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The New Viscosity Approximation Methods for Nonexpansive Nonself-Mappings

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Abstract

In this paper, to find the fixed points of the nonexpansive nonself-mappings, we introduced two new viscosity approximation methods, and then we prove the iterative sequences defined by above viscosity approximation methods which converge strongly to the fixed points of nonexpansive nonself-mappings. The results presented in this paper extend and improve the results of Song-Chen [1] and Song-Li [2].

Keywords

Fixed Points, Nonexpansive Nonself-Mappings, Viscosity Approximation Methods, Real Banach Space

1. Introduction

Let C be a closed convex subset of a Hilbert space H and $T: C \to C$ a nonexpansive mapping (*i.e.*, $\|Tx - Ty\| \le \|x - y\|$ for any $x, y \in C$). Let $u \in C$ be a fixed point of *T*. Then for any initial $x_0 \in C$ and real sequence $\{\lambda_n\} \subset (0,1)$, we define a sequence $\{x_n\}$ by

$$x_{n+1} = \lambda_{n+1} u + (1 - \lambda_{n+1}) T x_n \ (n \ge 0) \tag{1}$$

Helpern [3] was the first to study the strong convergence of the iteration process (1). In 1992, Albert [4] studied the convergence of the Ishikawa iteration process in Banach space, which was extended the results of Mann iteration process [5]. But the mappings in these results must be self-mapping and continuous. It is more useful to get some results for nonself-mappings.

In 2006, Yisheng Song and Rudong Chen [1] studied viscosity approximation methods for nonexpansive nonself-mappings by the following iterative sequence $\{x_n\}$.

$$x_{n+1} = P\left(\alpha_n f\left(x_n\right) + \left(1-\alpha_n\right)Tx_n\right) \quad \left(0<\alpha_n<1\right)$$

where X is a real reflexive Banach space, and C is a closed subset of X which is also a sunny nonexpansive retract of X. $T:C\to X$ is a nonexpansive mapping, $f:C\to C$ is a fixed contractive mapping and P is a sunny nonexpansive retraction of X onto C.

In 2007, Yisheng Song and Qingchun Li [2] found a new viscosity approximation method for nonexpansive nonself-mappings as follows

$$x_{n+1} = \alpha_n f\left(x_n\right) + \left(1-\alpha_n\right)PTx_n \quad \left(0<\alpha_n<1\right)$$

where X is a real reflexive Banach space, and C is a closed subset of X which is also a sunny nonexpansive retract of X. $T:C\to X$ is a nonexpansive mapping, $f:C\to C$ is a fixed contractive mapping and P is a sunny nonexpansive retraction of X onto C.

In this paper, we will study two new viscosity approximation methods for nonexpansive nonself-mappings in reflexive Banach space X, which can extend the results of Song-Chen [1] and Song-Li [2] on the two-dimensional space.

Let us start by making some basic definitions.

2. Preliminary Notes

Let X be a real Banach space with the norm $\|\cdot\|$, and X^* be its dual space. When $\{x_n\}$ is a sequence in X, the $x_n \to x$ (respectively $x_n \xrightarrow{w} x$, $x_n \xrightarrow{w^*} x$) will denote the strong (respectively the weak, the weak star) convergence of the sequence x_n to x.

Definition 2.1. Let X be a real Banach space and J denote the normalized duality mapping from X into 2^{E^*} given by

$$J\left(X\right) = \left\{f \in E^* : \left\langle x, f\right\rangle = \|x\|\|f\|, \|x\| = \|f\|\right\} \text{ for all } x\in X ,$$

where X^* denotes the dual space of X and $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing.

Let $F\left(T\right)$ denotes set of the fixed point of T.

Definition 2.2. Let X ba a real Banach space and T a mapping with domain $D\left(T\right)$ and range $R\left(T\right)$ in T. T is called nonexpansive if for any $x,y\in D\left(T\right)$, such that $\|Tx-Ty\|\le\|x-y\|$ (respectively T is called contractive if for any $x,y\in D\left(T\right)$, such that $\|Tx-Ty\|\le\beta\|x-y\|$), where $0<\beta<1$.

Definition 2.3. Let X be a Banach space, C and D be nonempty subsets of X, $D\subset C$. A mapping $P:C\to D$ is called a retraction from C to D, if P is continuous with $F\left(P\right)=D$. A mapping $P:C\to D$ is called a sunny, if $P\left(Px+t\left(x-Px\right)\right)=Px$, for all $x\in C$, $t>0$, whenever $Px+t\left(x-Px\right)\in C$. And a subset D of C is said to be a sunny nonexpansive retract of C, if there exists a sunny nonexpansive retraction of C onto D.

