Retraction Notice

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☐ no

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The paper does not meet the standards of "Journal of Quantum Information Science".

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Generation of Bright Squeezed Light from N Three-Level Atoms Pumped by a Coherent Light: Open Quantum System

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Abstract

The manuscript investigated the steady-state analysis of the squeezing and statistical properties of the light generated by *N* three-level atoms available in an open cavity pumped a coherent light and the cavity coupled to a two-mode vacuum reservoir. The results indicate that as the frequency increases, the local quadrature squeezing of the two-mode cavity light approaches the global quadrature squeezing. The effect of the spontaneous emission leads to an increase in the quadrature squeezing, but to a decrease in the mean photon number of the system. It is also found that, unlike the mean photon number and the variance of the photon number, the quadrature squeezing does not depend on the number of atoms. This implies that the quadrature squeezing of the two-mode cavity light is independent of the number of photons.

Keywords

Spontaneous Emission, Operator Dynamics, Photon Statistics, Power Spectrum, Quadrature squeezing, Quadrature Fluctuations

1. Introduction

Squeezed states of light has played a crucial role in the development of quantum physics. Squeezing is one of the nonclassical features of light that have been extensively studied by several authors [1]-[8]. In a squeezed state the quantum noise in one quadrature is below the vacuum-state level or the coherent-state level at the expense of enhanced fluctuations in the conjugate quadrature, with the product of the uncertainties in the two quadratures satisfying the uncertainty relation [1] [2] [4] [9]. Because of the quantum noise reduction achievable below the vacuum level, squeezed light has potential applications in the detection of week signals and in

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low-noise communications [1] [3]. Squeezed light can be generated by various quantum optical processes such as subharmonic generations [1]-[5] [10]-[12], four-wave mixing [13] [14], resonance fluorescence [6] [7], second harmonic generation [8] [15], and three-level laser under certain conditions [1] [3] [4] [9] [16]-[27]. Hence it proves useful to find some convenient means of generating a bright squeezed light.

A three-level laser is a quantum optical device in which light is generated by three-level atoms in a cavity usually coupled to a vacuum reservoir via a single-port mirror. In one model of a three-level laser, three-level atoms initially prepared in a coherent superposition of the top and bottom levels are injected into a cavity and then removed from the cavity after they have decayed due to spontaneous emission [9] [16]-[21]. In another model of a three-level laser, the top and bottom levels of the three-level atoms injected into a cavity are coupled by coherent light [22]-[27]. It is found that a three-level laser in either model generates squeezed light under certain conditions. The superposition or the coupling of the top and bottom levels is responsible for the squeezed of the generated light. It appears to be quite difficult to prepare the atoms in a coherent superposition of the top and bottom levels before they are injected into the cavity. In addition, it should certainly be hard to find out that the atoms have decayed spontaneously before they are removed from the cavity.

In order to avoid the aforementioned problems, Fesseha [28] have considered that N two-level atoms available in a closed cavity are pumped to the top level by means of electron bombardment. He has shown that the light generated by this laser operating well above threshold is coherent and the light generated by the same laser operating below threshold is chaotic light. In addition, Fesseha [29] has studied the squeezing and the statistical properties of the light produced by a degenerate three-level laser with the atoms in a closed cavity and pumped by electron bombardment. He has shown that the maximum quadrature squeezing of the light generated by the laser, operating far below threshold, is 50% below the vacuum-state level. Alternatively, the three-level atoms available in a closed cavity and pumped by coherent light also generated squeezed light under certain conditions, with the maximum global quadrature squeezing is being 43% below the vacuum-state level [1]. It appears to be practically more convenient to pump the atoms by coherent light than electron bombardment.

In this paper, we investigate the steady-state analysis of the squeezing and statistical properties of the light generated by a coherently pumped degenerate three-level laser with open cavity which is coupled to a two-mode vacuum reservoir via a single-port mirror. We carry out our calculation by putting the noise operators associated with the vacuum reservoir in normal order and by taking into consideration the interaction of the three-level atoms with the vacuum reservoir outside the cavity.

2. Model and Dynamics of Atomic and Cavity Mode Operators

Let us consider a system of N degenerate three-level atoms in cascade configuration are available in an open cavity and interacting with the two (degenerate) cavity modes. The top and bottom levels of the three-level atoms are coupled by coherent light. When a degenerate three-level atom in cascade configuration decays from the top level to the bottom levels via the middle level, two photons of the same frequency are emitted. For the sake of convenient, we denote the top, middle, and bottom levels of these atoms by $|a\rangle_k$, $|b\rangle_k$, and $|c\rangle_k$, respectively. We wish to represent the light emitted from the top level by $\hat{a}_1$ and the light emitted from the middle by $\hat{a}_2$. In addition, we assume that the two cavity modes a_1 and a_2 to be at resonance with the two transitions $|a\rangle_k \to |b\rangle_k$ and $|b\rangle_k \to |c\rangle_k$, with direct transitions between levels $|a\rangle_k$ and $|c\rangle_k$ to be dipole forbidden.

The interaction of one of the three-level atoms with light modes a_1 and a_2 can be described at resonance by the Hamiltonian

$$\hat{H}_1(t) = ig\left[\hat{\sigma}_a^{\dagger k}\hat{a}_1 - \hat{a}_1^{\dagger}\hat{\sigma}_a^k + \hat{\sigma}_b^{\dagger k}\hat{a}_2 - \hat{a}_2^{\dagger}\hat{\sigma}_b^k\right], \tag{1}$$

where

$$\hat{\sigma}_a^k = |b\rangle_{kk}\langle a| \tag{2}$$

and

$$\hat{\sigma}_b^k = |c\rangle_{kk}\langle b| \tag{3}$$

are lowering atomic operators, $\hat{a}_1$ and $\hat{a}_2$ are the annihilation operators for light modes a_1 and a_2, and g is the coupling constant between the atom and the light mode a_1 or light mode a_2. And the interaction of the three-level atom with the driving coherent light can be described at resonance by the Hamiltonian

$$\hat{H}_2(t) = \frac{i\Omega}{2}\left[\hat{\sigma}_c^{\dagger k} - \hat{\sigma}_c^{k}\right], \tag{4}$$

in which

$$\hat{\sigma}_c^{k} = |c\rangle_{kk}\langle a| \tag{5}$$

and

$$\Omega = 2g'\varepsilon. \tag{6}$$

Here ε is the amplitude of the driving coherent light and g' is coupling constant between the atom and coherent light. Thus upon combining Equations (1) and (4), the interaction of the three-level atom with the driving coherent light and cavity modes a_1 and a_2 is described at resonance by the Hamiltonian

$$\hat{H}_s(t) = ig\left[\hat{\sigma}_a^{\dagger k}\hat{a}_1 - \hat{a}_1^{\dagger}\hat{\sigma}_a^{k} + \hat{\sigma}_b^{\dagger k}\hat{a}_2 - \hat{a}_2^{\dagger}\hat{\sigma}_b^{k}\right] + \frac{i\Omega}{2}\left[\hat{\sigma}_c^{\dagger k} - \hat{\sigma}_c^{k}\right]. \tag{7}$$

On the other hand, the degenerate three-level atoms available in an open cavity are coupled to a two-mode vacuum reservoir. The master equation for the three-level atom interacting with a two-mode vacuum reservoir has the form [1] [3]

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}(t) = &-i\left[\hat{H}_s(t),\hat{\rho}(t)\right] + \frac{\gamma}{2}\left[2\hat{\sigma}_b^{k}\hat{\rho}\hat{\sigma}_b^{\dagger k} - \hat{\sigma}_b^{\dagger k}\hat{\sigma}_b^{k}\hat{\rho} - \hat{\rho}\hat{\sigma}_b^{\dagger k}\hat{\sigma}_b^{k}\right]\\ &+\frac{\gamma}{2}\left[2\hat{\sigma}_c^{k}\hat{\rho}\hat{\sigma}_c^{\dagger k} - \hat{\sigma}_c^{\dagger k}\hat{\sigma}_c^{k}\hat{\rho} - \hat{\rho}\hat{\sigma}_c^{\dagger k}\hat{\sigma}_c^{k}\right],\end{aligned} \tag{8}$$

where γ is the spontaneous emission decay constant associated with the two modes a_1 and a_2. Hence with the aid of Equation (7), the master equation describing the two-mode cavity light of a coherently pumped degenerate three-level atom would be

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}(t) = &\, g\left[\hat{\sigma}_a^{\dagger k}\hat{a}_1\hat{\rho} - \hat{a}_1^{\dagger}\hat{\sigma}_a^{k}\hat{\rho} + \hat{\sigma}_b^{\dagger k}\hat{a}_2\hat{\rho} - \hat{a}_2^{\dagger}\hat{\sigma}_b^{k}\hat{\rho} - \hat{\rho}\hat{\sigma}_a^{\dagger k}\hat{a}_1 + \hat{\rho}\hat{a}_1^{\dagger}\hat{\sigma}_a^{k} - \hat{\rho}\hat{\sigma}_b^{\dagger k}\hat{a}_2 + \hat{\rho}\hat{a}_2^{\dagger}\hat{\sigma}_b^{k}\right]\\ &+\frac{\Omega}{2}\left[\hat{\sigma}_c^{\dagger k}\hat{\rho} - \hat{\sigma}_c^{k}\hat{\rho} + \hat{\rho}\hat{\sigma}_c^{k} - \hat{\rho}\hat{\sigma}_c^{\dagger k}\right] + \frac{\gamma}{2}\left[2\hat{\sigma}_b^{k}\hat{\rho}\hat{\sigma}_b^{\dagger k} - \hat{\sigma}_b^{\dagger k}\hat{\sigma}_b^{k}\hat{\rho} - \hat{\rho}\hat{\sigma}_b^{\dagger k}\hat{\sigma}_b^{k}\right]\\ &+\frac{\gamma}{2}\left[2\hat{\sigma}_c^{k}\hat{\rho}\hat{\sigma}_c^{\dagger k} - \hat{\sigma}_c^{\dagger k}\hat{\sigma}_c^{k}\hat{\rho} - \hat{\rho}\hat{\sigma}_c^{\dagger k}\hat{\sigma}_c^{k}\right].\end{aligned} \tag{9}$$

