B(G) \ne \varnothing$ then

1) $H_2(\mathcal{V}_G(\rho), \mathbb{R}) \ne 0$, for any ρ,

2) $H_{2n-4}(\mathcal{V}_G(\rho), \mathbb{R}) \ne 0$, for any ρ,

3) $IH_2^{\bar{t}}(\mathcal{V}_G(\rho), \mathbb{R}) \ne 0$, for any ρ, where $\bar{t}$ is the total perversity.

7. Examples

Example 7.1. We give here an example to illustrate the calculations of the set $\mathcal{V}_G$ in the case of a polynomial mapping $G:\mathbb{C}^2 \to \mathbb{C}$ where $K_0(G) = \varnothing$, $B(G) \ne \varnothing$ and there exists a very good projection with respect to any point of $B(G)$. In general, the calculations of the set $\mathcal{V}_G$ are enough complicate, but the software *Maple* may support us. That is what we do in this example.

Let us consider the Broughton's example [9]:

$$G:\mathbb{C}^2 \to \mathbb{C}, \qquad G(z,w) = z + z^2 w.$$

We have $K_0(G) = \varnothing$ and $B(G) \ne \varnothing$. In fact, since the system of equations $\frac{\delta G}{\delta z} = \frac{\delta G}{\delta w} = 0$ has no solutions, then $K_0(G) = \varnothing$. Moreover,

$$G^{-1}(0)=\{(z,w): z=0 \text{ or } zw=-1\}\cong\mathbb{C}\sqcup(\mathbb{C}\setminus\{0\}),$$

and for any $\epsilon\neq 0$, we have

$$G^{-1}(\epsilon)=\{(z,w): z\neq 0 \text{ and } w=(\epsilon-z)/z^2\}\cong\mathbb{C}\setminus\{0\}.$$

So $G^{-1}(0)$ is not homeomorphic to $G^{-1}(\epsilon)$ for any $\epsilon\neq 0$. Hence $B(G)=\{0\}$. We determine now all the possible *very good projections* of G with respect to $t^0=0\in B(G)$. In fact, for any $\delta>0$ and for any $t\in D_\delta(0)$, we have

$$G^{-1}(t)=\{(z,w)\in\mathbb{C}^2: z+z^2w=t\neq 0\}=\left\{(z,w)\in\mathbb{C}^2: z\neq 0 \text{ and } w=\frac{t-z}{z^2}\right\}.$$

Assume that $\{(z_k,w_k)\}$ is a sequence in $G^{-1}(t)$ tending to infinity. If z_k tends to infinity then w_k tends to zero. If w_k tends to infinity then z_k tends to zero. If L is a very good projection with respect to $t^0=0$ then, by definition, the restriction $L_t:=L|_{G^{-1}(t)}\to\mathbb{C}$ is proper. Then $L=az+bw$, where $a\neq 0$ and $b\neq 0$. We check now the cardinal $\#L^{-1}(\lambda)$ of $L^{-1}(\lambda)$ where λ is a regular value of L. Let us replace $w=\dfrac{t-z}{z^2}$ in the equation $az+bw=\lambda$, we have the following equation

$$az+b\frac{t-z}{z^2}=0,$$

where $z\neq 0$. This equation always has three (complex) solutions. Thus, the number $\#L^{-1}(\lambda)$ does not depend on λ. Hence, any linear function of the form $L=az+bw$, where $a\neq 0$ and $b\neq 0$, is a very good projection of G with respect to $t^0=0$. It is easy to see that the set of very good projections of G with respect to $t^0=0$ is dense in the set of linear functions.