Definition 2.4. Let X be a real reflexive Banach space, which admits a weakly sequentially continuous duality mapping from X to X^*, and C be a closed convex subset of X, which is also a sunny nonexpansive retract of X, and $T:C\to X$ be nonexpansive mapping satisfying the weakly inward condition and $F\left(T\right)\ne\phi$, and $f:C\to C$ is called contractive mapping. For a given $x_0\in C$ and $n\in \mathrm{N}$, let us define $\{x_n\}$ and $\{y_n\}$ by the following iterative scheme:

$$\begin{cases} x_{n+1} = P\left(\alpha_n f\left(x_n\right)+\left(1-\alpha_n\right)Ty_n\right) \\ y_n = \beta_n x_n + \left(1-\beta_n\right)Tx_n \end{cases} \tag{2}$$

where $\alpha_n,\beta_n\in\left(0,1\right)$, $\lim\limits_{n\to\infty}\alpha_n=0$, $\lim\limits_{n\to\infty}\beta_n=1$.

$$\begin{cases} x_{n+1} = \alpha_n f\left(x_n\right)+\left(1-\alpha_n\right)PTy_n \\ y_n = \beta_n x_n + \left(1-\beta_n\right)Tx_n \end{cases} \tag{3}$$

where $\alpha_n,\beta_n\in\left(0,1\right)$, $\lim\limits_{n\to\infty}\alpha_n=0$, $\lim\limits_{n\to\infty}\beta_n=1$.

We call (2) the first type viscosity approximation method for nonexpansive nonself-mapping and call (3) the second type viscosity approximation method for nonexpansive nonself-mapping.

Let us introduce some lemmas, which play important roles in our results.

Lemma 2.1. ([6]) Let X be a real Banachspace, then for each $x, y \in X$, the following inequality holds:

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x+y)\rangle, \text{ for } j(x+y) \in J(x+y)$$

Lemma 2.2. ([7]) Let $\{a_n\},\{b_n\},\{c_n\}$ be three nonnegative real sequences satisfying

$$a_{n+1} \le (1-t_n)a_n + b_n + c_n \quad \text{with} \quad \{t_n\} \subset [0,1], \ \sum_{n=0}^{\infty} c_n < \infty .$$

Then $a_n \to 0$ as $n \to \infty$.

Lemma 2.3. ([1]) Let X be a real smooth Banach space, and C be nonempty closed convex subset of X, which is also a sunny nonexpansive retract of X and $T: C \to X$ be mapping satisfying the weakly inward condition, and P be a sunny nonexpansive retraction of X onto C, then $F(T) = F(PT)$.

Lemma 2.4. ([1]) Let C be nonempty closed convex subset of a reflexive Banach space X which satisfies Opial's condition, and suppose $T: C \to X$ is nonexpansive. Then the mapping I-T is demiclosed at zero, *i.e.*, $x_n \xrightarrow{w} x$, $x_n - Tx_n \to 0$ implies $x = Tx$.

3. Main Results

First of all, let us study the first type viscosity approximation for nonexpansive nonself-mappings.

Lemma 3.1. ([1]) Let X be a reflexive Banach space which admits a weakly sequentially continuous duality mapping J from X to X^*. Suppose C is a nonexpansive retract of X which is also a sunny nonexpansive retract of X, and $T: C \to X$ is a nonexpansive mapping satisfying the weakly inward condition and $F(T) \ne \phi$, let $f: C \to C$ be a fixed contractive mapping from C to C. Let $x_t \in C$ be the unique fixed point of T, that is,

$$x_t = P\left(tf(x_t) + (1-t)Tx_t\right), \text{ for any } t \in (0,1),$$

where P is a sunny nonexpansive retract of X onto C. Then as $t \to 0$, $\{x_t\}$ converges strongly to some fixed point p of T. And p is the unique solution in $F(T)$ to the following variational inequality

$$\langle (I-f)p, j(p-u)\rangle \le 0$$

For all $u \in F(T)$.