We recall that the laser cavity is coupled to a two-mode vacuum reservoir via a single-port mirror. In addition, we carry out our analysis by putting the noise operators associated with the vacuum reservoir in normal order. Thus the noise operators will not have any effect on the dynamics of the cavity mode operators [1] [28]. In view of this, we can drop the noise operators and write the quantum Langevin equation for the operators $\hat{a}_1$ and $\hat{a}_2$ as

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{a}_1(t) = -\frac{k}{2}\hat{a}_1(t) - i\left[\hat{a}_1(t),\hat{H}_s(t)\right], \tag{10}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{a}_2(t) = -\frac{k}{2}\hat{a}_2(t) - i\left[\hat{a}_2(t),\hat{H}_s(t)\right], \tag{11}$$

where k is the cavity damping constant. Then with the aid of Equation (7), we easily find

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{a}_1(t) = -\frac{k}{2}\hat{a}_1(t) - g\hat{\sigma}_a^{k}(t), \tag{12}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{a}_2(t) = -\frac{k}{2}\hat{a}_2(t) - g\hat{\sigma}_b^{k}(t). \tag{13}$$

The procedure of normal ordering the noise operators renders the vacuum reservoir to be a noiseless physical entity. We uphold the view point that the notion of a noiseless vacuum reservoir would turn out to be compatible with observation [30]. Furthermore, making use of the master equation and the fact that $\frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{A} \rangle = Tr\left(\frac{\mathrm{d}\hat{\rho}(t)}{\mathrm{d}t}\hat{A}\right)$ (where $\hat{A}$ is an operator), it is not difficult to verify that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{\sigma}_a^k \rangle = g\left[\langle \hat{\eta}_b^k \hat{a}_1 \rangle - \langle \hat{\eta}_a^k \hat{a}_1 \rangle + \langle \hat{a}_2^\dagger \hat{\sigma}_c^k \rangle\right] + \frac{\Omega}{2}\langle \hat{\sigma}_b^{\dagger k} \rangle - \gamma \langle \hat{\sigma}_a^k \rangle, \tag{14}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{\sigma}_b^k \rangle = g\left[\langle \hat{\eta}_c^k \hat{a}_2 \rangle - \langle \hat{\eta}_b^k \hat{a}_2 \rangle - \langle \hat{a}_1^\dagger \hat{\sigma}_c^k \rangle\right] - \frac{\Omega}{2}\langle \hat{\sigma}_a^{\dagger k} \rangle - \frac{\gamma}{2}\langle \hat{\sigma}_b^k \rangle, \tag{15}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{\sigma}_c^k \rangle = g\left[\langle \hat{\sigma}_b^k \hat{a}_1 \rangle - \langle \hat{\sigma}_a^k \hat{a}_2 \rangle\right] + \frac{\Omega}{2}\left[\langle \hat{\eta}_c^k \rangle - \langle \hat{\eta}_a^k \rangle\right] - \frac{\gamma}{2}\langle \hat{\sigma}_c^k \rangle, \tag{16}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{\eta}_a^k \rangle = g\left[\langle \hat{\sigma}_a^{\dagger k} \hat{a}_1 \rangle + \langle \hat{a}_1^\dagger \hat{\sigma}_a^k \rangle\right] + \frac{\Omega}{2}\left[\langle \hat{\sigma}_c^{\dagger k} \rangle + \langle \hat{\sigma}_c^k \rangle\right] - \gamma \langle \hat{\eta}_a^k \rangle, \tag{17}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{\eta}_b^k \rangle = g\left[\langle \hat{\sigma}_b^{\dagger k} \hat{a}_2 \rangle + \langle \hat{a}_2^\dagger \hat{\sigma}_b^k \rangle - \langle \hat{\sigma}_a^{\dagger k} \hat{a}_1 \rangle - \langle \hat{a}_1^\dagger \hat{\sigma}_a^k \rangle\right] - \gamma \langle \hat{\eta}_b^k \rangle, \tag{18}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{\eta}_c^k \rangle = -g\left[\langle \hat{\sigma}_b^{\dagger k} \hat{a}_1 \rangle + \langle \hat{a}_1^\dagger \hat{\sigma}_b^k \rangle\right] - \frac{\Omega}{2}\left[\langle \hat{\sigma}_c^{\dagger k} \rangle + \langle \hat{\sigma}_c^k \rangle\right] + \gamma\left[\langle \hat{\eta}_a^k \rangle + \langle \hat{\eta}_b^k \rangle\right], \tag{19}$$

where

$$\hat{\eta}_a^k = |a\rangle_{kk}\langle a|, \tag{20}$$

$$\hat{\eta}_b^k = |b\rangle_{kk}\langle b|, \tag{21}$$

$$\hat{\eta}_c^k = |c\rangle_{kk}\langle c|. \tag{22}$$

We see that Equations (14)-(19) are nonlinear and coupled differential equations. Therefore, it is not possible to obtain the exact time-dependent solutions. We intend to overcome this problem by applying the large-time approximation [28]. Then using this approximation scheme, we get from Equations (12) and (13) the approximately valid relations

$$\hat{a}_1 = -\frac{2g}{k}\hat{\sigma}_a^k \tag{23}$$

and

$$\hat{a}_2 = -\frac{2g}{k}\hat{\sigma}_b^k. \tag{24}$$

Upon substituting (23) and (24) into Equations (14)-(19), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{\sigma}_a^k \rangle = -[\gamma_c + \gamma]\langle \hat{\sigma}_a^k \rangle + \frac{\Omega}{2}\langle \hat{\sigma}_b^{\dagger k} \rangle, \tag{25}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{\sigma}_b^k \rangle = -\frac{1}{2}[\gamma_c + \gamma]\langle \hat{\sigma}_b^k \rangle - \frac{\Omega}{2}\langle \hat{\sigma}_a^{\dagger k} \rangle, \tag{26}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{\sigma}_c^k \rangle = -\frac{1}{2}[\gamma_c + \gamma]\langle \hat{\sigma}_c^k \rangle + \frac{\Omega}{2}\left[\langle \hat{\eta}_c^k \rangle - \langle \hat{\eta}_a^k \rangle\right], \tag{27}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{\eta}_a^k \rangle = -[\gamma_c + \gamma]\langle \hat{\eta}_a^k \rangle + \frac{\Omega}{2}\left[\langle \hat{\sigma}_c^{\dagger k} \rangle + \langle \hat{\sigma}_c^k \rangle\right], \tag{28}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{\eta}_b^k \rangle = -[\gamma_c + \gamma]\langle \hat{\eta}_b^k \rangle + \gamma_c \langle \hat{\eta}_a^k \rangle, \tag{29}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{\eta}_c^k \rangle = [\gamma_c + \gamma]\langle \hat{\eta}_b^k \rangle + \gamma \langle \hat{\eta}_a^k \rangle - \frac{\Omega}{2}\left[\langle \hat{\sigma}_c^{\dagger k} \rangle + \langle \hat{\sigma}_c^k \rangle\right], \tag{30}$$

where

$$\gamma_c = \frac{4g^2}{k} \tag{31}$$

is the stimulated emission decay constant. We next sum Equations (25)-(30) over the N three-level atoms. We then see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{m}_a \rangle = -[\gamma_c + \gamma]\langle \hat{m}_a \rangle + \frac{\Omega}{2}\langle \hat{m}_b^\dagger \rangle, \tag{32}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{m}_b \rangle = -\frac{1}{2}[\gamma_c + \gamma]\langle \hat{m}_b \rangle - \frac{\Omega}{2}\langle \hat{m}_a^\dagger \rangle, \tag{33}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{m}_c \rangle = -\frac{1}{2}[\gamma_c + \gamma]\langle \hat{m}_c \rangle + \frac{\Omega}{2}\left[\langle \hat{N}_c \rangle - \langle \hat{N}_a \rangle\right], \tag{34}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{N}_a \rangle = -[\gamma_c + \gamma]\langle \hat{N}_a \rangle + \frac{\Omega}{2}\left[\langle \hat{m}_c^\dagger \rangle + \langle \hat{m}_c \rangle\right], \tag{35}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{N}_b \rangle = -[\gamma_c + \gamma]\langle \hat{N}_b \rangle + \gamma_c\langle \hat{N}_a \rangle, \tag{36}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{N}_c \rangle = [\gamma_c + \gamma]\langle \hat{N}_b \rangle - \frac{\Omega}{2}\left[\langle \hat{m}_c^\dagger \rangle + \langle \hat{m}_c \rangle\right] + \gamma\langle \hat{N}_a \rangle, \tag{37}$$

in which

$$\hat{m}_a = \sum_{k=1}^{N} \hat{\sigma}_a^k, \tag{38}$$

$$\hat{m}_b = \sum_{k=1}^{N} \hat{\sigma}_b^k, \tag{39}$$

$$\hat{m}_c = \sum_{k=1}^{N} \hat{\sigma}_c^k, \tag{40}$$

$$\hat{N}_a = \sum_{k=1}^{N} \hat{\eta}_a^k, \tag{41}$$

$$\hat{N}_b = \sum_{k=1}^{N} \hat{\eta}_b^k, \tag{42}$$

$$\hat{N}_c = \sum_{k=1}^{N} \hat{\eta}_c^k, \tag{43}$$

with the operators $\hat{N}_a$, $\hat{N}_b$, and $\hat{N}_c$ representing the number of atoms in the top, middle, and bottom levels. In addition, employing the completeness relation

$$\hat{\eta}_a^k + \hat{\eta}_b^k + \hat{\eta}_c^k = \hat{I}, \tag{44}$$

we easily arrive at

$$\langle \hat{N}_a \rangle + \langle \hat{N}_b \rangle + \langle \hat{N}_c \rangle = N. \tag{45}$$