We choose $L=z+w$ and we compute the variety $\mathcal{V}_G$ associated to (G,ρ) where $\rho=|z|^2+|w|^2$. Let us denote

$$z=x_1+ix_2,\quad w=x_3+ix_4,$$

where $x_1,x_2,x_3,x_4\in\mathbb{R}$. Consider G as a real polynomial mapping, we have

$$G(x_1,x_2,x_3,x_4)=\left(x_1+x_1^2x_3-x_2^2x_3-2x_1x_2x_4,\ x_2+2x_1x_2x_3+x_1^2x_4-x_2^2x_4\right),$$

and

$$\varphi=\frac{1}{1+\rho}=\frac{1}{|z|^2+|w|^2}=\frac{1}{x_1^2+x_2^2+x_3^2+x_4^2}.$$

The set $\mathcal{N}_G=Sing(G,\rho)$ is the set of the solutions of the determinant of the minors 3×3 of the matrix

$$D_{\mathbb{R}}(G,\rho)=\begin{pmatrix}1+2x_1x_3-2x_2x_4 & -2x_2x_3-2x_1x_4 & x_1^2-x_2^2 & -2x_1x_2\\ 2x_2x_3+2x_1x_4 & 1+2x_1x_3-2x_2x_4 & 2x_1x_2 & x_1^2-x_2^2\\ x_1 & x_2 & x_3 & x_4\end{pmatrix}.$$

Using *Maple*, we:

A) Calculate the determinants of the minors 3×3 of the matrix $D_{\mathbb{R}}(G,\rho)$:

1) Calculate the determinant of the minor defined by the columns 1, 2 and 3:

with(*linalg*) :
with(*LinearAlgebra*) :

$\mathrm{M}_{123} := \mathrm{Matrix}\Big(\big[\big[1 + 2\cdot x_1\cdot x_3 - 2\cdot x_2\cdot x_4, -2\cdot x_2\cdot x_3 - 2\cdot x_1\cdot x_4, x_1^2 - x_2^2\big], \big[2\cdot x_2\cdot x_3 + 2\cdot x_1\cdot x_4, 1 + 2\cdot x_1\cdot x_3 - 2\cdot x_2\cdot x_4, 2\cdot x_1\cdot x_2\big], [x_1, x_2, x_3]\big]\Big);$

$$\begin{bmatrix} 1 + 2 x_1 x_3 - 2 x_2 x_4 & -2 x_2 x_3 - 2 x_1 x_4 & x_1^2 - x_2^2 \\ 2 x_2 x_3 + 2 x_1 x_4 & 1 + 2 x_1 x_3 - 2 x_2 x_4 & 2 x_1 x_2 \\ x_1 & x_2 & x_3 \end{bmatrix} \tag{1}$$

$\mathrm{Determinant}(\mathrm{M}_{123}, \mathrm{method} = \mathrm{multivar});$

$$-2 x_3 x_1^4 - x_1^3 - 4 x_1^2 x_2^2 x_3 + 4 x_1^2 x_3^3 + 4 x_1^2 x_3 x_4^2 + 4 x_1 x_3^2 - x_1 x_2^2 - 4 x_3 x_2 x_4 + 4 x_3 x_2^2 x_4^2 + x_3 - 2 x_2^4 x_3 + 4 x_3^3 x_2^2 \tag{2}$$

2) Calculate the determinant of the minor defined by the columns 1, 2 and 4:

with(*linalg*) :
with(*LinearAlgebra*) :

$\mathrm{M}_{124} := \mathrm{Matrix}\Big(\big[\big[1 + 2\cdot x_1\cdot x_3 - 2\cdot x_2\cdot x_4, -2\cdot x_2\cdot x_3 - 2\cdot x_1\cdot x_4, -2\cdot x_1\cdot x_2\big], \big[2\cdot x_2\cdot x_3 + 2\cdot x_1\cdot x_4, 1 + 2\cdot x_1\cdot x_3 - 2\cdot x_2\cdot x_4, x_1^2 - x_2^2\big], [x_1, x_2, x_4]\big]\Big);$

$$\begin{bmatrix} 1 + 2 x_1 x_3 - 2 x_2 x_4 & -2 x_2 x_3 - 2 x_1 x_4 & -2 x_1 x_2 \\ 2 x_2 x_3 + 2 x_1 x_4 & 1 + 2 x_1 x_3 - 2 x_2 x_4 & x_1^2 - x_2^2 \\ x_1 & x_2 & x_4 \end{bmatrix} \tag{1}$$

