Applied Mathematics, 2011, 2, 1019-1026
doi:10.4236/am.2011.28141 Published Online August 2011 (http://www.SciRP.org/journal/am)

Intersection Curves of Implicit and Parametric Surfaces in $\mathbb{R}^3$

Mohamed Abdel-Latif Soliman, Nassar Hassan Abdel-All, Soad Ali Hassan, Sayed Abdel-Naeim Badr
Department of Mathematics, *Faculty of Science*, *Assiut University*, *Assiut*, *Egypt*
E-mail: *sayed_badr@ymail.com*
Received June 9, 2011; *revised June* 30, 2011; *accepted July* 7, 2011

Abstract

We present algorithms for computing the differential geometry properties of Frenet apparatus (t, n, b, κ, τ) and higher-order derivatives of intersection curves of implicit and parametric surfaces in $\mathbb{R}^3$ for transversal and tangential intersection. This work is considered as a continuation to Ye and Maekawa [1]. We obtain a classification of the singularities on the intersection curve. Some examples are given and plotted.

Keywords: Geometric Properties, Frenet Frame, Frenet Apparatus, Frenet-Serret Formulas, Surface-Surface Intersection, Transversal Intersection, Tangential Intersection, Dupin Indicatrices

1. Introduction

The intersection problem is a fundamental process needed in modeling complex shapes in CAD/CAM system. It is useful in the representation of the design of complex objects, in computer animation and in NC machining for trimming off the region bounded by the self-intersection curves of offset surfaces. It is also essential to Boolean operations necessary in the creation of boundary representation in solid modeling [1]. The numerical marching method is the most widely used method for computing intersection curves in $\mathbb{R}^3$. The Marching method involves generation of sequences of points of an intersection curve in the direction prescribed by the local differential geometry [2,3]. Willmore [4] described how to obtain the unit tangent, the unit principal normal, the unit binormal, the curvature and the torsion of the transversal intersection curve of two implicit surfaces [5]. Kruppa [6] explained that the tangential direction of the intersection curve at a tangential intersection point corresponds to the direction from the intersection point towards the intersection of the Dupin indicatrices of the two surfaces. Hartmann [7] provided formulas for computing the curvature of the transversal intersection curves for all types of intersection problems in Euclidean 2-space. Kriezis *et al.* [8] determined the marching direction for tangential intersection curves based on the fact that the determinant of the Hessian matrix of the oriented distance function is zero. Luo *et al.* [9] presented a method to trace such tangential intersection curves for parametric-parametric surfaces employing the marching method. The marching direction is obtained by solving an undetermined system based on the equilibrium of the differentiation of the two normal vectors and the projection of the Taylor expansion of the two surfaces onto the normal vector at the intersection point. Ye and Maekawa [1] presented algorithms for computing all the differential geometry properties of both transversal and tangentially intersection curves of two parametric surfaces. They described how to obtain these properties for two implicit surfaces or parametric-implicit surfaces. They also gave algorithms to evaluate the higher-order derivative of the intersection curves. Aléssio [10] studied the differential geometry properties of intersection curves of three implicit surfaces in $\mathbb{R}^4$ for transversal intersection, using the implicit function theorem.

In this study, we present algorithms for computing the deferential geometry properties of both transversal and tangentially intersection curves of implicit and Parametric surfaces in $\mathbb{R}^3$ as an extension to the works of [1].

This paper is organized as follows: Section 2 briefly introduces some notations, definitions and reviews of differential geometry properties of curves and surfaces in $\mathbb{R}^3$. Section 3 derives the formulas to compute the properties for the transversal intersection. Section 4 derives the formulas to compute the properties for the tangential intersection. Some examples of transversal and tangentially intersection are given and plotted in Section 5. Finally, conclusion is given in Section 6.

Copyright © 2011 SciRes.

2. Geometric Preliminaries [1, 11-13]

Let us first introduce some notation and definitions. The scalar product and cross product of two vectors $\boldsymbol{a}$ and $\boldsymbol{c}$ are expressed as $\langle \boldsymbol{a},\boldsymbol{c}\rangle$ and $\boldsymbol{a}\times\boldsymbol{c}$, respectively. The length of the vector $\boldsymbol{a}$ is $\|\boldsymbol{a}\| = \sqrt{\langle \boldsymbol{a},\boldsymbol{a}\rangle}$.

2.1. Differential Geometry of the Curves in $\mathbb{R}^3$

Let $\alpha : \mathrm{I} \to \mathbb{R}^3$ be a regular curve in $\mathbb{R}^3$ with arc-length parameterization,

$$\boldsymbol{\alpha}(s) = \left(x_1(s), x_2(s), x_3(s)\right) \tag{2.1}$$

The notation for the differentiation of the curve α in relation to the arc length s is $\boldsymbol{\alpha}'(s) = \frac{\mathrm{d}\alpha}{\mathrm{d}s}$, $\boldsymbol{\alpha}''(s) = \frac{\mathrm{d}^2\alpha}{\mathrm{d}s^2}$, $\boldsymbol{\alpha}'''(s) = \frac{\mathrm{d}^3\alpha}{\mathrm{d}s^3}$. Then from elementary differential geometry, we have

$$\boldsymbol{\alpha}'(s) = \boldsymbol{t} \tag{2.2}$$

$$\boldsymbol{\alpha}''(s) = \kappa \boldsymbol{n} \tag{2.3}$$

$$\kappa^2(s) = \langle \boldsymbol{\alpha}'', \boldsymbol{\alpha}''\rangle \tag{2.4}$$

where $\boldsymbol{t}$ is the unit tangent vector field and $\boldsymbol{\alpha}''$ is the curvature vector. The factor κ is the curvature and $\boldsymbol{n}$ is the unit principal normal vector. The unit binormal vector $\boldsymbol{b}$ is defined as

$$\boldsymbol{b}(s) = \boldsymbol{t}\times\boldsymbol{n} \tag{2.5}$$

The vectors $\boldsymbol{t},\boldsymbol{n},\boldsymbol{b},$ are called collectively the Frenet-Serret frame. The Frenet–Serret formulas along $\boldsymbol{\alpha}$ are given by

$$\begin{aligned} \boldsymbol{t}'(s) &= \kappa\boldsymbol{n}, \\ \boldsymbol{n}'(s) &= -\kappa\boldsymbol{t} + \tau\boldsymbol{b}, \\ \boldsymbol{b}'(s) &= -\tau\boldsymbol{n}. \end{aligned} \tag{2.6}$$

where τ is the torsion which is given by

$$\tau = \frac{\langle \boldsymbol{b}, \boldsymbol{\alpha}'''\rangle}{\kappa} \tag{2.7}$$

provided that the curvature does not vanish.