Lemma 3.2. Let X be a reflexive Banach space which admits a weakly sequentially continuous duality mapping J from X to X^*. Suppose C is a nonexpansive retract of X, which is also a sunny nonexpansive retract of X and $T: C \to X$ is a nonexpansive mapping satisfying the weakly inward condition and $F(T) \ne \phi$, let $f: C \to C$ be a fixed contractive mapping from C to C. And $\{x_n\}_{n=0}^{\infty}$ is a sequence by definition 2.4 (2), then the sequence $\{x_n\}$ is bounded.

Proof. Let $p \in F(T)$, so we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \left\|P\left(\alpha_n f(x_n) + (1-\alpha_n)Ty_n\right) - p\right\| \\
&\le \left\|\alpha_n f(x_n) + (1-\alpha_n)Ty_n - p\right\| \\
&= \alpha_n \left\|f(x_n) - f(p)\right\| + (1-\alpha_n)\|Ty_n - p\| + \alpha_n \left\|f(p) - p\right\| \\
&\le \alpha_n \left\|f(x_n) - f(p)\right\| + (1-\alpha_n)\|y_n - p\| + \alpha_n \left\|f(p) - p\right\| \\
&\le \alpha_n \beta \|x_n - p\| + (1-\alpha_n)\|y_n - p\| + \left\|f(p) - p\right\|
\end{aligned}$$

while,

$$\begin{aligned}
\|y_n - p\| &= \left\|\beta_n x_n + (1-\beta_n)Tx_n - p\right\| \\
&\le \left\|\beta_n(x_n - p) + (1-\beta_n)(Tx_n - p)\right\| \\
&\le \beta_n \|x_n - p\| + (1-\beta_n)\|Tx_n - p\| \\
&\le \|x_n - p\|
\end{aligned}$$

therefore,

$$\begin{aligned}\|x_{n+1}-p\| &\le \alpha_n\beta\|x_n-p\|+(1-\alpha_n)\|x_n-p\|+\|f(p)-p\| \\ &=(1-\alpha_n+\alpha_n\beta)\|x_n-p\|+\|f(p)-p\| \\ &=(1-\alpha_n(1-\beta))\|x_n-p\|+\|f(p)-p\|\end{aligned}$$

since $(1-\alpha_n(1-\beta))\in(0,1)$,

therefore $\|x_{n+1}-p\|\le\max\{\|x_0-p\|,\|f(p)-p\|\}$, then $\{x_n\}$ is bounded.

Lemma 3.3. Let X be a reflexive Banach space which admits a weakly sequentially continuous duality mapping J from X to X^*. Suppose C is a nonexpansive retract of X which is also a sunny nonexpansive retract of X, and $T:C\to X$ is a nonexpansive mapping satisfying the weakly inward condition and $F(T)\ne\phi$, let $f:C\to C$ be a fixed contractive mapping from C to C. And $\{x_n\}_{n=0}^{\infty}$ is a sequence by definition 2.4 (2). Let us assume that there are two sequences $\{\alpha_n\}$, $\{\beta_n\}$ in $[0,1]$ satisfying the following conditions:

$$\sum_{n=1}^{\infty}\left(|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|\right)<\infty$$

then

1) $\lim_{n\to\infty}\|x_n-x_{n+1}\|=0$

2) $\lim_{n\to\infty}\|x_n-PTx_n\|=0$

Proof by lemma 3.2, we know that the sequence $\{x_n\}$ is bounded. So the sequences $\{f(x_n)\}$, $\{y_n\}$, Tx_n are also bounded. Therefore, we have

$$\begin{aligned}\|y_n-y_{n-1}\| &= \|\beta_n x_n+(1-\beta_n)Tx_n-\beta_{n-1}x_{n-1}-(1-\beta_{n-1})Tx_{n-1}\| \\ &= \|\beta_n(x_n-x_{n-1})+(1-\beta_n)(Tx_n-Tx_{n-1})+(\beta_n-\beta_{n-1})(x_{n-1}-Tx_{n-1})\| \\ &\le \beta_n\|x_n-x_{n-1}\|+(1-\beta_n)\|Tx_n-Tx_{n-1}\|+|\beta_n-\beta_{n-1}|\|x_{n-1}-Tx_{n-1}\| \\ &\le \|x_n-x_{n-1}\|+|\beta_{n-1}-\beta_n|\|x_{n-1}-Tx_{n-1}\|\end{aligned} \tag{4}$$