$\mathrm{Determinant}(\mathrm{M}_{124}, \mathrm{method} = \mathrm{multivar});$

$$-2 x_4 x_1^4 + 4 x_1^2 x_4 x_3^2 - 4 x_1^2 x_2^2 x_4 + 4 x_1^2 x_4^3 + x_1^2 x_2 + 4 x_4 x_1 x_3 + 4 x_4 x_3^2 x_2^2 + 4 x_2^2 x_4^3 - 2 x_2^4 x_4 - 4 x_2 x_4^2 + x_2^3 + x_4 \tag{2}$$

3) Calculate the determinant of the minor defined by the columns 1, 3 and 4:

with(*linalg*) :
with(*LinearAlgebra*) :

$\mathrm{M}_{134} := \mathrm{Matrix}\Big(\big[\big[1 + 2\cdot x_1\cdot x_3 - 2\cdot x_2\cdot x_4, x_1^2 - x_2^2, -2\cdot x_1\cdot x_2\big], \big[2\cdot x_2\cdot x_3 + 2\cdot x_1\cdot x_4, 2\cdot x_1\cdot x_2, x_1^2 - x_2^2\big], [x_1, x_3, x_4]\big]\Big);$

$$\begin{bmatrix} 1 + 2 x_1 x_3 - 2 x_2 x_4 & x_1^2 - x_2^2 & -2 x_1 x_2 \\ 2 x_2 x_3 + 2 x_1 x_4 & 2 x_1 x_2 & x_1^2 - x_2^2 \\ x_1 & x_3 & x_4 \end{bmatrix} \tag{1}$$

$\mathrm{Determinant}(\mathrm{M}_{134}, \mathrm{method} = \mathrm{multivar});$

$$x_1^5 + 2 x_1^3 x_2^2 - 2 x_1^3 x_3^2 - 2 x_1^3 x_4^2 - x_3 x_1^2 - 2 x_1 x_2^2 x_3^2 + 2 x_1 x_2 x_4 - 2 x_1 x_2^2 x_4^2 + x_1 x_2^4 + x_2^2 x_3 \tag{2}$$

4) Calculate the determinant of the minor defined by the columns 2, 3 and 4:

$with(linalg):$
$with(LinearAlgebra):$

$M_{234} := \mathrm{Matrix}\left(\left[\left[-2\cdot x_2\cdot x_3 - 2\cdot x_1\cdot x_4, x_1^2 - x_2^2, -2\cdot x_1\cdot x_2,\right], \left[1 + 2\cdot x_1\cdot x_3 - 2\cdot x_2\cdot x_4, 2\cdot x_1\cdot x_2, x_1^2 - x_2^2\right], \left[x_2, x_3, x_4\right]\right]\right);$

$$\begin{bmatrix} -2\,x_2\,x_3 - 2\,x_1\,x_4 & x_1^2 - x_2^2 & -2\,x_1\,x_2 \\ 1 + 2\,x_1\,x_3 - 2\,x_2\,x_4 & 2\,x_1\,x_2 & x_1^2 - x_2^2 \\ x_2 & x_3 & x_4 \end{bmatrix} \quad \mathbf{(1)}$$

$\mathrm{Determinant}(M_{234}, \mathrm{method} = \mathrm{multivar});$

$$x_2^5 - 2\,x_2^3\,x_3^2 - 2\,x_2^3\,x_4^2 + 2\,x_2^3\,x_1^2 + x_2^2\,x_4 - 2\,x_2\,x_1^2\,x_3^2 - 2\,x_2\,x_4^2\,x_1^2 - 2\,x_1\,x_2\,x_3 + x_2\,x_1^4 - x_1^2\,x_4 \quad \mathbf{(2)}$$