2.2. Differential Geometry of the Parametric Surfaces in $\mathbb{R}^3$

Assume that $\boldsymbol{R}(u_1, v_2)$ is a regular parametric surface. In other words $\boldsymbol{R}_1\times\boldsymbol{R}_2 \neq 0,$ where $\boldsymbol{R}_r = \frac{\partial \boldsymbol{R}}{\partial u_r} (r = 1,2)$ denote to partial derivatives of the surface R. The unit surface normal vector field of the surface $\boldsymbol{R}$ is given by

$$\boldsymbol{N} = \frac{\boldsymbol{R}_1\times\boldsymbol{R}_2}{\|\boldsymbol{R}_1\times\boldsymbol{R}_2\|} \tag{2.8}$$

The first fundamental form coefficients of the surface $\boldsymbol{R}$ are given by

$$g_{pq} = \langle \boldsymbol{R}_p, \boldsymbol{R}_q\rangle;\ p,q = 1,2 \tag{2.9}$$

The second fundamental form coefficients of the surface $\boldsymbol{R}$ are given by

$$L_{11} = \langle \boldsymbol{R}_{11}, \boldsymbol{N}\rangle, L_{12} = \langle \boldsymbol{R}_{12}, \boldsymbol{N}\rangle, L_{22} = \langle \boldsymbol{R}_{22}, \boldsymbol{N}\rangle \tag{2.10}$$

Let $u_r(s)$, $r = 1,2$ in the u_1u_2-plane defines a curve on the surface $\boldsymbol{R}$ which can be written as

$$\boldsymbol{\alpha}(s) = \boldsymbol{R}\left(u_1(s), u_2(s)\right) \tag{2.11}$$

Then the three derivatives of the curve $\boldsymbol{\alpha}$ are given by

$$\boldsymbol{\alpha}' = \boldsymbol{R}_1 u_1' + \boldsymbol{R}_2 u_2' \tag{2.12}$$

$$\boldsymbol{\alpha}'' = \boldsymbol{R}_{11}(u_1')^2 + 2\boldsymbol{R}_{12}u_1'u_2' + \boldsymbol{R}_{22}(u_2')^2 + \boldsymbol{R}_1 u_1'' + \boldsymbol{R}_2 u_2'' \tag{2.13}$$

$$\begin{aligned} \boldsymbol{\alpha}''' &= \boldsymbol{R}_1 u_1''' + \boldsymbol{R}_2 u_2''' + \boldsymbol{R}_{111}(u_1')^3 + \boldsymbol{R}_{222}(u_2')^3 \\ &+ 3\left(\boldsymbol{R}_{11}u_1'u_1'' + \boldsymbol{R}_{12}\left(u_1''u_2' + u_1'u_2''\right) + \boldsymbol{R}_{22}u_2'u_2''\right) \\ &+ 3\boldsymbol{R}_{112}(u_1')^2 u_2' + 3\boldsymbol{R}_{122}u_1'(u_2')^2 \end{aligned} \tag{2.14}$$

The projection of the curvature vector $\boldsymbol{\alpha}''$ onto the unit normal vector field of the surface $\boldsymbol{R}$ is given by

$$\left\langle \boldsymbol{\alpha}'', \frac{\boldsymbol{R}_1\times\boldsymbol{R}_2}{\|\boldsymbol{R}_1\times\boldsymbol{R}_2\|}\right\rangle = L_{11}(u_1')^2 + 2L_{12}u_1'u_2' + L_{22}(u_2')^2 \tag{2.15}$$

2.3. Differential Geometry of the Implicit Surfaces in $\mathbb{R}^3$

Assume that $f(x_1, x_2, x_3) = 0$ is a regular implicit surface. In other words $\nabla \boldsymbol{f} \neq 0$, where $\nabla \boldsymbol{f} = (f_1, f_2, f_3)$ is the gradient vector of the surface f, $f_p = \frac{\partial f}{\partial x_p}$, then the unit surface normal vector field of the surface f is given by

$$\boldsymbol{N} = \frac{\nabla \boldsymbol{f}}{\|\nabla \boldsymbol{f}\|} \tag{2.16}$$

Let

$$\boldsymbol{a}(s) = \left(x_1(s), x_2(s), x_3(s)\right) \tag{2.17}$$

be a curve on the surface f with constraint $f(x_1, x_2, x_3) = 0$ then we have

Copyright © 2011 SciRes.

$$\boldsymbol{\alpha}' = (x_1', x_2', x_3'),$$
$$\boldsymbol{\alpha}'' = (x_1'', x_2'', x_3''), \tag{2.18}$$
$$\boldsymbol{\alpha}''' = (x_1''', x_2''', x_3''').$$

$$\frac{\mathrm{d}f}{\mathrm{d}s} = f_1 x_1' + f_2 x_2' + f_3 x_3' = 0 \tag{2.19}$$

$$\begin{aligned}\frac{\mathrm{d}^2 f}{\mathrm{d}s^2} &= f_{11}(x_1')^2 + f_{22}(x_2')^2 + f_{33}(x_3')^2 \\ &+ 2\left(f_{12}x_1'x_2' + f_{13}x_1'x_3' + f_{23}x_2'x_3'\right) \\ &+ f_1 x_1'' + f_2 x_2'' + f_3 x_3'' = 0\end{aligned} \tag{2.20}$$

The projection of the curvature vector $\boldsymbol{\alpha}''$ onto the unit normal vector field of the surface f is given by

$$\left\langle \boldsymbol{\alpha}'', \frac{\nabla \boldsymbol{f}}{\|\nabla \boldsymbol{f}\|} \right\rangle = \frac{-\eta}{\sqrt{f_1^2 + f_2^2 + f_3^2}} \tag{2.21}$$

where

$$\begin{aligned}\eta &= f_{11}(x_1')^2 + f_{22}(x_2')^2 + f_{33}(x_3')^2 \\ &+ 2\left(f_{12}x_1'x_2' + f_{13}x_1'x_3' + f_{23}x_2'x_3'\right)\end{aligned}$$

3. Transversal Intersection Curves

Consider the intersecting implicit and parametric surfaces $f(x_1, x_2, x_3) = 0$ and $\boldsymbol{R} = R(u_1, u_2)$; $c_1 < u_1 < c_2$, $c_3 < u_2 < c_4$ such that, $f \neq 0$, $\boldsymbol{R}_1 \times \boldsymbol{R}_2 \neq 0$. Then the intersection curve of these surfaces can be viewed as a curve on both surfaces as

$$\boldsymbol{\alpha}(s) = \left(x_1(\mathrm{s}), x_2(\mathrm{s}), x_3(\mathrm{s})\right); \quad f(x_1, x_2, x_3) = 0,$$

$$\boldsymbol{\alpha}(s) = \boldsymbol{R}\left(u_1(\mathrm{s}), u_2(\mathrm{s})\right); \quad c_1 < u_1 < c_2, c_3 < u_2 < c_4.$$