Furthermore, applying the definition given by (2) and setting for any k

$$\hat{\sigma}_a^k = |b\rangle\langle a|, \tag{46}$$

we have

$$\hat{m}_a = N|b\rangle\langle a|. \tag{47}$$

Following the same procedure, one can easily find

$$\hat{m}_b = N|c\rangle\langle b|, \tag{48}$$

$$\hat{m}_c = N|c\rangle\langle a|, \tag{49}$$

$$\hat{N}_a = N|a\rangle\langle a|, \tag{50}$$

$$\hat{N}_b = N|b\rangle\langle b|, \tag{51}$$

$$\hat{N}_c = N|c\rangle\langle c|. \tag{52}$$

Moreover, using the definition

$$\hat{m} = \hat{m}_a + \hat{m}_b \tag{53}$$

and taking into account Equations (47)-(52), it can be readily established that

$$\hat{m}^\dagger \hat{m} = N\left(\hat{N}_a + \hat{N}_b\right), \tag{54}$$

$$\hat{m}\hat{m}^\dagger = N\left(\hat{N}_b + \hat{N}_c\right), \tag{55}$$

$$\hat{m}^2 = N\hat{m}_c. \tag{56}$$

Upon adding Equations (12) and (13), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{a}(t) = -\frac{k}{2}\hat{a}(t) - g\left(\hat{\sigma}_a^k(t) + \hat{\sigma}_b^k(t)\right), \tag{57}$$

where

$$\hat{a}(t) = \hat{a}_1(t) + \hat{a}_2(t). \tag{58}$$

In the presence of N three-level atoms, we can rewrite Equation (57) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{a}(t) = -\frac{k}{2}\hat{a}(t) + \lambda\hat{m}(t), \tag{59}$$

in which λ is a constant whose value remains to be fixed. The steady-state solution of Equation (57) is

$$\hat{a}(t) = -\frac{2g}{k}\left(\hat{\sigma}_a^k(t) + \hat{\sigma}_b^k(t)\right). \tag{60}$$

Taking into account of (60) and its adjoint, the commutation relation for the cavity mode operator is found to be

$$\left[\hat{a}, \hat{a}^\dagger\right]_k = \frac{\gamma_c}{k}\left(\hat{\eta}_c^k - \hat{\eta}_a^k\right) \tag{61}$$

and on summing over all atoms, we have

$$\left[\hat{a}, \hat{a}^\dagger\right] = \frac{\gamma_c}{k}\left(\hat{N}_c - \hat{N}_a\right), \tag{62}$$

where

$$\left[\hat{a}, \hat{a}^\dagger\right] = \sum_{k=1}^{N}\left[\hat{a}, \hat{a}^\dagger\right]_k, \tag{63}$$

stands for the commutator of $\left(\hat{a}, \hat{a}^\dagger\right)$ when the superposed light mode a is interacting with all the N three-level atoms. On the other hand, using the steady-state solution of Equation (59), one can verify that

$$\left[\hat{a}, \hat{a}^\dagger\right] = N\left[\frac{2\lambda}{k}\right]^2\left(\hat{N}_c - \hat{N}_a\right). \tag{64}$$

Comparison of Equations (62) and (64) shows that

$$\lambda = \pm\frac{g}{\sqrt{N}}. \tag{65}$$

On account of (65), one can put Equation (59) in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{a}(t)=-\frac{k}{2}\hat{a}(t)+\frac{g}{\sqrt{N}}\hat{m}(t). \tag{66}$$

3. Photon Statistics

Here we seek to obtain the global (local) mean photon number and the global (local) variance of the photon number for the two-mode cavity light beam at steady state.

3.1. The Global Mean Photon Number

We wish to calculate the mean photon number of the two-mode cavity light in the entire frequency interval. The steady-state solution of Equation (66) is given by

$$\hat{a}(t)=\frac{2g}{k\sqrt{N}}\hat{m}(t). \tag{67}$$

On account of (67) together with (54), the mean photon number of the two-mode cavity light is expressible as

$$\bar{n}=\frac{\gamma_c}{k}\left\{\left\langle \hat{N}_a\right\rangle+\left\langle \hat{N}_b\right\rangle\right\}, \tag{68}$$

in which $\bar{n}=\left\langle \hat{a}^{\dagger}(t)\hat{a}(t)\right\rangle$. With the aid of (147) and (148), the mean photon number of the two-mode cavity light turns out to be

$$\bar{n}=\frac{\gamma_c}{k}\left\{\frac{\Omega^2(2\gamma_c+\gamma)}{(\gamma_c+\gamma)^3+\Omega^2(3\gamma_c+2\gamma)}\right\}N. \tag{69}$$

We note that the global mean photon number takes for $\Omega \gg \gamma_c$ the form

$$\bar{n}=\frac{\gamma_c}{k}\left\{\frac{2\gamma_c+\gamma}{3\gamma_c+2\gamma}\right\}N. \tag{70}$$

We observe from the plots in Figure 1 that the presence of spontaneous emission leads to a decrease in the global mean photon number of the two-mode cavity light beam.

3.2. Local Mean Photon Number

We seek to determine the mean photon number in a given frequency interval, employing the power spectrum for the two-mode cavity light. The power spectrum of a two-mode cavity light with central common frequency ω_0 is defined as

$$P(\omega)=\frac{1}{\pi}Re\int_0^{\infty}\mathrm{d}\tau\, \mathrm{e}^{i(\omega-\omega_0)\tau}\left\langle \hat{a}^{\dagger}(t)\hat{a}(t+\tau)\right\rangle_{ss}. \tag{71}$$

On introducing (162) into Equation (71) and carrying out the integration, we readily get

$$P(\omega)=\bar{n}\left\{\left[\frac{k}{k-\nu}\right]\left[\frac{\nu/2\pi}{(\omega-\omega_0)^2+(\nu/2)^2}\right]-\left[\frac{\nu}{k-\nu}\right]\left[\frac{k/2\pi}{(\omega-\omega_0)^2+(k/2)^2}\right]\right\}. \tag{72}$$

The mean photon number in the frequency interval between $\omega'=-\lambda$ and $\omega'=+\lambda$ is expressible as

$$\bar{n}_{\pm\lambda}=\int_{-\lambda}^{+\lambda}P(\omega')\mathrm{d}\omega', \tag{73}$$

in which $\omega'=\omega-\omega_0$. Thus upon substituting (72) into Equation (73), we find

$$\bar{n}_{\pm\lambda}=\left[\frac{k\bar{n}}{k-\nu}\right]\int_{-\lambda}^{+\lambda}\left[\frac{\nu/2\pi}{\omega'^2+(\nu/2)^2}\right]\mathrm{d}\omega'-\left[\frac{\nu\bar{n}}{k-\nu}\right]\int_{-\lambda}^{+\lambda}\left[\frac{k/2\pi}{\omega'^2+(k/2)^2}\right]\mathrm{d}\omega' \tag{74}$$

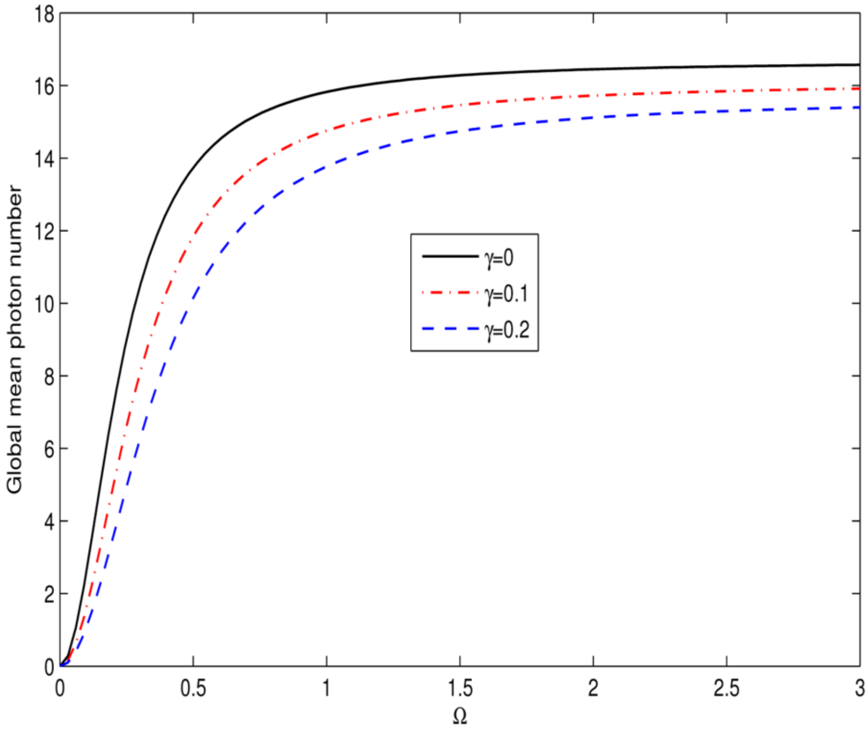

Figure 1. Plots of the global mean photon number [Equation (69)] versus Ω at steady state for $\gamma_c = 0.4$, $k = 0.8$, $N = 50$, and different values of γ.

and on carrying out the integration over ω', applying the relation

$$\int_{-\lambda}^{+\lambda} \frac{\mathrm{d}x}{x^2 + a^2} = \frac{2}{a}\tan^{-1}\left(\frac{\lambda}{a}\right), \tag{75}$$

we arrive at

$$\bar{n}_{\pm\lambda} = \bar{n} z(\lambda), \tag{76}$$

where

$$z(\lambda) = \left[\frac{2k/\pi}{k-\nu}\right]\tan^{-1}\left(\frac{2\lambda}{\nu}\right) - \left[\frac{2\nu/\pi}{k-\nu}\right]\tan^{-1}\left(\frac{2\lambda}{k}\right). \tag{77}$$

One can readily get from **Figure 2** that $z(0.5) = 0.7671$, $z(1) = 0.9362$, and $z(2) = 0.9889$ for $\gamma = 0$. And for $\gamma = 0.2$, we find $z(0.5) = 0.7102$, $z(1) = 0.9015$, and $z(2) = 0.979$. Then combination of these results with Equation (76) yields $\bar{n}_{\pm 0.5} = 0.7671\bar{n}$, $\bar{n}_{\pm 1} = 0.9362\bar{n}$, and $\bar{n}_{\pm 2} = 0.9889\bar{n}$ for $\gamma = 0$. And we have $\bar{n}_{\pm 0.5} = 0.7102\bar{n}$, $\bar{n}_{\pm 1} = 0.9015\bar{n}$, and $\bar{n}_{\pm 2} = 0.979\bar{n}$ for $\gamma = 0.2$. We therefore observe that a large part of the total mean photon number is confined in a relatively small frequency interval.