B) Solve now the system of equations of the above determinants:

$with(linalg):$
$with(LinearAlgebra):$

$> solve\left(\left\{-2\cdot x_3\cdot x_1^4 - x_1^3 - 4\cdot x_1^2\cdot x_2^2\cdot x_3 + 4\cdot x_1^2\cdot x_3^3 + 4\cdot x_1^2\cdot x_3\cdot x_4^2 + 4\cdot x_1\cdot x_3^2 - x_1\cdot x_2^2 - 4\cdot x_3\cdot x_2\cdot x_4 + 4\cdot x_3\cdot x_2^2\cdot x_4^2 + x_3 - 2\cdot x_2^4\cdot x_3 + 4\cdot x_3^3\cdot x_2^2 = 0, -2\cdot x_4\cdot x_1^4 + 4\cdot x_1^2\cdot x_4\cdot x_3^2 - 4\cdot x_1^2\cdot x_2^2\cdot x_4 + 4\cdot x_1^2\cdot x_4^3 + x_1^2\cdot x_2 + 4\cdot x_4\cdot x_1\cdot x_3 + 4\cdot x_4\cdot x_3^2\cdot x_2^2 + 4\cdot x_2^2\cdot x_4^3 - 2\cdot x_2^4\cdot x_4 - 4\cdot x_2\cdot x_4^2 + x_2^3 + x_4 = 0, x_1^5 + 2\cdot x_1^3\cdot x_2^2 - 2\cdot x_1^3\cdot x_3^2 - 2\cdot x_1^3\cdot x_4^2 - x_1^2\cdot x_3 - 2\cdot x_1\cdot x_2^2\cdot x_3^2 + 2\cdot x_1\cdot x_2\cdot x_4 - 2\cdot x_1\cdot x_2^2\cdot x_4^2 + x_1\cdot x_2^4 + x_2^2\cdot x_3 = 0, x_2^5 - 2\cdot x_2^3\cdot x_3^2 - 2\cdot x_2^3\cdot x_4^2 + 2\cdot x_2^3\cdot x_1^2 + x_2^2\cdot x_4 - 2\cdot x_2\cdot x_1^2\cdot x_3^2 - 2\cdot x_2\cdot x_4^2\cdot x_1^2 - 2\cdot x_1\cdot x_2\cdot x_3 + x_2\cdot x_1^4 - x_1^2\cdot x_4 = 0\right\}, \{x_1, x_2, x_3, x_4\}\right);$

$$\left\{x_1 = x_1, x_2 = 0, x_3 = RootOf\left(\_Z - x_1^3 + 2\,x_1\,\_Z^2\right), x_4 = 0\right\}, \left\{x_1 = x_1, x_2 = RootOf\left(\_Z^2 + 1\right)x_1, x_3 = 0, x_4 = 0\right\}, \left\{x_1 = x_1, x_2 = x_2, x_3 = RootOf\left(\left(2\,x_1^2 + 2\,x_2^2\right)\_Z^2 - x_1^2 - x_2^2 + \_Z\right)x_1, x_4 = -RootOf\left(\left(2\,x_1^2 + 2\,x_2^2\right)\_Z^2 - x_1^2 - x_2^2 + \_Z\right)x_2\right\} \quad \mathbf{(1)}$$

We conclude that $\mathcal{N}_G = N_1 \cup N_2 \cup N_3$, where

$$N_1 = \left\{x_1 = x_1, x_2 = 0, x_3 = RootOf\left(\_Z - x_1^3 + 2x_1\_Z^2\right), x_4 = 0\right\},$$

$$N_2 = \left\{x_1 = x_1, x_2 = RootOf\left(\_Z^2 + 1\right)x_1, x_3 = 0, x_4 = 0\right\},$$

$$N_3 = \left\{x_1 = x_1, x_2 = x_2, x_3 = RootOf\left(\left(2x_1^2 + 2x_2^2\right)\_Z^2 - x_1^2 - x_2^2 + \_Z\right)x_1, x_4 = -RootOf\left(\left(2x_1^2 + 2x_2^2\right)\_Z^2 - x_1^2 - x_2^2 + \_Z\right)x_2\right\}$$

C) In order to calculate $\mathcal{V}_G$, we have to calculate and draw $(G,\varphi)(N_i)$, for $i = 1, 2, 3$.