Then we have

$$\boldsymbol{x}_i(s) = \boldsymbol{R}^i\left(\boldsymbol{u}_1(\mathrm{s}), \boldsymbol{u}_2(\mathrm{s})\right), \boldsymbol{i} = 1,2,3$$

where $\boldsymbol{R}\left(u_1(\mathrm{s}), u_2(\mathrm{s})\right) = \left(R^1, R^2, R^3\right)$. Then the surface f can be expressed as

$$h(u_1, u_2) = f\left(R^1, R^2, R^3\right) = 0 \tag{3.1}$$

Thus the intersection curve is given by

$$\begin{gathered}\boldsymbol{\alpha}(s) = \boldsymbol{R}\left(u_1(\mathrm{s}), u_2(\mathrm{s})\right); \; h(u_1, u_2) = 0, \\ c_1 < u_1 < c_2, c_3 < u_2 < c_4\end{gathered} \tag{3.2}$$

3.1. Tangential Direction

Differentiation (3.1) yields

$$h_1 u_1' + h_2 u_2' = 0 \tag{3.3}$$

where $h_i = \dfrac{\partial h}{\partial u_i}$, then we have

$$u_2' = \frac{-h_1}{h_2} u_1', \quad h_2 \neq 0 \tag{3.4}$$

Since $\boldsymbol{\alpha}'$ is the unit tangent vector field of the curve (3.2), then we have

$$\|\boldsymbol{\alpha}'\| = \left\langle \boldsymbol{R}_1 u_1' \boldsymbol{R}_2 u_2', \boldsymbol{R}_1 u_1' \boldsymbol{R}_2 u_2' \right\rangle = 1 \tag{3.5}$$

which can be written as

$$\mathrm{g}_{11}(u_1')^2 + 2\mathrm{g}_{12}u_1'u_2' + \mathrm{g}_{22}(u_2')^2 = 1 \tag{3.6}$$

Substituting (3.4) into (3.6) yields

$$\begin{aligned}u_1' &= h_2\left((h_2)^2 \mathrm{g}_{11} - 2h_1h_2g_{12} + (h_1)^2 g_{22}\right)^{-\frac{1}{2}}, \\ u_2' &= -h_1\left((h_2)^2 \mathrm{g}_{11} - 2h_1h_2g_{12} + (h_1)^2 g_{22}\right)^{-\frac{1}{2}}.\end{aligned} \tag{3.7}$$

The unit tangent vector field of the intersection curve is given by substituting (3.7) into (2.12) as follows

$$\boldsymbol{t} = \frac{\boldsymbol{\zeta}}{\|\zeta\|}; \quad \boldsymbol{\zeta} = h_2\boldsymbol{R}_1 - h_1\boldsymbol{R}_2 \tag{3.8}$$

3.2. Curvature and Curvature Vector

The curvature vector is given by differentiation (3.8) with respect to s as follows

$$\begin{aligned}\boldsymbol{\alpha}'' &= \frac{\|\zeta\|^2 \zeta' - \langle \zeta', \zeta \rangle \zeta}{\|\zeta\|^3} \\ \zeta' &= \|\zeta\|^{-1}\Big((h_2)^2 \boldsymbol{R}_{11} + (h_1)^2 \boldsymbol{R}_{22} - 2h_1h_2\boldsymbol{R}_{12} \\ &+ \left(h_2h_{12} - h_1h_{22}\right)\boldsymbol{R}_1 + \left(h_1h_{12} - h_2h_{11}\right)\boldsymbol{R}_2\Big)\end{aligned} \tag{3.9}$$

The unit principal normal vector field, the curvature and the unit binormal vector are given by using (2.3) (2.4) and (2.5) as follows

$$\begin{aligned}\boldsymbol{n} &= \frac{\|\zeta\|^2 \zeta' - \langle \zeta', \zeta \rangle \zeta}{\left\|\|\zeta\|^2 \zeta' - \langle \zeta', \zeta \rangle \zeta\right\|}, \\ \kappa &= \frac{\left\|\|\zeta\|^2 \zeta' - \langle \zeta', \zeta \rangle \zeta\right\|}{\|\zeta\|^3}, \\ \boldsymbol{b} &= \frac{\zeta}{\|\zeta\|} \times \frac{\|\zeta\|^2 \zeta' - \langle \zeta', \zeta \rangle \zeta}{\left\|\|\zeta\|^2 \zeta' - \langle \zeta', \zeta \rangle \zeta\right\|}.\end{aligned} \tag{3.10}$$

3.3. Torsion and Higher-Order Derivatives

Equation (3.7) can be written as

$$u_1' = \frac{h_2}{\|\zeta\|}, \quad u_2' = \frac{-h_1}{\|\zeta\|} \tag{3.11}$$

Differentiation (3.13) we obtain

$$u_1'' = \left(\frac{h_{12}}{\|\zeta\|} - \frac{\langle \zeta', \zeta \rangle}{\|\zeta\|^2}\right) u_1' - \frac{h_{22}}{\|\zeta\|} u_2',$$

$$u_2'' = -\frac{h_{11}}{\|\zeta\|} u_1' - \left(\frac{h_{12}}{\zeta} + \frac{\langle \zeta', \zeta \rangle}{\|\zeta\|^2}\right) u_2', \tag{3.12}$$

$$\begin{aligned} \boldsymbol{\zeta}' &= \left(h_{12}\boldsymbol{R}_1 + h_2\boldsymbol{R}_{11} - h_{11}\boldsymbol{R}_2 - h_1\boldsymbol{R}_{12}\right) u_1' \\ &\quad + \left(h_{22}\boldsymbol{R}_1 + h_2\boldsymbol{R}_{12} - h_{12}\boldsymbol{R}_2 - h_1\boldsymbol{R}_{22}\right) u_2'. \end{aligned}$$