$$\begin{aligned}\|x_{n+1}-x_n\| &= \|P(\alpha_n f(x_n)+(1-\alpha_n)Ty_n)-P(\alpha_{n-1}f(x_{n-1})+((1-\alpha_{n-1})Ty_{n-1})\| \\ &\le \|\alpha_n f(x_n)+(1-\alpha_n)Ty_n-\alpha_{n-1}f(x_{n-1})-(1-\alpha_{n-1})Ty_{n-1}\| \\ &= \|\alpha_n(f(x_n)-f(x_{n-1}))+(\alpha_n-\alpha_{n-1})(f(x_{n-1})-Ty_{n-1})+(1-\alpha_n)(Ty_n-Ty_{n-1})\| \\ &= \alpha_n\beta\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})-Ty_{n-1}\|+(1-\alpha_n)\|y_n-y_{n-1}\|\end{aligned}$$

by (4), we have

$$\begin{aligned}\|x_{n+1}-x_n\| &\le \alpha_n\beta\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})-Ty_{n-1}\| \\ &\quad +(1-\alpha_n)(\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\|x_{n-1}-Tx_{n-1}\|) \\ &\le \alpha_n\beta\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})-Ty_{n-1}\| \\ &\quad +(1-\alpha_n)\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\|x_{n-1}-Tx_{n-1}\|\end{aligned}$$

Set $M_1=\max\{\|x_{n-1}-Tx_{n-1}\|,\|f(x_{n-1})-Ty_{n-1}\|\}$

$$\begin{aligned}\|x_{n+1}-x_n\| &\le (1-\alpha_n+\alpha_n\beta)\|x_n-x_{n-1}\|+(|\alpha_{n-1}-\alpha_n|+|\beta_{n-1}-\beta_n|)M_1 \\ &\le (1-\alpha_n(1-\beta))\|x_n-x_{n-1}\|+(|\alpha_{n-1}-\alpha_n|+|\beta_{n-1}-\beta_n|)M_1\end{aligned}$$

Set $a_{n+1}=\|x_{n+1}-x_n\|$, $t_n=1-\alpha_n(1-\beta)$, $b_n=0$, $c_n=(|\alpha_{n-1}-\alpha_n|+|\beta_{n-1}-\beta_n|)M_1$

by the lemma 2.2 we have

$$\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0.$$

Now we will proof $\|x_n - PTx_n\| \to 0$ as $n \to \infty$.

$$\begin{aligned}\|x_n - PTx_n\| &= \|x_n - x_{n+1}\| + \|x_{n+1} - PTx_n\| \\ &\le \|x_n - x_{n+1}\| + \|P(\alpha_n f(x_n) + (1-\alpha_n)Tx_n) - PTx_n\| \\ &\le \|x_n - x_{n+1}\| + \|\alpha_n f(x_n) + (1-\alpha_n)Tx_n - Tx_n\| \\ &\le \|x_n - x_{n+1}\| + \alpha_n \|f(x_n) - Tx_n\|\end{aligned} \tag{5}$$

as $n\to\infty$, $\alpha_n \to 0$ therefore

$$\lim_{n\to\infty}\|x_n - PTx_n\| = 0. \tag{6}$$

Remark 3.1. From the lemma 3.1 we know that p is the unique solution in $F(T)$ to the following variational inequality:

$$\langle (I-F)p, j(p-u)\rangle \le 0 \text{ for all } u \in F(T). \tag{7}$$

Now, we can take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle f(p)-p, j(x_n - p)\rangle = \limsup_{k\to\infty}\langle f(p)-p, j(x_{n_k} - p)\rangle$$

we may assume that $x_{n_k} \to x^*$ by X is reflexive and $\{x_n\}$ is bounded. It follows from Lemma 2.3, Lemma 2.4, and (3.3), we have $x^* \in F(T) = F(PT)$, by (7) we have

$$\limsup_{n\to\infty}\langle f(p)-p, j(x_n - p)\rangle = \limsup_{k\to\infty}\langle f(p)-p, j(x_{n_k} - p)\rangle \le 0.$$

Theorem 3.4. Let X be a reflexive Banach space which admits a weakly sequentially continuous duality mapping J from X to X^*. Suppose C is a nonexpansive retract of X which is also a sunny nonexpansive retract of X, and $T: C \to X$ is a nonexpansive mapping satisfying the weakly inward condition and $F(T) \ne \phi$, let $f: C \to C$ be a fixed contractive mapping from C to C. And $\{x_n\}_{n=0}^{\infty}$ is the sequence by definition 2.4 (2). Let us assume there are two sequences $\{\alpha_n\}$, $\{\beta_n\}$ in $[0,1]$ satisfying the following conditions:

$$\sum_{n=1}^{\infty}\left(|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|\right) < \infty$$

then the sequence $\{x_n\}$ converges strongly to the unique solution p of the variational inequality:

$$p \in F(T) \text{ and } \langle (I-f)p, j(p-u)\rangle \le 0 \text{ for all } u \in F(T).$$