1) Calculate and draw $(G,\varphi)(N_1)$:

> $solve\left(\left\{x_1 = x_1, x_2 = 0, x_3 = RootOf\left(\_Z - x_1^3 + 2\,x_1\,\_Z^2\right), x_4 = 0, a = x_1 + x_1^2 \cdot x_3 - x_2^2 \cdot x_3 - 2\,x_1 \cdot x_2 \cdot x_4, b = x_2 + 2\,x_1 \cdot x_2 \cdot x_3 + x_1^2 \cdot x_4 - x_2^2 \cdot x_4,\ c = \frac{1}{1 + x_1^2 + x_2^2 + x_3^2 + x_4^2}\right\}, \{x_1, x_2, x_3, x_4, a, b, c\}\right);$

$$\left\{a = x_1 + x_1^2\,RootOf\left(\_Z - x_1^3 + 2\,x_1\,\_Z^2\right), b = 0, c = RootOf\left(\left(1 + 5\,x_1^2 + 12\,x_1^4 + 9\,x_1^6\right)\_Z^2 + \left(-8\,x_1^2 - 1 - 12\,x_1^4\right)\_Z + 4\,x_1^2\right), x_1 = x_1, x_2 = 0, x_3 = RootOf\left(\_Z - x_1^3 + 2\,x_1\,\_Z^2\right), x_4 = 0\right\} \qquad \textbf{(1)}$$

> $plot3d\left(\left[x_1 + x_1^3\,RootOf\left(\_Z - x_1^3 + 2\,x_1\,\_Z^2\right), 0, RootOf\left(\left(1 + 5\,x_1^2 + 12\,x_1^4 + 9\,x_1^6\right)\_Z^2 + \left(-8\,x_1^2 - 1 - 12\,x_1^4\right)\_Z + 4\,x_1^2\right)\right], x_1 = -5..5, x_1 = -5..5\right);$

2) Calculate and draw $(G,\varphi)(N_2)$:

> $solve\left(\left\{x_1 = x_1, x_2 = RootOf\left(\_Z^2 + 1\right)x_1, x_3 = 0, x_4 = 0, a = x_1 + x_1^2 \cdot x_3 - x_2^2 \cdot x_3 - 2\,x_1 \cdot x_2 \cdot x_4, b = x_2 + 2\,x_1 \cdot x_2 \cdot x_3 + x_1^2 \cdot x_4 - x_2^2 \cdot x_4,\ c = \frac{1}{1 + x_1^2 + x_2^2 + x_3^2 + x_4^2}\right\}, \{x_1, x_2, x_3, x_4, a, b, c\}\right);$

$$\left\{a = x_1, b = RootOf\left(\_Z^2 + 1\right)x_1, c = 1, x_1 = x_1, x_2 = RootOf\left(\_Z^2 + 1\right)x_1, x_3 = 0, x_4 = 0\right\} \qquad \textbf{(1)}$$

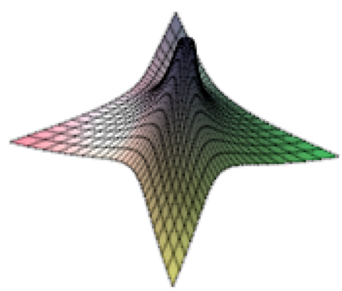
> $plot3d\left(\left[0, x_2, 1\right], x_2 = -2..2, x_2 = -2..2.\right);$

+ Calculate and draw $(G,\varphi)(N_3)$:

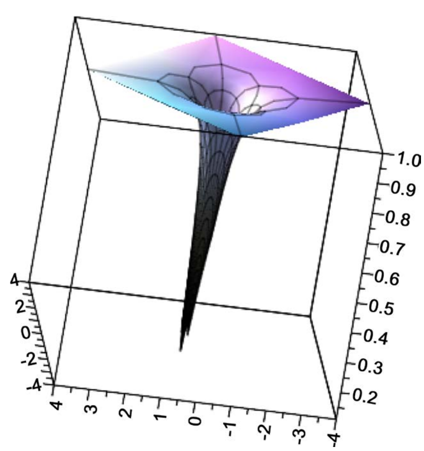
> $solve\left(\left\{x_1 = x_1, x_2 = x_2, x_3 = RootOf\left(\left(2\,x_1^2 + 2\,x_2^2\right)\_Z^2 - x_1^2 - x_2^2 + \_Z\right)x_1, x_4 = -RootOf\left(\left(2\,x_1^2 + 2\,x_2^2\right)\_Z^2 - x_1^2 - x_2^2 + \_Z\right)x_2, a = x_1 + x_1^2\cdot x_3 - x_2^2\cdot x_3 - 2\,x_1\cdot x_2\cdot x_4, b = x_2 + 2\,x_1\cdot x_2\cdot x_3 + x_1^2\cdot x_4 - x_2^2\cdot x_4,\ c = \frac{1}{1 + x_1^2 + x_2^2 + x_3^2 + x_4^2}\right\}, \{x_1, x_2, x_3, x_4, a, b, c\}\right);$

$$\left\{a = x_1 + x_1^3\,RootOf\left(\left(2\,x_1^2 + 2\,x_2^2\right)\_Z^2 - x_1^2 - x_2^2 + \_Z\right) + x_2^2\,RootOf\left(\left(2\,x_1^2 + 2\,x_2^2\right)\_Z^2 - x_1^2 - x_2^2 + \_Z\right)x_1, b = x_2 + x_1^2\,x_2\,RootOf\left(\left(2\,x_1^2 + 2\,x_2^2\right)\_Z^2 - x_1^2 - x_2^2 + \_Z\right) + x_2^3\,RootOf\left(\left(2\,x_1^2 + 2\,x_2^2\right)\_Z^2 - x_1^2 - x_2^2 + \_Z\right), c = RootOf\left(\left(27\,x_1^4\,x_2^2 + 24\,x_1^2\,x_2^2 + 27\,x_1^2\,x_2^4 + 1 + 5\,x_1^2 + 5\,x_2^2 + 12\,x_1^4 + 12\,x_2^4 + 9\,x_1^6 + 9\,x_2^6\right)\_Z^2 + \left(-12\,x_1^4 - 12\,x_2^4 - 8\,x_1^2 - 8\,x_2^2 - 24\,x_1^2\,x_2^2 - 1\right)\_Z + 4\,x_1^2 + 4\,x_2^2\right), x_1 = x_1, x_2 = x_2, x_3 = RootOf\left(\left(2\,x_1^2 + 2\,x_2^2\right)\_Z^2 - x_1^2 - x_2^2 + \_Z\right)x_1, x_4 = -RootOf\left(\left(2\,x_1^2 + 2\,x_2^2\right)\_Z^2 - x_1^2 - x_2^2 + \_Z\right)x_2\right\} \tag{1}$$