3.3. The Global Variance of the Photon Number

The variance of the photon number for the two-mode cavity light is expressible as

$$(\Delta n)^2 = \left\langle \left(\hat{a}^\dagger \hat{a}\right)^2 \right\rangle - \left\langle \hat{a}^\dagger \hat{a} \right\rangle^2 \tag{78}$$

and using the fact that $\hat{a}(t)$ is a Gaussian variable with zero mean, we arrive at

$$(\Delta n)^2 = \left\langle \hat{a}^\dagger \hat{a} \right\rangle \left\langle \hat{a} \hat{a}^\dagger \right\rangle + \left\langle \hat{a}^{\dagger 2} \right\rangle \left\langle \hat{a}^2 \right\rangle. \tag{79}$$

Employing once more (67) and taking into account (55), we readily get

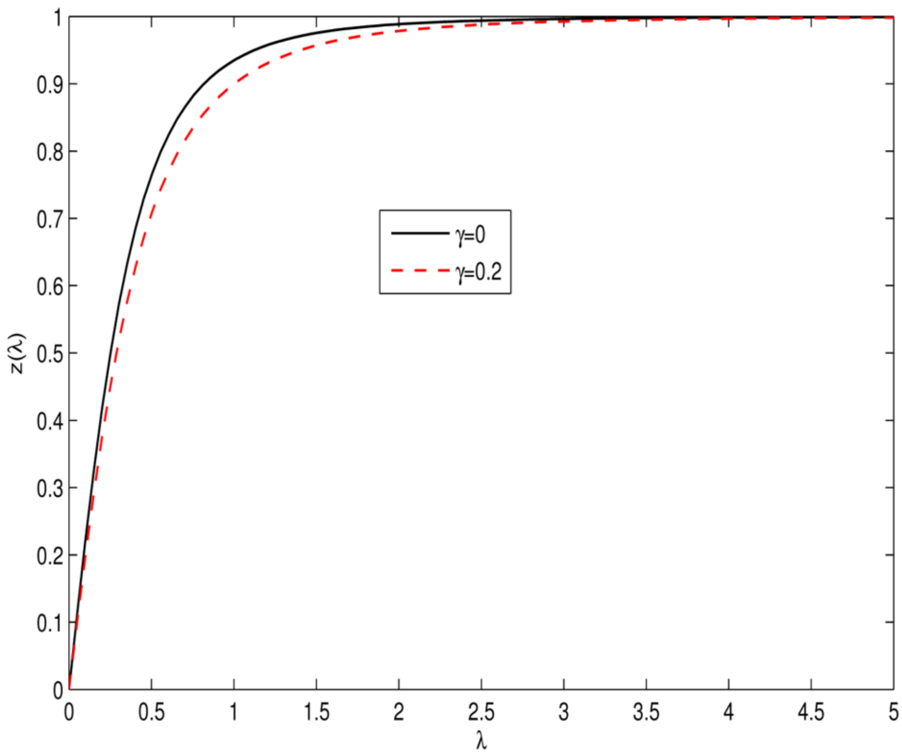

Figure 2. Plot of $z(\lambda)$ [Equation (77)] versus λ for $\gamma_c = 0.4$, $k = 0.8$, $\Omega = 3$, and different values of γ.

$$\langle \hat{a}\hat{a}^\dagger \rangle = \frac{\gamma_c}{k}\left[\langle \hat{N}_b \rangle + \langle \hat{N}_c \rangle\right]. \quad (80)$$

On account of (45), one can be put Equation (80) in the form

$$\langle \hat{a}\hat{a}^\dagger \rangle = \frac{\gamma_c}{k}\left[N - \langle \hat{N}_a \rangle\right], \quad (81)$$

with the aid of (68) and (145), we arrive at

$$\langle \hat{a}\hat{a}^\dagger \rangle = \frac{\gamma_c}{k} N - \bar{n}\left[\frac{\gamma_c + \gamma}{2\gamma_c + \gamma}\right]. \quad (82)$$

On the other hand, using (67) along with (56), we easily obtain

$$\langle \hat{a}^2 \rangle = \frac{\gamma_c}{k}\langle \hat{m}_c \rangle, \quad (83)$$

so that in view of (146) and (69), there follows

$$\langle \hat{a}^2 \rangle = \bar{n}\left[\frac{(\gamma_c + \gamma)^2}{\Omega(2\gamma_c + \gamma)}\right]. \quad (84)$$

Now on account of Equations (68), (83), and (84), we readily find Equation (79) to be

$$(\Delta n)^2 = \bar{n}\left[\frac{\gamma_c}{k} N - \bar{n}\left(\frac{\gamma_c + \gamma}{2\gamma_c + \gamma}\right)\right] + \bar{n}^2\left[\frac{(\gamma_c + \gamma)^4}{\Omega^2 (2\gamma_c + \gamma)^2}\right]. \quad (85)$$

This can be put in the form

$$\left(\Delta n\right)^2 = \left[\frac{\gamma_c}{k}N - \left\{\frac{\gamma_c+\gamma}{2\gamma_c+\gamma}\right\}\left\{\frac{\Omega^2\left(2\gamma_c+\gamma\right)-\left(\gamma_c+\gamma\right)^3}{\Omega^2\left(2\gamma_c+\gamma\right)}\right\}\bar{n}\right]\bar{n}. \tag{86}$$

In view of (69), we arrive at

$$\left(\Delta n\right)^2 = \left(\frac{\gamma_c}{k}N\right)^2\left\{\frac{\Omega^2\left(3\gamma_c+2\gamma\right)\left(\gamma_c+\gamma\right)^3}{\left(\left(\gamma_c+\gamma\right)^3+\Omega^2\left(3\gamma_c+2\gamma\right)^2\right)^2} + \frac{\Omega^4\left(2\gamma_c+\gamma\right)^2}{\left(\left(\gamma_c+\gamma\right)^3+\Omega^2\left(3\gamma_c+2\gamma\right)^2\right)^2}\right\}. \tag{87}$$

We immediately see from the plots in **Figure 3** that the presence of spontaneous emission leads to a decrease in the global variance of the photon number of the two-mode cavity light beam. In addition, the global variance of the photon number of the two-mode cavity light increases with increasing Ω.

Finally, we note that the variance of the photon number takes for $\Omega \gg \gamma_c$ the form

$$= \bar{n}^2, \tag{88}$$

in which $\bar{n}$ is given by (70). This represents the normally ordered variance of the photon number for a chaotic state.

3.4. Local Variance of the Photon Number

Here we wish to obtain the variance of the photon number in a given frequency interval, employing the spectrum of the photon number fluctuations for the superposition of light modes a_1 and a_2. We denote the central common frequency of these modes by ω_0. The spectrum of the photon number fluctuations for the superposed light modes can be expressed as

$$R\left(\omega\right) = \frac{1}{\pi}Re\int_0^{+\infty}\mathrm{d}\tau\,\mathrm{e}^{i\left(\omega-\omega_0\right)\tau}\left\langle\hat{n}\left(t\right),\hat{n}\left(t+\tau\right)\right\rangle_{ss}, \tag{89}$$

where

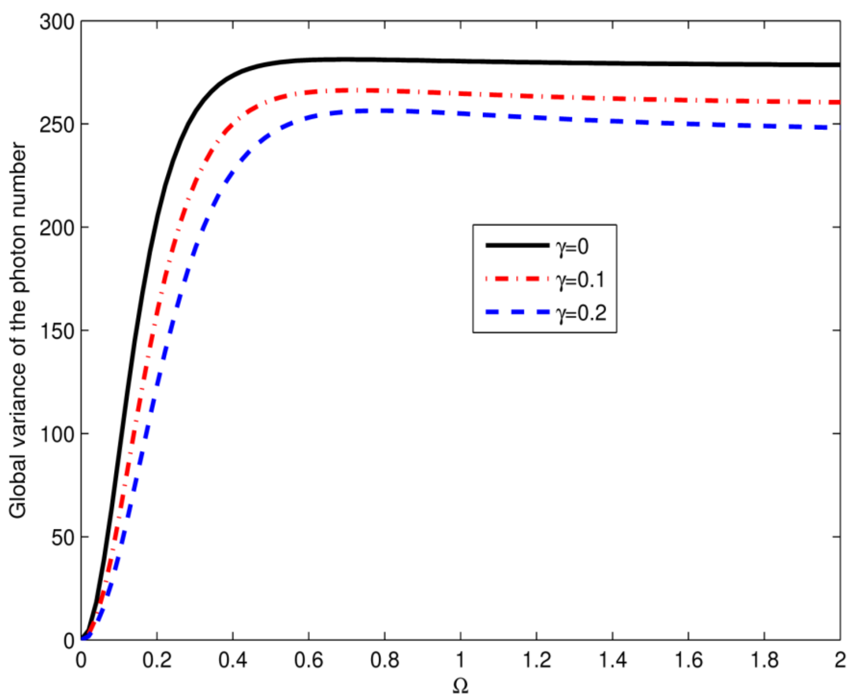

Figure 3. Plots of the global variance of the photon number [Equation (87)] versus Ω at steady state for $\gamma_c = 0.4$, $k = 0.8$, $N = 50$, and different values of γ.