Differentiation (3.12) we obtain

$$\begin{aligned} u_1''' =& \left(\frac{h_{12}}{\|\zeta\|} - \frac{\langle \zeta', \zeta \rangle}{\|\zeta\|^2}\right) u_1'' - \frac{h_{22}}{\|\zeta\|} u_2'' + \frac{h_{112}}{\|\zeta\|} \left(u_1'\right)^2 \\ &+ \left(2\frac{\langle \zeta', \zeta \rangle^2}{\|\zeta\|^4} - \frac{\langle \zeta', \zeta \rangle h_{12}}{\|\zeta\|^3} - \frac{\langle \zeta'', \zeta \rangle}{\|\zeta\|^2} - \frac{\|\zeta'\|^2}{\|\zeta\|^2}\right) u_1' \\ &+ \frac{2h_{122}}{\|\zeta\|} u_1' u_2' + \frac{h_{222}}{\|\zeta\|} \left(u_2'\right)^2 + \frac{\langle \zeta', \zeta \rangle h_{22}}{\|\zeta\|^3} u_2', \\ u_2''' =& \frac{-h_{11}}{\|\zeta\|} u_1'' - \left(\frac{h_{12}}{\|\zeta\|} + \frac{\langle \zeta', \zeta \rangle}{\|\zeta\|^2}\right) u_2'' \\ &+ \frac{h_{111}}{\|\zeta\|} \left(u_1'\right)^2 - \frac{h_{122}}{\|\zeta\|} \left(u_2'\right)^2 - \frac{\langle \zeta', \zeta \rangle h_{11}}{\|\zeta\|^3} u_1' \\ &+ \left(2\frac{\langle \zeta', \zeta \rangle^2}{\|\zeta\|^4} - \frac{\langle \zeta', \zeta \rangle h_{12}}{\|\zeta\|^3} - \frac{\langle \zeta'', \zeta \rangle}{\|\zeta\|^2} - \frac{\|\zeta'\|^2}{\|\zeta\|^2}\right) u_2', \\ \boldsymbol{\zeta}'' =& u_1''\left(h_{12}\boldsymbol{R}_1 + h_2\boldsymbol{R}_{11} - h_{11}\boldsymbol{R}_2 - h_1\boldsymbol{R}_{12}\right) \\ &+ u_2''\left(h_{22}\boldsymbol{R}_1 + h_2\boldsymbol{R}_{12} - h_{12}\boldsymbol{R}_2 - h_1\boldsymbol{R}_{22}\right) \\ &+ \left(u_1'\right)^2 \begin{pmatrix} h_{112}\boldsymbol{R}_1 + 2h_{12}\boldsymbol{R}_{11} + h_2\boldsymbol{R}_{111} \\ -h_{111}\boldsymbol{R}_2 - 2h_{11}\boldsymbol{R}_{12} - h_1\boldsymbol{R}_{112} \end{pmatrix} \\ &+ \left(u_2'\right)^2 \begin{pmatrix} h_{222}\boldsymbol{R}_1 + 2h_{22}\boldsymbol{R}_{12} + h_2\boldsymbol{R}_{122} \\ -h_{122}\boldsymbol{R}_2 - 2h_{12}\boldsymbol{R}_{22} - h_1\boldsymbol{R}_{222} \end{pmatrix} \\ &+ 2u_1'u_2' \begin{pmatrix} h_{122}\boldsymbol{R}_1 - 2h_{112}\boldsymbol{R}_2 + h_{22}\boldsymbol{R}_{11} \\ -h_{11}\boldsymbol{R}_{22} + 2h_2\boldsymbol{R}_{112} - h_1\boldsymbol{R}_{122} \end{pmatrix}. \end{aligned} \tag{3.13}$$

Substituting $u_1', u_1'', u_1''', u_2', u_2''$ and u_2''' into (2.14) we obtain the third-order derivative vector of the intersection curve. Hence the torsion can be obtained by (2.7).

We can compute all higher-order derivatives of the intersection curve by a similar way.

4. Tangentially Intersection Curves

Assume that the surfaces $f(x_1, x_2, x_3) = 0$ and $\boldsymbol{R} = R(u_1, u_2)$; $c_1 < u_1 < c_2, c_3 < u_2 < c_4$ are intersecting tangentially at a point P on the curve (3.2) then the unit surface normal vector field of both surfaces are parallel to each other. In other words

$$\frac{\nabla \boldsymbol{f}}{\|\nabla \boldsymbol{f}\|} = \pm \frac{\boldsymbol{R}_1 \times \boldsymbol{R}_2}{\|\boldsymbol{R}_1 \times \boldsymbol{R}_2\|}$$

which can be written as

$$\nabla \boldsymbol{f} = A\left(\boldsymbol{R}_1 \times \boldsymbol{R}_2\right),\ A = \pm \frac{\|\nabla \boldsymbol{f}\|}{\|\boldsymbol{R}_1 \times \boldsymbol{R}_2\|} \tag{4.1}$$

Then we can write

$$\begin{aligned} \boldsymbol{f}_1 &= A\left(\boldsymbol{R}_1^2\boldsymbol{R}_2^3 - \boldsymbol{R}_1^3\boldsymbol{R}_2^2\right), \\ \boldsymbol{f}_2 &= A\left(\boldsymbol{R}_1^3\boldsymbol{R}_2^1 - \boldsymbol{R}_1^1\boldsymbol{R}_2^3\right), \\ \boldsymbol{f}_3 &= A\left(\boldsymbol{R}_1^1\boldsymbol{R}_2^2 - \boldsymbol{R}_1^2\boldsymbol{R}_2^1\right). \end{aligned} \tag{4.2}$$

Since $\boldsymbol{x}_i(s) = \boldsymbol{R}^i\left(u_1(s), u_2(s)\right)$, $i = 1, 2, 3$, then we have

$$\boldsymbol{x}_i' = \boldsymbol{R}_1^i u_1' + \boldsymbol{R}_2^i u_2' \tag{4.3}$$

4.1. Tangential Direction

Projecting the curvature vector $\boldsymbol{\alpha}''$ onto the two unit normal vectors of both surfaces yields

$$\left\langle \boldsymbol{\alpha}'', \frac{\nabla \boldsymbol{f}}{\|\nabla \boldsymbol{f}\|} \right\rangle = \pm \left\langle \boldsymbol{\alpha}'', \frac{\boldsymbol{R}_1 \times \boldsymbol{R}_2}{\|\boldsymbol{R}_1 \times \boldsymbol{R}_2\|} \right\rangle \tag{4.4}$$

Using (2.15) (2.21) and (4.4) we obtain

$$\begin{aligned} &\begin{pmatrix} f_{11}\left(x_1'\right)^2 + f_{22}\left(x_2'\right)^2 + f_{33}\left(x_3'\right)^2 \\ +2\left(f_{12}x_1'x_2' + \mathrm{f}_{13}x_1'x_3' + \mathrm{f}_{23}x_2'x_3'\right) \end{pmatrix} \\ &= -A\left\|\boldsymbol{R}_1 \times \boldsymbol{R}_2\right\|\left(L_{11}\left(u_1'\right)^2 + 2L_{12}u_1'\mathrm{u}_2' + L_{22}\left(u_2'\right)^2\right) \end{aligned} \tag{4.5}$$