> $plot3d\left(\left[x_1 + x_1^3\,RootOf\left(\left(2\,x_1^2 + 2\,x_2^2\right)\_Z^2 - x_1^2 - x_2^2 + \_Z\right) + x_2^2\,RootOf\left(\left(2\,x_1^2 + 2\,x_2^2\right)\_Z^2 - x_1^2 - x_2^2 + \_Z\right)x_1, x_2 + x_1^2\,x_2\,RootOf\left(\left(2\,x_1^2 + 2\,x_2^2\right)\_Z^2 - x_1^2 - x_2^2 + \_Z\right) + x_2^3\,RootOf\left(\left(2\,x_1^2 + 2\,x_2^2\right)\_Z^2 - x_1^2 - x_2^2 + \_Z\right), RootOf\left(\left(27\,x_1^4\,x_2^2 + 24\,x_1^2\,x_2^2 + 27\,x_1^2\,x_2^4 + 1 + 5\,x_1^2 + 5\,x_2^2 + 12\,x_1^4 + 12\,x_2^4 + 9\,x_1^6 + 9\,x_2^6\right)\_Z^2 + \left(-12\,x_1^4 - 12\,x_2^4 - 8\,x_1^2 - 8\,x_2^2 - 24\,x_1^2\,x_2^2 - 1\right)\_Z + 4\,x_1^2 + 4\,x_2^2\right)\right], x_1 = -2..2, x_2 = -2..2\right);$



Since $\mathcal{V}_G$ is the closure of $\bigcup_{i=1}^{3}(G,\varphi)(N_i)$ then $\mathcal{V}_G$ is connected and has a pure dimension, then V_F is a cone:



Example 7.2. If we take *the suspension* of the Broughton's example

$$G:\mathbb{C}^3 \to \mathbb{C}^2, \quad G(z,w,\eta) = (z + z^2 w, \eta),$$

then, similarly to the example 7.1, the variety $\mathcal{V}_G$ is a cone as in the example 7.1 but it has dimension 4, in the space $\mathbb{R}^6$. We can check easily that the intersection homology in dimension 2 of the variety $\mathcal{V}_G$ of this example is non-trivial. We get an example to illustrate the corollary 5.1.

Example 7.3. If we take the Broughton example for $G:\mathbb{C}^3 \to \mathbb{C}$ such that $G(z,w,\eta) = z + z^2 w$, then similarly to the example 7.1, we get an example of varieties $\mathcal{V}_G$ for the case $G:\mathbb{C}^n \to \mathbb{C}^m$ where $m \le n-2$. This example illustrates the remark 6.5.

References

[1] Nguyen, T.B.T. and Ruas, M.A.S. (2015) On a Singular Variety Associated to a Polynomial Mapping. ArXiv:1503.08079.

[2] Ha, V.H. and Nguyen, T.T. (2011) On the Topology of Polynomial Mappings from $\mathbb{C}^n$ to $\mathbb{C}^{n-1}$. *International Journal of Mathematics*, **22**, 435-448. http://dx.doi.org/10.1142/S0129167X11006842

[3] Goresky, M. and MacPherson, R. (1980) Intersection Homology Theory. *Topology*, **19**, 135-162. http://dx.doi.org/10.1016/0040-9383(80)90003-8

[4] Brasselet, J.-P. (2015) Introduction to Intersection Homology and Perverse Sheaves. Instituto de Matemática Pura e Aplicada, Rio de Janeiro.

[5] Kurdyka, K., Orro, P. and Simon, S. (2000) *Semialgebraic Sard Theorem for Generalized Critical Values*. *Journal of Differential Geometry*, **56**, 67-92.

[6] Jelonek, Z. (1993) The Set of Point at Which Polynomial Map Is Not Proper. *Annales Polonici Mathematici*, **58**, 259-266.

[7] Mostowski, T. (1976) Some Properties of the Ring of Nash Functions. *Annali della Scuola normale superiore di Pisa*, **2**, 245-266.

[8] Dias, L.R.G., Ruas, M.A.S. and Tibar, M. (2012) Regularity at Infinity of Real Mappings and a Morse-Sard Theorem. *Journal of Topology*, **5**, 323-340. http://dx.doi.org/10.1112/jtopol/jts005

[9] Broughton, A. (1981) On the Topology of Polynomial Hypersurfaces. *Proc. Sympos. Pure Math.*, Vol. 40, 167-178.