$$\hat{n}(t)=\hat{a}^{\dagger}(t)\hat{a}(t), \tag{90}$$

$$\hat{n}(t+\tau)=\hat{a}^{\dagger}(t+\tau)\hat{a}(t+\tau) \tag{91}$$

and we have used the notation $\langle\hat{n}(t),\hat{n}(t+\tau)\rangle=\langle\hat{n}(t)\hat{n}(t+\tau)\rangle-\langle\hat{n}(t)\rangle\langle\hat{n}(t+\tau)\rangle$. With the aid of (90) and (91) and Equation (142), the photon number fluctuation can be expressed as

$$\begin{aligned}R(\omega)=\frac{1}{\pi}Re\int_0^{+\infty}\mathrm{d}\tau\,\mathrm{e}^{i(\omega-\omega_0)\tau}\Big[&\langle\hat{a}^{\dagger}(t)\hat{a}(t+\tau)\rangle\langle\hat{a}(t)\hat{a}^{\dagger}(t+\tau)\rangle\\&+\langle\hat{a}^{\dagger}(t)\hat{a}^{\dagger}(t+\tau)\rangle\langle\hat{a}(t)\hat{a}(t+\tau)\rangle\Big].\end{aligned} \tag{92}$$

Upon introducing (162)-(165) into Equation (92) and on carrying out the integration over τ, the spectrum of the photon number fluctuations for the two-mode cavity light is found to be

$$\begin{aligned}R(\omega)=(\Delta n)^2\Bigg\{&\left[\frac{k^2}{(k-\nu)^2}\right]\left[\frac{\nu/\pi}{(\omega-\omega_0)^2+\nu^2}\right]+\left[\frac{\nu^2}{(k-\nu)^2}\right]\left[\frac{k/\pi}{(\omega-\omega_0)^2+k^2}\right]\\&-\left[\frac{2k\nu}{(k-\nu)^2}\right]\left[\frac{(k+\nu)/2\pi}{(\omega-\omega_0)^2+(k+\nu)^2/4}\right]\Bigg\},\end{aligned} \tag{93}$$

where $(\Delta n)^2$ is given by (79).

Upon integrating both sides of (93) over ω, one easily finds

$$\int_{-\infty}^{+\infty}R(\omega)\mathrm{d}\omega=(\Delta n)_{ss}^2. \tag{94}$$

On the basis of Equation (94), we observe that $R(\omega)\mathrm{d}\omega$ represents the steady-state variance of the photon number for the two-mode cavity light in the interval between ω and $\omega+\mathrm{d}\omega$. We thus realize that the photon-number variance in the interval between $\omega'=-\lambda$ and $\omega'=+\lambda$ can be written as

$$(\Delta n)_{\pm\lambda}^2=\int_{-\lambda}^{+\lambda}R(\omega')\mathrm{d}\omega', \tag{95}$$

in which $\omega'=\omega-\omega_0$. Thus upon substituting (93) into Equation (95), we find

$$\begin{aligned}(\Delta n)_{\pm\lambda}^2=(\Delta n)_{ss}^2\Bigg\{&\left[\frac{k^2}{(k-\nu)^2}\right]\int_{-\lambda}^{+\lambda}\left[\frac{\nu/\pi}{\omega'^2+\nu^2}\right]\mathrm{d}\omega'+\left[\frac{\nu^2}{(k-\nu)^2}\right]\int_{-\lambda}^{+\lambda}\left[\frac{k/\pi}{\omega'^2+k^2}\right]\mathrm{d}\omega'\\&-\left[\frac{2k\nu}{(k-\nu)^2}\right]\int_{-\lambda}^{+\lambda}\left[\frac{(k+\nu)/2\pi}{\omega'^2+(k+\nu)^2/4}\right]\mathrm{d}\omega'\Bigg\},\end{aligned} \tag{96}$$

so on carrying out the integration over ω', applying the relation described by Equation (75), we readily get

$$(\Delta n)_{\pm\lambda}^2=(\Delta n)^2 z'(\lambda), \tag{97}$$

where $z'(\lambda)$ is given by

$$z'(\lambda)=\left[\frac{2k^2/\pi}{(k-\nu)^2}\right]\tan^{-1}\left(\frac{\lambda}{\nu}\right)+\left[\frac{2\nu^2/\pi}{(k-\nu)^2}\right]\tan^{-1}\left(\frac{\lambda}{k}\right)-\left[\frac{4k\nu/\pi}{(k-\nu)^2}\right]\tan^{-1}\left(\frac{2\lambda}{k+\nu}\right). \tag{98}$$

From the plots in **Figure 4** that we easily find $z'(0.5)=0.7625$, $z'(1)=0.9483$, and $z'(2)=0.9934$ for $\gamma=0$. And for $\gamma=0.2$, we find $z'(0.5)=0.5204$, $z'(1)=0.7893$, and $z'(2)=0.9496$. Then combination of these results with Equation (97) yields $(\Delta n)_{\pm0.5}^2=0.7625(\Delta n)^2$, $(\Delta n)_{\pm1}^2=0.9483(\Delta n)^2$, and $(\Delta n)_{\pm2}^2=0.9934(\Delta n)^2$ for $\gamma=0$. And for $\gamma=0.2$, we find $(\Delta n)_{\pm0.5}^2=0.5204(\Delta n)^2$, $(\Delta n)_{\pm1}^2=0.7893(\Delta n)^2$, and $(\Delta n)_{\pm2}^2=0.9496(\Delta n)^2$. We immediately observe that a large part of the total variance of the photon number is confined in a relatively small frequency interval.

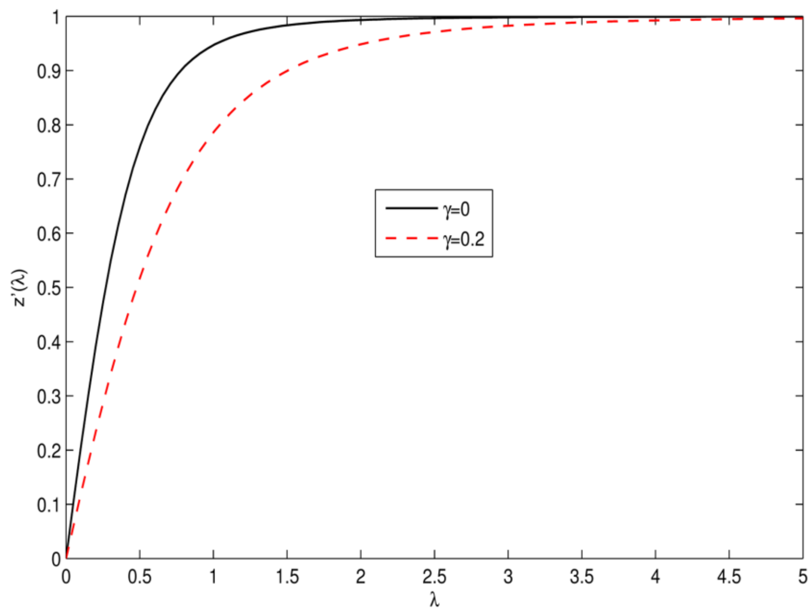

Figure 4. Plots of $z'(\lambda)$ [Equation (98)] versus λ for $\gamma_c = 0.4$, $k = 0.8$, $\Omega = 2.0$, and different values of γ.

4. Quadrature Squeezing

In this section we seek to calculate the quadrature squeezing of the two-mode cavity light in any frequency interval.

4.1. The Global Quadrature Squeezing

The squeezing properties of the two-mode cavity light are described by two quadrature operators defined as

$$\hat{a}_+(t) = \hat{a}^\dagger(t) + \hat{a}(t), \tag{99}$$

$$\hat{a}_-(t) = i\left(\hat{a}^\dagger(t) - \hat{a}(t)\right). \tag{100}$$

It can be readily established that

$$[\hat{a}_-, \hat{a}_+] = 2i\frac{\gamma_c}{k}\left(\hat{N}_a - \hat{N}_c\right). \tag{101}$$

It then follows that

$$\Delta a_+ \Delta a_- \geq \frac{\gamma_c}{k}\left|\langle \hat{N}_a \rangle - \langle \hat{N}_c \rangle\right|. \tag{102}$$

Now upon replacing the atomic operators that appear in Equation (62) by their expectation values, the commutation relation for the two-mode light can write as

$$\left[\hat{a}, \hat{a}^\dagger\right] = \lambda, \tag{103}$$

in which

$$\lambda = \frac{\gamma_c}{k}\left(\langle \hat{N}_c \rangle - \langle \hat{N}_a \rangle\right). \tag{104}$$

The variance of the quadrature operators is expressible as

$$(\Delta a_\pm)^2 = \lambda + 2\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle \pm \langle \hat{a}^{\dagger 2}(t)\rangle \pm \langle \hat{a}^2(t)\rangle \mp \langle \hat{a}^\dagger(t)\rangle^2 \mp \langle \hat{a}(t)\rangle^2 - 2\langle \hat{a}^\dagger(t)\rangle\langle \hat{a}(t)\rangle. \tag{105}$$

In view of Equation (142), one can put Equation (105) in the form

$$(\Delta a_{\pm})^2 = \lambda + 2\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle \pm \langle \hat{a}^{\dagger 2}(t)\rangle \pm \langle \hat{a}^2(t)\rangle. \tag{106}$$

With the aid of (68), (83), and (104) together with (45), we obtain

$$(\Delta a_+)^2 = \frac{\gamma_c}{k}\left\{N + \langle \hat{N}_b\rangle + 2\langle \hat{m}_c\rangle\right\}, \tag{107}$$

$$(\Delta a_-)^2 = \frac{\gamma_c}{k}\left\{N + \langle \hat{N}_b\rangle - 2\langle \hat{m}_c\rangle\right\}. \tag{108}$$

Finally, on account of (146) and (148), the global quadrature variance of the two-mode cavity light turns out at steady state to be

$$(\Delta a_+)^2 = \frac{\gamma_c}{k}\left\{1 + \frac{\Omega^2\gamma_c + 2\Omega(\gamma_c+\gamma)^2}{(\gamma_c+\gamma)^3 + \Omega^2(3\gamma_c+2\gamma)}\right\}N \tag{109}$$

and

$$(\Delta a_-)^2 = \frac{\gamma_c}{k}\left\{1 + \frac{\Omega^2\gamma_c - 2\Omega(\gamma_c+\gamma)^2}{(\gamma_c+\gamma)^3 + \Omega^2(3\gamma_c+2\gamma)}\right\}N. \tag{110}$$

It is then not difficult to observe that the two-mode cavity light beam is in a squeezed state and the squeezing occurs in the minus quadrature.