Substituting (4.3) into (4.5) yields

$$a_{11}\left(\frac{u_1'}{u_2'}\right)^2 + 2\mathrm{a}_{12}\left(\frac{u_1'}{u_2'}\right) + \mathrm{a}_{22} = 0,\ u_2' \neq 0 \tag{4.6}$$

where

$$\begin{aligned} a_{11} &= A\boldsymbol{R}_1 \times \boldsymbol{R}_2 L_{11} + f_{11}\left(\boldsymbol{R}_1^1\right)^2 + f_{22}\left(\boldsymbol{R}_1^2\right)^2 \\ &\quad + f_{33}\left(\boldsymbol{R}_1^3\right)^2 + 2\left(f_{12}\boldsymbol{R}_1^1\boldsymbol{R}_1^2 + f_{23}\boldsymbol{R}_1^2\boldsymbol{R}_1^3 + f_{13}\boldsymbol{R}_1^1\boldsymbol{R}_1^3\right), \\ a_{22} &= A\boldsymbol{R}_1 \times \boldsymbol{R}_2 L_{22} + f_{11}\left(\boldsymbol{R}_2^1\right)^2 + f_{22}\left(\boldsymbol{R}_2^2\right)^2 \\ &\quad + f_{33}\left(\boldsymbol{R}_2^3\right)^2 + 2\left(f_{12}\boldsymbol{R}_2^1\boldsymbol{R}_2^2 + f_{23}\boldsymbol{R}_2^2\boldsymbol{R}_2^3 + f_{13}\boldsymbol{R}_2^1\boldsymbol{R}_2^3\right), \\ a_{12} &= A\boldsymbol{R}_1 \times \boldsymbol{R}_2 L_{12} + f_{11}\boldsymbol{R}_1^1\boldsymbol{R}_2^1 + f_{22}\boldsymbol{R}_1^2\boldsymbol{R}_2^2 \\ &\quad + f_{33}\boldsymbol{R}_1^3\boldsymbol{R}_2^3 + f_{12}\left(\boldsymbol{R}_1^1\boldsymbol{R}_2^2 + \boldsymbol{R}_1^2\boldsymbol{R}_2^1\right) \\ &\quad + f_{23}\left(\boldsymbol{R}_1^2\boldsymbol{R}_2^3 + \boldsymbol{R}_1^3\boldsymbol{R}_2^2\right) + \mathrm{f}_{13}\left(\boldsymbol{R}_1^1\boldsymbol{R}_2^3 + \boldsymbol{R}_1^3\boldsymbol{R}_2^1\right). \end{aligned}$$

Copyright © 2011 SciRes.

This can be written in a matrix form as follows

$$\boldsymbol{a}_{ij} = \nabla \boldsymbol{f} \boldsymbol{R}_{ij} + \boldsymbol{R}_i^{\mathrm{T}} \boldsymbol{H} \boldsymbol{R}_j \tag{4.7}$$

where $\nabla \boldsymbol{f} = [f_1 \quad f_2 \quad f_3]$, $\boldsymbol{R}_{ij} = [R_{ij}^1 \quad R_{ij}^2 \quad R_{ij}^3]^{\mathrm{T}}$, $\boldsymbol{R}_i = [R_i^1 \quad R_i^2 \quad R_i^3]^{\mathrm{T}}$ and $\boldsymbol{H} = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{12} & f_{22} & f_{23} \\ f_{13} & f_{23} & f_{33} \end{bmatrix}$ is the Hessian matrix of the surface f. Solving (4.6) for $\frac{u_1'}{u_2'}$ yields

$$u_1' = Bu_2', \quad B = \frac{-a_{12} \pm \sqrt{(\mathrm{a}_{12})^2 - a_{11}a_{22}}}{a_{11}} \tag{4.8}$$

Substituting (3.7) and (4.7) into (4.8) we obtain

$$\begin{aligned} u_1' &= B\left(B^2 g_{11} + 2Bg_{12} + g_{22}\right)^{-\frac{1}{2}} \\ u_2' &= \left(B^2 g_{11} + 2Bg_{12} + g_{22}\right)^{-\frac{1}{2}}. \end{aligned} \tag{4.9}$$

Then the unit tangent vector field of the intersection curve is given by

$$\boldsymbol{t} = \frac{B\boldsymbol{R}_1 + \boldsymbol{R}_2}{\left\| B\boldsymbol{R}_1 + \boldsymbol{R}_2 \right\|} \tag{4.10}$$

From the previous formulas, it is easy to see that, there are four distinct cases for the solution of (4.6) depending upon the discriminant $\Delta = \left(a_{12}\right)^2 - a_{11}a_{22}$, these cases are as the following [1]

Lemma 1. The point P is a branch point of the intersection curve (3.2) if $\Delta > 0$ and there is another intersection branch crossing the curve (3.2) at that point.

Lemma 2. The surfaces f and h intersect at the point P and at its neighborhood, if $\Delta = 0$ and $\left(a_{11}\right)^2 + \left(a_{12}\right)^2 + \left(a_{22}\right)^2 \neq 0$.(Tangential intersection curve).

Lemma 3. The point P is an isolated contact point of the surfaces f and h, if $\Delta < 0$.

Lemma 4. The surfaces f and h have contact of at least second order at the point P, if $a_{11} = a_{12} = a_{22} = 0$. (Higher-order contact point).

4.2. Curvature and Curvature Vector

Differentiation (4.6) and using (4.9) we obtain

$$\begin{aligned} & u_1'' - Bu_2'' = a_1, \\ & a_1 = -u_2' \frac{\left(a_{11}' B^2 + 2a_{12}' B + a_{22}'\right)}{\left(a_{11}B + a_{12}\right)}; \quad a_{11}B + a_{12} \neq 0, \end{aligned} \tag{4.11}$$

where

$$\begin{aligned} a_{ij}' &= \boldsymbol{t}^{\mathrm{T}} \boldsymbol{H}\boldsymbol{R}_{ij} + (\nabla \boldsymbol{f}\boldsymbol{R}_{1ij} + \boldsymbol{R}_{1i}^{\mathrm{T}} \boldsymbol{H}\boldsymbol{R}_j + \boldsymbol{R}_1^{\mathrm{T}} \boldsymbol{H}\boldsymbol{R}_{1j}) u_1' \\ &+ \left(\nabla \boldsymbol{f}\boldsymbol{R}_{2ij} + \boldsymbol{R}_{2i}^{\mathrm{T}} \boldsymbol{H}\boldsymbol{R}_j + \boldsymbol{R}_i^{\mathrm{T}} \boldsymbol{H}\boldsymbol{R}_{2j}\right) u_2' + \boldsymbol{R}_i^{\mathrm{T}} \boldsymbol{Q}\boldsymbol{R}_j, \end{aligned}$$

$$\boldsymbol{Q} = \left[\mathrm{H}_1 \quad \mathrm{H}_2 \quad \mathrm{H}_3\right]\boldsymbol{t}, \quad \boldsymbol{H}_i = \begin{bmatrix} f_{11i} & f_{12i} & f_{13i} \\ f_{12i} & f_{22i} & f_{23i} \\ f_{13i} & f_{23i} & f_{33i} \end{bmatrix}. \tag{4.11}$$