Proof. Since C is closed, by lemma 3.2, $\{x_n\}$ is bounded, so $\{f(x_n)\}$, $\{y_n\}$, $\{Tx_n\}$ are also bounded. Let $\{x_t\}$ be the sequence defined by

$$x_t = P(tf(x_t) + (1-t)Tx_t)$$

by the lemma 3.1 as $t \to 0$ we have $\{x_t\}$ converges strongly to a fixed point p of T and p is also the unique solution in $F(T)$ to the following variational inequality

$$\langle (I-f)p, j(p-u)\rangle \le 0 \text{ for all } u \in F(T)$$

using the remark 3.1, we have

$$\limsup_{n\to\infty}\langle f(p)-p, j(x_n - p)\rangle \le 0.$$

By the definition 2.4 (2), we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \left\|P\left(\alpha_n f(x_n)+(1-\alpha_n)Ty_n\right)-p\right\|^2 \\ &\le \left\|\alpha_n f(x_n)+(1-\alpha_n)Ty_n-p\right\|^2 \\ &\le \left\|\alpha_n\left(f(x_n)-p\right)+(1-\alpha_n)(Ty_n-p)\right\|^2 \\ &\le (1-\alpha_n)^2\|y_n-p\|^2+2\alpha_n\left\langle f(x_n)-f(p), j(x_{n+1}-p)\right\rangle \\ &\quad +2\alpha_n\left\langle f(p)-p, j(x_{n+1}-p)\right\rangle \\ &\le (1-\alpha_n)^2\|y_n-p\|^2+2\alpha_n\beta\|x_n-p\|\|x_{n+1}-p\| \\ &\quad +2\alpha_n\left\langle f(p)-p, j(x_{n+1}-p)\right\rangle \\ &\le (1-\alpha_n)^2\|y_n-p\|^2+\alpha_n\beta\left(\|x_n-p\|^2+\|x_{n+1}-p\|^2\right) \\ &\quad +2\alpha_n\left\langle f(p)-p, j(x_{n+1}-p)\right\rangle\end{aligned}$$

While

$$\begin{aligned}\|y_n-p\| &= \|\beta_n x_n+(1-\beta_n)Tx_n-p\| \le \beta_n\|x_n-p\|+(1-\beta_n)\|Tx_n-p\| \\ &\le \beta_n\|x_n-p\|+(1-\beta_n)\|x_n-p\| = \|x_n-p\|\end{aligned}$$

therefore,

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \frac{(1-\alpha_n)^2+\alpha_n\beta}{1-\alpha_n\beta}\|x_n-p\|^2+\frac{2\alpha_n}{1-\alpha_n\beta}\left\langle f(p)-p, j(x_{n+1}-p)\right\rangle \\ &\le \left(1-\frac{1-2\alpha_n\beta}{1-\alpha_n\beta}\right)\|x_n-p\|^2+\frac{(1-\alpha_n)^2}{1-\alpha_n\beta}\|x_n-p\|^2+\frac{2\alpha_n}{1-\alpha_n\beta}r_{n+1}\end{aligned}$$

where $r_{n+1}=\max\left\{\left\langle f(p)-p, j(x_{n+1}-p)\right\rangle, 0\right\}$.

Setting $a_n=\|x_n-p\|$, $t_n=1-\dfrac{1-2\alpha_n\beta}{1-\alpha_n\beta}$, $b_n=\dfrac{(1-\alpha_n)^2}{1-\alpha_n\beta}\|x_n-p\|^2$, $c_n=\dfrac{2\alpha_n}{1-\alpha_n\beta}r_{n+1}$ and applying Lemma 2.1, we conclude that $x_n\to p$.

Let us prove p is the unique fixed point of T.

We assume that p^* is another solution of (7) in $F(T)$, then $\left\langle f(p)-p, j\left(p^*-p\right)\right\rangle\le 0$ and $\left\langle f\left(p^*\right)-p^*, j\left(p-p^*\right)\right\rangle\le 0$, so we have $(1-\alpha)\left\|p-p^*\right\|\le 0$, which implies the equality $p=p^*$.