We next proceed to calculate the quadrature squeezing of the two-mode cavity light relative to the quadrature variance of the two-mode cavity vacuum state. We define the quadrature squeezing of the two-mode cavity light by

$$S = 1 - \frac{(\Delta a_-)^2}{(\Delta a_-)_v^2}, \tag{111}$$

Moreover, upon setting $\Omega = 0$ in Equation (110), we see that

$$(\Delta a_-)_v^2 = \frac{\gamma_c}{k}N, \tag{112}$$

which represents the quadrature variance of the two-mode cavity vacuum state. Hence on account of Equations (110) and (112), we arrive at

$$S = \left\{\frac{2\Omega(\gamma_c+\gamma)^2}{(\gamma_c+\gamma)^3 + \Omega^2(3\gamma_c+2\gamma)} - \frac{\Omega^2\gamma_c}{(\gamma_c+\gamma)^3 + \Omega^2(3\gamma_c+2\gamma)}\right\}. \tag{113}$$

We note that, unlike the mean photon number, the quadrature squeezing does not depend on the number of atoms. This implies that the quadrature squeezing of the two-mode cavity light is independent of the number of photons. We see from the plots in **Figure 5** that the maximum global quadrature squeezing of the two-mode cavity light for $\gamma = 0$ is 43.42% (and occurs at $\Omega = 0.1717$) and for $\gamma = 0.1$ is found 47.15% (and occurs at $\Omega = 0.2323$). And for $\gamma = 0.2$, the maximum global quadrature squeezing is observed to be 50% below the vacuum-state level and this occurs when the three-level laser is operating at $\Omega = 0.303$. Moreover, upon setting $\varepsilon = 0$ in Equation (113), we note that

$$S = \frac{2\chi - \chi^2}{1 + 3\chi^2}, \tag{114}$$

where $\chi = \Omega/\gamma_c$. Equation (114) is indicates that the quadrature squeezing of the light produced by degenerate three-level laser with the N three-level atoms available inside a closed cavity pumped to the top level by electron bombardment which has been reported by Fesseha [1].

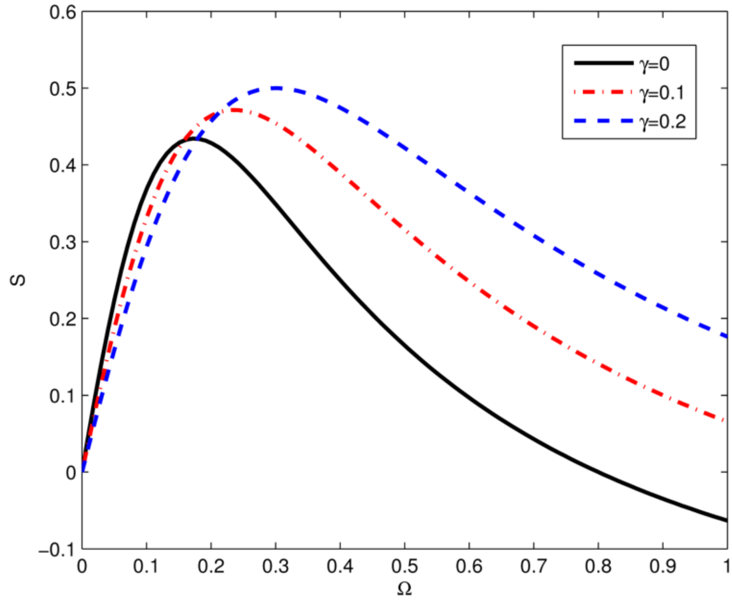

Figure 5. Plots of the global quadrature squeezing [Equation (113)] versus Ω at steady state for $\gamma_c = 0.4$, $k = 0.8$, and different values of γ.

4.2. Local Quadrature Squeezing

Here we wish to obtain the quadrature squeezing of the two-mode cavity light in a given frequency interval. To this end, we first obtain the spectrum of the quadrature fluctuations of the superposition of light modes a_1 and a_2. We define this spectrum for the two-mode cavity light by

$$S_{\pm}(\omega) = \frac{1}{\pi} Re\int_0^{\infty} \mathrm{d}\tau \mathrm{e}^{i(\omega-\omega_0)\tau} \left\langle \hat{a}_{\pm}(t), \hat{a}_{\pm}(t+\tau)\right\rangle_{ss}, \tag{115}$$

in which

$$\hat{a}_{+}(t+\tau) = \hat{a}^{\dagger}(t+\tau) + \hat{a}(t+\tau), \tag{116}$$

$$\hat{a}_{-}(t+\tau) = i\left(\hat{a}^{\dagger}(t+\tau) - \hat{a}(t+\tau)\right) \tag{117}$$

and ω_0 is the central frequency of the modes a_1 and a_2. In view of Equation (142), we obtain

$$\left\langle \hat{a}_{\pm}(t), \hat{a}_{\pm}(t+\tau)\right\rangle = \left\langle \hat{a}_{\pm}(t)\hat{a}_{\pm}(t+\tau)\right\rangle. \tag{118}$$

Then on account of Equations (99), (100), (116), and (117), one can write Equation (118) as

$$\left\langle \hat{a}_{\pm}(t), \hat{a}_{\pm}(t+\tau)\right\rangle = \left\langle \hat{a}^{\dagger}(t)\hat{a}(t+\tau)\right\rangle + \left\langle \hat{a}(t)\hat{a}^{\dagger}(t+\tau)\right\rangle \pm \left\langle \hat{a}^{\dagger}(t)\hat{a}^{\dagger}(t+\tau)\right\rangle \pm \left\langle \hat{a}(t)\hat{a}(t+\tau)\right\rangle. \tag{119}$$

Upon substituting of (162)-(165) into Equation (119), we arrive at

$$\left\langle \hat{a}_{\pm}(t), \hat{a}_{\pm}(t+\tau)\right\rangle = \left\{\left\langle \hat{a}^{\dagger}(t)\hat{a}(t)\right\rangle + \left\langle \hat{a}(t)\hat{a}^{\dagger}(t)\right\rangle \pm \left\langle \hat{a}^{\dagger 2}(t)\right\rangle \pm \left\langle \hat{a}^{2}(t)\right\rangle\right\}\left\{\frac{k}{k-\nu}\mathrm{e}^{-\nu\tau/2} - \frac{\nu}{k-\nu}\mathrm{e}^{-k\tau/2}\right\}. \tag{120}$$

This can be put in the form

$$\left\langle \hat{a}_{+}(t), \hat{a}_{+}(t+\tau)\right\rangle = (\Delta a_{+})^2 \left\{\frac{k}{k-\nu}\mathrm{e}^{-\nu\tau/2} - \frac{\nu}{k-\nu}\mathrm{e}^{-k\tau/2}\right\} \tag{121}$$

and

$$\left\langle \hat{a}_{-}(t), \hat{a}_{-}(t+\tau)\right\rangle = (\Delta a_{-})^2 \left\{\frac{k}{k-\nu}\mathrm{e}^{-\nu\tau/2} - \frac{\nu}{k-\nu}\mathrm{e}^{-k\tau/2}\right\}. \tag{122}$$

Now introducing (122) into Equation (115) and on carrying out the integration over τ, we find the spectrum of the minus quadrature fluctuations for a two-mode cavity light to be

$$S_{-}(\omega)=(\Delta a_{-})_{ss}^{2}\left\{\left[\frac{k}{k-\nu}\right]\left[\frac{\nu/2\pi}{(\omega-\omega_0)^2+(\nu/2)^2}\right]-\left[\frac{\nu}{k-\nu}\right]\left[\frac{k/2\pi}{(\omega-\omega_0)^2+(k/2)^2}\right]\right\}. \tag{123}$$

Upon integrating both sides of (123) over ω, we get

$$\int_{-\infty}^{+\infty}S_{-}(\omega)\mathrm{d}\omega=(\Delta a_{-})^{2}. \tag{124}$$

On the basis of Equation (124), we observe that $S_{-}(\omega)\mathrm{d}\omega$ is the steady-state variance of the minus quadrature in the interval between ω and $\omega+\mathrm{d}\omega$. We thus realize that the variance of the minus quadrature in the interval between $\omega'=-\lambda$ and $\omega'=\lambda$ is expressible as

$$(\Delta a_{-})_{\pm\lambda}^{2}=\int_{-\lambda}^{+\lambda}S_{-}(\omega')\mathrm{d}\omega', \tag{125}$$

in which $\omega'=\omega-\omega_0$. On introducing (123) into Equation (125) and on carrying out the integration over ω', employing the relation described by Equation (75), we find

$$(\Delta a_{-})_{\pm\lambda}^{2}=(\Delta a_{-})^{2}z(\lambda), \tag{126}$$

where

$$z(\lambda)=\left[\frac{2k/\pi}{k-\nu}\right]\tan^{-1}\left(\frac{2\lambda}{\nu}\right)-\left[\frac{2\nu/\pi}{k-\nu}\right]\tan^{-1}\left(\frac{2\lambda}{k}\right). \tag{127}$$