Since the curvature vector is perpendicular to the tangent vector, then we have $\left\langle \boldsymbol{\alpha}', \boldsymbol{\alpha}'' \right\rangle = 0$. Using (2.12) (2.13) and (4.9) we obtain

$$a_2 u_1'' + a_3 u_2'' = a_4 \tag{4.13}$$

where

$$\begin{aligned} & a_2 = Bg_{11} + g_{12}, \quad a_3 = Bg_{12} + g_{22}, \\ & a_4 = -\left(u_2'\right)^2 \left(B^3 \left\langle \boldsymbol{R}_{11}, \boldsymbol{R}_1 \right\rangle + 2\mathrm{B}^2 \left\langle \boldsymbol{R}_{12}, \boldsymbol{R}_1 \right\rangle \right. \\ & \left. + 2B\left\langle \boldsymbol{R}_{12}, \boldsymbol{R}_2 \right\rangle + B\left\langle \boldsymbol{R}_{22}, \boldsymbol{R}_1 \right\rangle + \left\langle \boldsymbol{R}_{22}, \boldsymbol{R}_2 \right\rangle + \left\langle \boldsymbol{R}_{11}, \boldsymbol{R}_2 \right\rangle \right) \end{aligned}$$

Solving the linear system (4.11) and (4.13) yields

$$\begin{aligned} u_1'' &= \frac{a_3 a_4 + a_4 B}{a_3 + a_2 B}, \\ u_2'' &= \frac{a_4 - a_1 a_2}{a_3 + a_2 \mathrm{B}} \end{aligned} \tag{4.14}$$

The curvature vector of the intersection curve is obtained by substituting u_1', u_1'', u_2', and u_2'' into (2.13).

4.3. Torsion

If we have a branch point, then we can compute the torsion by taking the limit of the torsion of transversal intersection curve at this point. If we have tangential intersection curve, then we can compute u_1''' and u_2''' by differentiation u_1'' and u_2''. Substituting $u_1', u_1'', u_1''', u_2', u_2''$, and u_2''' into (2.14) we obtain the third-order derivative vector of the intersection curve. Then we can obtain the torsion by using (2.7).

5. Examples

Example 1. Consider the intersection of the implicit and the parametric surfaces

$$\begin{aligned} & f = x_1^2 + x_2^2 - 9 = 0, \\ & \boldsymbol{R} = \left(u_1, 3\sin u_2, 3\cos u_2\right); \; 0 < u_2 < 2\pi. \end{aligned} \tag{5.1}$$

as shown in **Figure 1**.

Transversal intersection: Using (3.1) yields

$$h = u_1^2 - 9\cos^2 u_2 = 0 \tag{5.2}$$

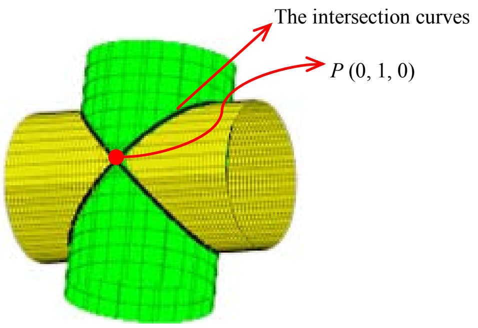

Figure 1. Transversal and tangential intersection.

Differentiation (5.1) and (5.2) we obtain

$$\begin{aligned}
&h_1 = 2u_2,\, h_2 = 9\sin 2u_2,\, \boldsymbol{R}_1 = (1,0,0),\\
&h_{22} = 18\cos 2u_2,\, h_{11} = h_{12} = h_{111} = 0,\\
&h_{222} = -36\sin 2u_2,\\
&\boldsymbol{R}_1 = (1,0,0), \quad \boldsymbol{R}_2 = 3(0,\cos u_2,-\sin u_2),\\
&\boldsymbol{R}_{22} = -3(0,\sin\sin \mathrm{u}_2,\cos u_2),\\
&\boldsymbol{R}_{222} = -3(0,\cos\cos \mathrm{u}_2,-\sin u_2).
\end{aligned} \tag{5.3}$$

Using (3.8) and (5.2), we obtain

$$\boldsymbol{t} = \left(\frac{\sin u_2}{\sqrt{1+\sin^2 \mathrm{u}_2}}, \frac{-u_1}{3\sqrt{1+\sin^2 u_2}}, \frac{u_1\tan u_2}{3\sqrt{1+\sin^2 u_2}}\right), \quad \cos u_2 \neq 0. \tag{5.4}$$

Using (3.12) and (5.2), hence

$$\begin{aligned}
&\zeta = (18\sin u_2\cos u_2, -6u_1\cos u_2, 6u_1\sin u_2),\\
&\zeta' = \left(\frac{-2u_1\cos 2u_2}{\cos u_2\sqrt{1+\sin^2 u_2}}, \frac{-6\sin 2u_2}{\sqrt{1+\sin^2 u_2}}, \frac{-6\cos 2u_2}{\sqrt{1+\sin^2 u_2}}\right),\\
&\|\zeta\| = 18\cos u_2\sqrt{1+\sin^2 u_2},\\
&\langle\zeta,\zeta'\rangle = \frac{72u_1\sin^3 u_2}{\sqrt{1+\sin^2 u_2}}.
\end{aligned} \tag{5.5}$$

Using (2.4), (2.5), (3.12), (3.13) and (5.4) then we have

$$\begin{aligned}
&\boldsymbol{\alpha}'' = \left(\frac{-u_1}{9\left(1+\sin^2 u_2\right)^2}, \frac{-2\sin u_2}{3\left(1+\sin^2 u_2\right)^2}, \frac{-\cos u_2}{3\left(1+\sin^2 u_2\right)^2}\right),\\
&\boldsymbol{n} = \left(\frac{-u_1}{3\sqrt{2}\sqrt{1+\sin^2 u_2}}, \frac{-\sqrt{2}\sin u_2}{\sqrt{1+\sin^2 u_2}}, \frac{-\cos u_2}{\sqrt{2}\sqrt{1+\sin^2 u_2}}\right),\\
&\kappa = \frac{\sqrt{2}}{3}\left(1+\sin^2 u_2\right)^{-\frac{3}{2}},\\
&\boldsymbol{b} = \left(\frac{u_1\left(\cos u_2 + 2\sin u_2\tan u_2\right)}{3\sqrt{2}\left(1+\sin^2 u_2\right)}, 0, \frac{-1}{\sqrt{2}}\right).
\end{aligned} \tag{5.6}$$

Using (3.15) and (3.16) hence

$$\begin{aligned}
&u_1' = \frac{\sin u_2}{\sqrt{1+\sin^2 \mathrm{u}_2}}, \quad u_2' = \frac{-u_1}{9\cos u_2\sqrt{1+\sin^2 u_2}},\\
&u_1'' = \frac{-u_1}{9(1+\sin^2 u_2)^2}, \quad u_2'' = \frac{-\sin u_2\cos u_2}{9(1+\sin^2 u_2)^2}.
\end{aligned} \tag{5.7}$$

Using (3.17) and (5.7) hence

$$\begin{aligned}
&u_1''' = \frac{-\sin u_2\left(2+3\cos^2 u_2\right)}{9\left(1+\sin^2 u_2\right)^{\frac{7}{2}}},\\
&u_2''' = \frac{-u_1(2\sin u_2\tan u_2 - \cos u_2\cos 2u_2)}{81\left(1+\sin^2 u_2\right)^{\frac{7}{2}}}.
\end{aligned} \tag{5.8}$$

Using (2.7) and (2.14) yields

$$\boldsymbol{\alpha}''' = \left(\frac{-3(2+3\cos^2 u_2)\sin u_2}{27(1+\sin^2 u_2)^{\frac{7}{2}}}, \frac{2\mathrm{u}_1 - 6\mathrm{u}_1\sin^2 u_2}{27(1+\sin^2 u_2)^{\frac{7}{2}}}, \frac{-\left(4+\sin^2 u_2\right)\cos u_2\sin u_2 - \left(1+\sin^2 u_2\right)\sin u_2}{27\left(1+\sin^2 u_2\right)^{\frac{7}{2}}}\right) \tag{5.9}$$

$$\begin{aligned}
\tau = \frac{-\sqrt{2}}{4-2\cos^2 \mathrm{u}_2}&(4u_1\tan u_2 - 4\sin u_2 - 10\cos u_2\sin u_2\\
&+4\cos^2 u_2\sin u_2 + 7\cos^3 u_2\sin u_2 - \cos^4 u_2\sin u_2\\
&-\cos^5 u_2\sin u_2 + 2u_1\cos u_2\sin u_2 + 3u_1\cos^3 u_2\sin u_2\\
&+2u_1\cos^2 u_2\tan u_2 - 6u_1\cos^4 u_2\tan u_2)
\end{aligned} \tag{5.10}$$

Tangentially intersection: The surfaces are intersecting tangentially at the points $P(0,\pm 1,0)$. Consider the point $P_1(0,1,0)$, using (4.7) (4.8) (4.9) and (5.3), then we have

$$\begin{aligned}
&a_1 = 2, \quad a_2 = 0, \quad a_3 = -18,\\
&B = \pm 3,\ u_1' = \pm\frac{1}{\sqrt{2}},\ u_2' = \frac{1}{3\sqrt{2}}.
\end{aligned} \tag{5.11}$$

Then $\Delta > 0$, this means that the point P_1 is a branch point (**Figure 1**). From (4.10) and (5.11), we obtain

$$\boldsymbol{t} = \left(\pm\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right) \tag{5.12}$$

Using (2.13) and (4.14) hence

$$u_1' = u_2' = 0, \quad \boldsymbol{\alpha}'' = \frac{-1}{6}(0,1,0),$$
$$\boldsymbol{n} = (0,-1,0), \quad \kappa = \frac{1}{6}, \tag{5.13}$$
$$\boldsymbol{b} = \left(\frac{-1}{\sqrt{2}}, 0, \mp\frac{1}{\sqrt{2}}\right).$$

Using (5.10) at $P_1(0,1,0)$, we obtain

$$\begin{aligned}\tau = \lim_{u_2 \to \frac{\pi}{2}} \frac{-1}{4-2\cos^2 u_2}\Big(&-4\sqrt{2}\sin u_2 - \sqrt{2}\cos^4 u_2 \sin u_2 \\ &+2\sqrt{2}u_1 \cos^2 u_2 \tan u_2 + 4\sqrt{2}\cos^2 u_2 \sin u_2 \\ &+3\sqrt{2}u_1 \cos^3 u_2 \sin u_2 + 7\sqrt{2}\cos^3 u_2 \sin u_2 \\ &-10\sqrt{2}\cos u_2 \sin u_2 - \sqrt{2}\cos^5 u_2 \sin u_2 4\sqrt{2}u_1 \tan u_2 \\ &+2\sqrt{2}u_1 \cos u_2 \sin u_2 - 6\sqrt{2}u_1 \cos^4 u_2 \tan u_2\Big) \\ =0&\end{aligned} \tag{5.14}$$

Example 2. Consider the intersection of the implicit and the parametric surfaces

$$f = x_1^2 + x_2^2 + x_3^2 - 9 = 0,$$
$$\mathrm{R} = (u_1, 3\sin u_2, 3\cos u_2),\ 0 < u_2 < 2\pi \tag{5.15}$$

as shown in **Figure 2**.

At $x_1 = 0$, $(\nabla \boldsymbol{f}) // (\boldsymbol{R}_1 \times \boldsymbol{R}_2)$. Using (4.7) and (5.15), we have $\Delta = 0$, this means that the surfaces are intersecting tangentially in a curve as (**Figure 2**). Then from (4.8) and (4.9), we have

$$B = 0, \quad u_1' = 0, \quad u_2' = \frac{1}{3} \tag{5.16}$$

Using (4.10) we have

$$\boldsymbol{t} = (0, \cos u_2, -\sin u_2) \tag{5.17}$$

Using (5.16) hence

$$u_1'' = u_1''' = 0, \quad u_2'' = u_2''' = 0 \tag{5.18}$$

Using (2.4) and (2.13) hence the curvature vector and the curvature are given by

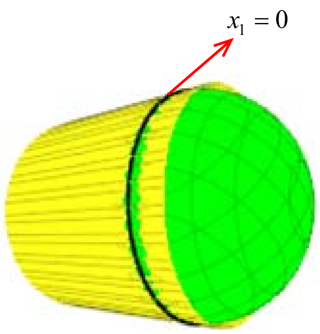

Figure 2. Tangential intersection.

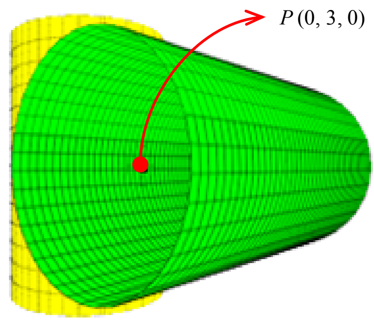

Figure 3. Tangential intersection.

$$\boldsymbol{\alpha}'' = -\frac{1}{3}(0, \sin u_2, \cos u_2),$$
$$\boldsymbol{n} = (0, -\sin u_2, -\cos u_2), \quad \kappa = \frac{1}{3} \tag{5.19}$$

Using (2.5) (2.7)and (2.14) hence

$$\boldsymbol{\alpha}''' = \frac{1}{9}(0, -\cos u_2, \sin u_2),$$
$$\boldsymbol{b} = (-1, 0, 0), \quad \tau = 0. \tag{5.20}$$

Example 3. Consider the intersection of the implicit and the parametric surfaces

$$f = x_1^2 + (x_2 - 6)^2 - 9 = 0,$$
$$\boldsymbol{R} = (u_1, 3 + 3\sin u_2, 3\cos u_2). \tag{5.21}$$

as shown in **Figure 3**.