We define the quadrature squeezing of the two-mode cavity light in the $\lambda_{\pm}$ frequency interval by

$$S_{\pm\lambda}=1-\frac{(\Delta a_{-})_{\pm\lambda}^{2}}{(\Delta a_{-})_{v\pm\lambda}^{2}}. \tag{128}$$

Furthermore, upon setting $\Omega=0$ in Equation (126), we see that the local quadrature variance of a two-mode cavity vacuum state in the same frequency is found to be

$$(\Delta a_{-})_{v\pm\lambda}^{2}=(\Delta a_{-})_{v}^{2}z_{v}(\lambda), \tag{129}$$

in which

$$z_{v}(\lambda)=\left[\frac{2k/\pi}{k-(\gamma_c+\gamma)}\right]\tan^{-1}\left(\frac{2\lambda}{\gamma_c+\gamma}\right)-\left[\frac{2(\gamma_c+\gamma)/\pi}{k-(\gamma_c+\gamma)}\right]\tan^{-1}\left(\frac{2\lambda}{k}\right) \tag{130}$$

and $(\Delta a_{-})_{v}^{2}$ is given by (112). Finally, on account of Equations (110), (112), and (129) along with (128), we readily get

$$S_{\pm\lambda}=\frac{1}{z_{v}(\lambda)}\left\{z_{v}(\lambda)-z(\lambda)-\left[\frac{\Omega^{2}\gamma_c-2\Omega(\gamma_c+\gamma)^{2}}{(\gamma_c+\gamma)^{3}+\Omega^{2}(3\gamma_c+2\gamma)}\right]z(\lambda)\right\}. \tag{131}$$

This shows that the local quadrature squeezing of the two-mode cavity light beams is not equal to that of the global quadrature squeezing. Moreover, we found from the plots in **Figure 6** that the maximum local quadrature squeezing for $\gamma=0$ is 71.73% (and occurs at $\lambda=0.06$) and for $\gamma=0.1$ is found 71.83% (and occurs at $\lambda=0.06$). And for $\gamma=0.2$, the maximum local quadrature squeezing is observed to be 71.88% (and occurs at $\lambda=0.06$). Furthermore, we note that the local quadrature squeezing approaches the global quadrature squeezing as λ increases.

5. Conclusions

The steady-state analysis of the squeezing and statistical properties of the light produced by coherently pumped degenerate three-level laser with open cavity and coupled to a two-mode vacuum reservoir is presented. We

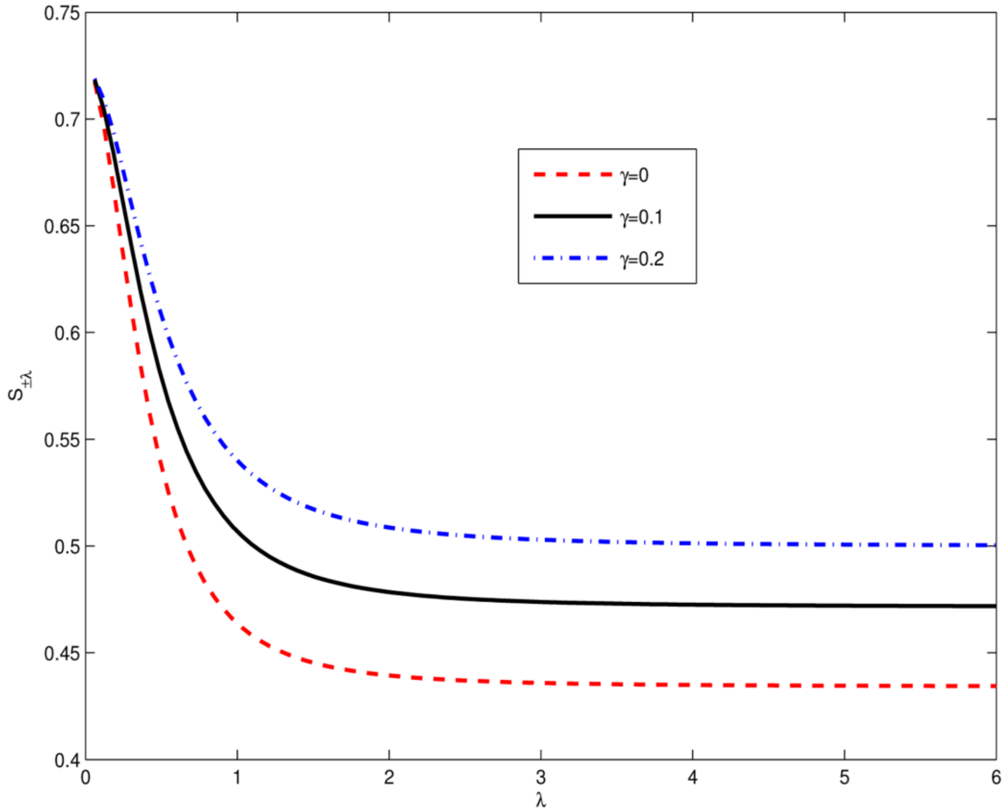

Figure 6. Plot of the local quadrature squeezing [Equations (131)] versus λ at steady state for $\gamma_c = 0.4$, $k = 0.8$, $\gamma = 0$ at $\Omega = 0.1717$, $\gamma = 0.1$ at $\Omega = 0.2323$, and $\gamma = 0.2$ at $\Omega = 0.303$.

carry out our analysis by putting the noise operators associated with the vacuum reservoir in normal order and by taking into consideration the interaction of the three-level atoms with the vacuum reservoir outside the cavity. We observe that a large part of the total mean photon number (variance of the photon number) is confined in a relatively small frequency interval. In addition, we find that the maximum global quadrature squeezing of the light produced by the system under consideration for $\gamma = 0$ operating at $\Omega = 0.1717$ is 43.42% and for $\gamma = 0.1$ operating at $\Omega = 0.2323$ is 47.15%. And for $\gamma = 0.2$, the maximum global quadrature squeezing is observed to be 50% below the vacuum-state level and this occurs when the three-level laser is operating at $\Omega = 0.303$. Furthermore, results show that the presence of spontaneous emission leads to a decrease in the mean photon number and to an increase in the quadrature squeezing.

Moreover, we find that the maximum local quadrature squeezing for $\gamma = 0$ is 71.73% (and occurs at $\lambda = 0.06$) and for $\gamma = 0.1$ is 71.83% (and occurs at $\lambda = 0.06$). And for $\gamma = 0.2$, the maximum local quadrature squeezing is observed to be 71.88% (and occurs at $\lambda = 0.06$). In addition, we note from the plots in **Figure 6** that as λ increases, the local quadrature squeezing approaches the global quadrature squeezing. We observe that the light generated by this laser operating under the condition Ω □ γ is in a chaotic light. And we have also established that the local quadrature squeezing of the two-mode light is not equal to the global quadrature squeezing.

Furthermore, we point out that unlike the mean photon number and the variance of the photon number, the quadrature squeezing does not depend on the number of atoms. This implies that the quadrature squeezing of the two-mode cavity light is independent of the number of photons.

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Appendix

1. Solutions of the Expectation Values of the Cavity (Atomic) Mode Operators

In order to determine the mean photon number and the variance of the photon number, and the quadrature squeezing of the two-mode cavity light in any frequency interval at steady state, we first need to calculate the solution of the equations of evolution of the expectation values of the atomic operators and cavity mode operators. To this end, the expectation values of the solution of Equation (66) is expressible as

$$\langle \hat{a}(t)\rangle = \langle \hat{a}(0)\rangle \mathrm{e}^{-\frac{1}{2}kt} + \frac{g}{\sqrt{N}}\mathrm{e}^{-\frac{1}{2}kt}\int_o^t \mathrm{d}t'\mathrm{e}^{\frac{1}{2}kt'}\langle \hat{m}(t')\rangle, \tag{132}$$

We next wish to obtain the expectation value of the expression of $\hat{m}(t)$ that appear in Equation (132). Thus applying the large-time approximation scheme to Equation (33), we get

$$\langle \hat{m}_b\rangle = -\left\{\frac{\Omega}{\gamma_c+\gamma}\right\}\langle \hat{m}_a^{\dagger}\rangle, \tag{133}$$

upon substituting the adjoint of this into Equation (32), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{m}_a(t)\rangle = -\nu\langle \hat{m}_a(t)\rangle, \tag{134}$$

in which

$$\nu = \frac{2(\gamma+\gamma_c)^2+\Omega^2}{2(\gamma+\gamma_c)}. \tag{135}$$

We notice that the solution of Equation (134) for ν different from zero at steady state is

$$\langle \hat{m}_a(t)\rangle = 0. \tag{136}$$

In a similar manner, applying the large-time approximation scheme to Equation (32), we obtain

$$\langle \hat{m}_a\rangle = \frac{1}{2}\left\{\frac{\Omega}{\gamma_c+\gamma}\right\}\langle \hat{m}_b^{\dagger}\rangle, \tag{137}$$

with the aid of the adjoint of Equation (137), one can put Equation (33) in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{m}_b(t)\rangle = -\frac{1}{2}\nu\langle \hat{m}_b(t)\rangle, \tag{138}$$

we also note that for ν different from zero, the solution of Equation (138) turns out at steady state to be

$$\langle \hat{m}_b(t)\rangle = 0. \tag{139}$$

Upon adding Equations (134) and (138), we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{m}(t)\rangle = -\frac{1}{2}\nu\langle \hat{m}(t)\rangle - \frac{1}{2}\nu\langle \hat{m}_a(t)\rangle. \tag{140}$$

We note that in view of Equation (136) with the assumption the atoms initially in the bottom level, the solution of Equation (140) turns out at steady state to be

$$\langle \hat{m}(t)\rangle = 0. \tag{141}$$

Now in view of (141) and with the assumption that the cavity light is initially in a vacuum state, Equation (132) goes over into

$$\langle \hat{a}(t)\rangle = 0. \tag{142}$$

On account of (142) together with Equation (66) that $\hat{a}(t)$ is a Gaussian variable with zero mean.