At the point $P(0,3,0)$, $(\nabla \boldsymbol{f}) // (\boldsymbol{R}_1 \times \boldsymbol{R}_2)$. Using (4.7) and (5.21), we have $\Delta < 0$, this means that the point P is an isolated tangential contact point (**Figure 3**).

Example 4. Consider the intersection of the implicit and the parametric surfaces

$$f = x_3^2 - x_1^2 - x_2^2 = 0,$$
$$\boldsymbol{R} = \left(u_1(1+u_2^2), -u_2(1+u_1^2), u_1^2 - u_2^3\right). \tag{5.22}$$

as shown in **Figure 4**.

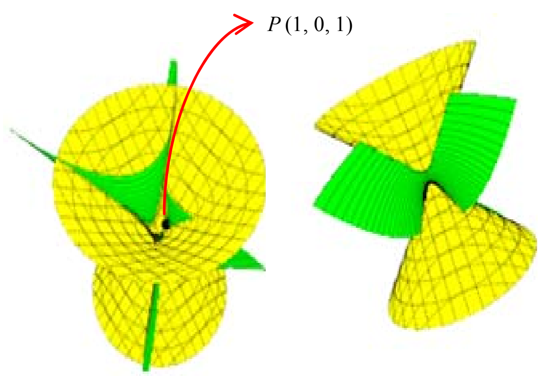

Figure 4. Transversal intersection.

At the point $P=(1,0,1)\in S^f\cap S^R$, on the intersection curve (**Figure 4**), we have

$$\begin{aligned}&\boldsymbol{R}_1=(1,0,2),\ \boldsymbol{R}_{11}=(0,0,2), h_{111}=24,\\&\boldsymbol{R}_2=(0,-2,0),\ \boldsymbol{R}_{22}=(2,0,0),\ \boldsymbol{R}_{12}=(0,-2,0),\\&\boldsymbol{R}_{112}=(0,-2,0),\ \boldsymbol{R}_{122}=(2,0,0),\ \boldsymbol{R}_{222}=(0,0,-6),\\&h_1=2,\ h_{11}=10,\ h_{22}=h_{222}=-12,\\&h_2=h_{12}=0,\ h_{122}=-24,\ h_{112}=0.\end{aligned}\tag{5.23}$$

Using (3.8) and (5.23), we obtain

$$\boldsymbol{t}=(0,1,0)\tag{5.24}$$

Using (3.12) (3.13) and (5.23) we obtain

$$\begin{aligned}&\boldsymbol{\alpha}''=(2,0,3),\\&\boldsymbol{n}=(\frac{2}{\sqrt{13}},0,\frac{3}{\sqrt{13}}),\ \kappa=\sqrt{13}.\end{aligned}\tag{5.25}$$

Using (2.5) (2.7) (2.14) (3.17) and (5.25) we obtain

$$\begin{aligned}&\boldsymbol{\alpha}'''=\left(\frac{-3}{4},-19,\frac{-3}{4}\right),\\&\boldsymbol{b}=\left(\frac{3}{\sqrt{13}},0,\frac{-2}{\sqrt{13}}\right),\quad \tau=-\frac{3}{52}.\end{aligned}\tag{5.26}$$

6. Conclusions

Algorithms for computing the differential geometry properties of intersection curves of implicit and parametric surfaces in $\mathbb{R}^3$ are given for transversal and tangential intersection. This paper is an extension to the works of Ye and Maekawa [1]. They gave algorithms to compute the differential geometry properties of intersection curves between two parametric surfaces then they applied it on a simple example for implicit and parametric surfaces intersection. This paper presented direct and simple formulas to compute all differential geometry properties, which may reduce the time it takes to calculate those properties. The types of singularities on the intersection curve are characterized. The questions of how to exploit and extend these algorithms to compute the differential geometry properties of intersection curves between three surfaces in $\mathbb{R}^4$, can be topics of future research.

7. Acknowledgements

The authors would like to thank the reviewers for their valuable comments and suggestions.

8. References

[1] X. Ye andT. Maekawa, “Differential Geometry of Intersection Curves of Two Surfaces,” *Computer-Aided Geometric Design*, Vol. 16, No. 8, September 1999, pp. 767-788. doi:10.1016/S0167-8396(99)00018-7

[2] C. L. Bajaj, C. M. Hoffmann, J. E. Hopcroft and R. E. Lynch, “Tracing Surface Intersections,” *Computer-Aided Geometric Design*, Vol. 5, No. 4, November 1988, pp. 285-307. doi:10.1016/0167-8396(88)90010-6

[3] N. M. Patrikalakis, “Surface-to-Surface Intersection,” *IEEE Computer Graphics & Applications*, Vol. 13, No. 1, January-February 1993, pp. 89-95. doi:10.1109/38.180122

[4] T. J. Willmore, “An Introduction to Differential Geometry,” Clarendon Press, Oxford, 1959.

[5] M. Düldül, “On the Intersection Curve of Three Parametric Hypersurfaces,” *Computer-Aided Geometric Design*, Vol. 27, No. 1, January 2010, pp. 118-127. doi:10.1016/j.cagd.2009.10.002

[6] E. Kruppa, “Analytische und Konstruktive Differentialgeometrie,” Springer, Wien, 1957.

[7] E. Hartmann, “G2 Interpolation and Blending on Surfaces,” *The Visual Computer*, Vol. 12, No. 4, 1996, pp. 181-192. doi: 10.1007/s003710050057

[8] G. A. Kriezis, N. M. Patrikalakis and F.-E. Wolter, “Topological and Differential Equation Methods for Surface Intersections,” *Computer-Aided Geometric Design*, Vol. 24, No. 1, January 1992, pp. 41-55. doi:10.1016/0010-4485(92)90090-W

[9] R. C. Luo, Y. Ma and D. F. McAllister, “Tracing Tangential Surface-Surface Intersections,” *Proceedings Third ACM Solid Modeling Symposium*, Salt Lake City, 1995, pp. 255-262. doi:10.1145/218013.218070

[10] O. Aléssio, “Differential Geometry of Intersection Curves in $\mathbb{R}^4$ of three Implicit Surfaces,” *Computer-Aided Geometric Design*, Vol. 26, No. 4, May 2009, pp. 455-471. doi:10.1016/j.cagd.2008.12.001

[11] M. P. do Carmo, “Differential Geometry of Curves and Surface,” Prentice Hall, Englewood Cliffs, NJ, 1976.

[12] J. J. Stoker, “Differential Geometry,” Wiley, New York, 1969.

[13] D. J. Struik, “Lectures on Classical Differential Geometry,” Addison-Wesley, Reading, 1950.

Copyright © 2011 SciRes.