We finally seek to determine the solution of the expectation values of the atomic operators at steady state. Moreover, the steady-state solution of Equations (34)-(36) yields

$$\langle \hat{m}_c \rangle = \left\{ \frac{\Omega}{(\gamma_c + \gamma)} \right\} \left\{ \langle \hat{N}_c \rangle - \langle \hat{N}_a \rangle \right\}, \tag{143}$$

$$\langle \hat{N}_a \rangle = \frac{1}{2} \left\{ \frac{\Omega}{\gamma_c + \gamma} \right\} \left\{ \langle \hat{m}_c^\dagger \rangle + \langle \hat{m}_c \rangle \right\}, \tag{144}$$

$$\langle \hat{N}_b \rangle = \left\{ \frac{\gamma_c}{\gamma_c + \gamma} \right\} \langle \hat{N}_a \rangle. \tag{145}$$

Solving these equations simultaneously, one can easily obtains

$$\langle \hat{m}_c \rangle = \left\{ \frac{\Omega(\gamma_c + \gamma)^2}{(\gamma_c + \gamma)^3 + \Omega^2 (3\gamma_c + 2\gamma)} \right\} N, \tag{146}$$

$$\langle \hat{N}_a \rangle = \left\{ \frac{\Omega^2 (\gamma_c + \gamma)}{(\gamma_c + \gamma)^3 + \Omega^2 (3\gamma_c + 2\gamma)} \right\} N, \tag{147}$$

$$\langle \hat{N}_b \rangle = \left\{ \frac{\Omega^2 \gamma_c}{(\gamma_c + \gamma)^3 + \Omega^2 (3\gamma_c + 2\gamma)} \right\} N. \tag{148}$$

Finally, on account of (147) and (148) along with Euation (45), we find

$$\langle \hat{N}_c \rangle = \left\{ \frac{\Omega^2 (\gamma_c + \gamma) + (\gamma_c + \gamma)^3}{(\gamma_c + \gamma)^3 + \Omega^2 (3\gamma_c + 2\gamma)} \right\} N. \tag{149}$$

2. Two-Time Correlation Functions

Here we seek to calculate the two-time correlation functions for the two-mode cavity light. To this end, we realize that the solution of Equation (66) can write as

$$\hat{a}(t+\tau) = \hat{a}(t) e^{-\frac{1}{2}k\tau} + \frac{g}{\sqrt{N}} e^{-\frac{1}{2}k\tau} \int_o^t d\tau' e^{\frac{1}{2}k\tau'} \hat{m}(t+\tau'). \tag{150}$$

On the other hand, one can put Equation (140) in the form

$$\frac{d}{dt}\hat{m}(t) = -\frac{1}{2}\nu \hat{m}(t) - \frac{1}{2}\nu \hat{m}_a(t) + \hat{F}_m(t), \tag{151}$$

in which $\hat{F}_m(t)$ is a noise operator with zero mean. The solution of this equation is expressible as

$$\hat{m}(t+\tau) = \hat{m}(t) e^{-\frac{1}{2}k\tau} + e^{-\frac{1}{2}k\tau} \int_o^\tau d\tau' e^{\frac{1}{2}k\tau'} \left[-\frac{1}{2}\nu \hat{m}_a(t+\tau') + \hat{F}_m(t+\tau') \right]. \tag{152}$$

In addition, one can rewrite Equation (134) as

$$\frac{d}{dt}\hat{m}_a(t) = -\nu \hat{m}_a(t) + \hat{F}_a(t), \tag{153}$$

where $\hat{F}_a(t)$ is a noise operator with vanishing mean. Employing the large-time approximation scheme to Equation (153), we see that

$$\hat{m}_a(t+\tau) = \frac{1}{\nu}\hat{F}_a(t+\tau), \tag{154}$$

on introducing this into Equation (152), we have

$$\hat{m}(t+\tau) = \hat{m}(t) e^{-\frac{1}{2}k\tau} + e^{-\frac{1}{2}k\tau} \int_o^\tau d\tau' e^{\frac{1}{2}k\tau'} \left[-\frac{1}{2}\hat{F}_a(t+\tau') + \hat{F}_m(t+\tau') \right]. \tag{155}$$

Now combination of Equations (150) and (155) yields

$$
\begin{aligned}
\hat{a}(t+\tau) = {} & \hat{a}(t)\mathrm{e}^{-k\tau/2} + \frac{g}{\sqrt{N}}\mathrm{e}^{-k\tau/2}\Bigg\{\hat{m}(t)\int_0^{\tau}\mathrm{d}\tau'\mathrm{e}^{(k-\nu)\tau'/2} \\
& + \int_0^{\tau}\mathrm{d}\tau'\mathrm{e}^{(k-\nu)\tau'/2}\int_0^{\tau'}\mathrm{d}\tau''\mathrm{e}^{\nu\tau''/2}\left[-\frac{1}{2}\hat{F}_a(t+\tau'') + \hat{F}_m(t+\tau'')\right]\Bigg\}.
\end{aligned} \tag{156}
$$

On multiplying both sides on the left by $\hat{a}^{\dagger}(t)$ and taking the expectation value of the resulting equation, we get

$$
\begin{aligned}
\left\langle \hat{a}^{\dagger}(t)\hat{a}(t+\tau)\right\rangle = {} & \left\langle \hat{a}^{\dagger}(t)\hat{a}(t)\right\rangle \mathrm{e}^{-k\tau/2} + \frac{g}{\sqrt{N}}\mathrm{e}^{-k\tau/2}\Bigg\{\left\langle \hat{a}^{\dagger}(t)\hat{m}(t)\right\rangle\int_0^{\tau}\mathrm{d}\tau'\mathrm{e}^{(k-\nu)\tau'/2} \\
& + \int_0^{\tau}\mathrm{d}\tau'\mathrm{e}^{(k-\nu)\tau'/2}\int_0^{\tau'}\mathrm{d}\tau''\mathrm{e}^{\nu\tau''/2}\left[-\frac{1}{2}\left\langle \hat{a}^{\dagger}(t)\hat{F}_a(t+\tau'')\right\rangle + \left\langle \hat{a}^{\dagger}(t)\hat{F}_m(t+\tau'')\right\rangle\right]\Bigg\}.
\end{aligned} \tag{157}
$$

Moreover, applying the large-time approximation scheme to Equation (66), we obtain

$$
\hat{m}(t) = \frac{k\sqrt{N}}{2g}\hat{a}(t). \tag{158}
$$

With this substituting into Equation (157), there follows

$$
\begin{aligned}
\left\langle \hat{a}^{\dagger}(t)\hat{a}(t+\tau)\right\rangle = {} & \left\langle \hat{a}^{\dagger}(t)\hat{a}(t)\right\rangle \mathrm{e}^{-k\tau/2} + \mathrm{e}^{-k\tau/2}\Bigg\{\frac{k}{2}\left\langle \hat{a}^{\dagger}(t)\hat{a}(t)\right\rangle\int_0^{\tau}\mathrm{d}\tau'\mathrm{e}^{(k-\nu)\tau'/2} \\
& + \frac{g}{\sqrt{N}}\int_0^{\tau}\mathrm{d}\tau'\mathrm{e}^{(k-\nu)\tau'/2}\int_0^{\tau'}\mathrm{d}\tau''\mathrm{e}^{\nu\tau''/2}\left[-\frac{1}{2}\left\langle \hat{a}^{\dagger}(t)\hat{F}_a(t+\tau'')\right\rangle + \left\langle \hat{a}^{\dagger}(t)\hat{F}_m(t+\tau'')\right\rangle\right]\Bigg\}.
\end{aligned} \tag{159}
$$

Since the cavity mode operator and the noise operator of the atomic modes are not correlated, we see that

$$
\left\langle \hat{a}^{\dagger}(t)\hat{F}_a(t+\tau'')\right\rangle = \left\langle \hat{a}^{\dagger}(t)\right\rangle\left\langle \hat{F}_a(t+\tau'')\right\rangle = 0, \tag{160}
$$

$$
\left\langle \hat{a}^{\dagger}(t)\hat{F}_m(t+\tau'')\right\rangle = \left\langle \hat{a}^{\dagger}(t)\right\rangle\left\langle \hat{F}_m(t+\tau'')\right\rangle = 0. \tag{161}
$$

On account of these results and on carrying out the integration of Equation (159) over τ', we readily get

$$
\left\langle \hat{a}^{\dagger}(t)\hat{a}(t+\tau)\right\rangle = \left\langle \hat{a}^{\dagger}(t)\hat{a}(t)\right\rangle\left\{\frac{k}{k-\nu}\mathrm{e}^{-\nu\tau/2} - \frac{\nu}{k-\nu}\mathrm{e}^{-k\tau/2}\right\}. \tag{162}
$$

It is not also difficult to verify that

$$
\left\langle \hat{a}(t)\hat{a}^{\dagger}(t+\tau)\right\rangle = \left\langle \hat{a}(t)\hat{a}^{\dagger}(t)\right\rangle\left\{\frac{k}{k-\nu}\mathrm{e}^{-\nu\tau/2} - \frac{\nu}{k-\nu}\mathrm{e}^{-k\tau/2}\right\}, \tag{163}
$$

$$
\left\langle \hat{a}(t)\hat{a}(t+\tau)\right\rangle = \left\langle \hat{a}^{2}(t)\right\rangle\left\{\frac{k}{k-\nu}\mathrm{e}^{-\nu\tau/2} - \frac{\nu}{k-\nu}\mathrm{e}^{-k\tau/2}\right\}, \tag{164}
$$

$$
\left\langle \hat{a}^{\dagger}(t)\hat{a}^{\dagger}(t+\tau)\right\rangle = \left\langle \hat{a}^{\dagger 2}(t)\right\rangle\left\{\frac{k}{k-\nu}\mathrm{e}^{-\nu\tau/2} - \frac{\nu}{k-\nu}\mathrm{e}^{-k\tau/2}\right\}. \tag{165}
$$