Remark 3.2. when $\beta_n=1$ for all $n\in\mathrm{N}$. The first type viscosity approximation methods for nonexpansive nonself-mappings (see definition 2.4) become the following iteration sequence:

$$x_{n+1}=P\left(\alpha_n f(x_n)+(1-\alpha_n)Tx_n\right).$$

So the theorem 3.4 improves the theorem 2.4 of Song-Chen [1].

Now let us study the second type viscosity approximation for nonexpansive nonself-mappings.

Lemma 3.5. Let X be a reflexive Banach space which admits a weakly sequentially continuous duality mapping J from X to X^*. Suppose C is a nonexpansive retract of X, which is also a sunny nonexpansive retract of X and $T:C\to X$ is a nonexpansive mapping satisfying the weakly inward condition and $F(T)\neq\phi$, let $f:C\to C$ be a fixed contractive mapping from C to C. And $\{x_n\}_{n=0}^{\infty}$ is a sequence by definition 2.4 (3), then the sequence $\{x_n\}$ is bounded.

Proof. Let $p\in F(T)$, so we have

$$\begin{aligned}\|x_{n+1}-p\| &= \|\alpha_n f(x_n)+(1-\alpha_n)PTy_n-p\| \\ &\le \|\alpha_n(f(x_n)-f(p))+\alpha_n(f(p)-p)+(1-\alpha_n)(PTy_n-p)\| \\ &\le \alpha_n\|f(x_n)-f(p)\|+(1-\alpha_n)\|y_n-p\|+\alpha_n\|f(p)-p\| \\ &\le \alpha_n\beta\|x_n-p\|+(1-\alpha_n)\|y_n-p\|+\alpha_n\|f(p)-p\|\end{aligned}$$

while,

$$\begin{aligned}\|y_n-p\| &= \|\beta_n x_n+(1-\beta_n)Tx_n-p\| \\ &\le \|\beta_n(x_n-p)+(1-\beta_n)(Tx_n-p)\| \\ &\le \beta_n\|x_n-p\|+(1-\beta_n)\|Tx_n-p\| \\ &\le \|x_n-p\|\end{aligned}$$

therefore,

$$\begin{aligned}\|x_{n+1}-p\| &\le (1-\alpha_n+\alpha_n\beta)\|x_n-p\|+\|f(p)-p\| \\ &= (1-\alpha_n(1-\beta))\|x_n-p\|+\|f(p)-p\|\end{aligned}$$

since $(1-\alpha_n(1-\beta))\in(0,1)$,

therefore $\|x_{n+1}-p\|\le\max\{\|x_0-p\|,\|f(p)-p\|\}$, then $\{x_n\}$ is bounded.

Lemma 3.6. ([2]) Let X be a reflexive Banach space which admits a weakly sequentially continuous duality mapping J from X to X^*. Suppose C is a nonexpansive retract of X which is also a sunny nonexpansive retract of X, and $T:C\to X$ is a nonexpansive mapping satisfying the weakly inward condition and $F(T)\ne\phi$, let $f:C\to C$ be a fixed contractive mapping from C to C. Let $x_t\in C$ be the unique fixed point of T, that is,

$$x_t=tf(x_t)+(1-t)PTx_t\text{, for any } t\in(0,1),$$

where P is a sunny nonexpansive retract of X onto C. Then as $t\to 0$, $\{x_t\}$ converges strongly to some fixed point p of T. And p is the unique solution in $F(T)$ to the following variational inequality:

$$\langle (I-f)p, j(p-u)\rangle\le 0$$

for all $u\in F(T)$.

Lemma 3.7. Let X be a reflexive Banach space which admits a weakly sequentially continuous duality mapping J from X to X^*. Suppose C is a nonexpansive retract of X which is also a sunny nonexpansive retract of X, and $T:C\to X$ is a nonexpansive mapping satisfying the weakly inward condition and $F(T)\ne\phi$, let $f:C\to C$ be a fixed contractive mapping from C to C. And $\{x_n\}_{n=0}^{\infty}$ is a sequence by definition 2.4 (3). Let us assume that there are two sequences $\{\alpha_n\}$, $\{\beta_n\}$ in $[0,1]$ satisfying the following conditions:

$$\sum_{n=1}^{\infty}(|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|)<\infty$$

then

1) $\lim_{n\to\infty}\|x_n-x_{n+1}\|=0$

2) $\lim_{n\to\infty}\|x_n-PTx_n\|=0$.

Proof by lemma 3.5, we know that the sequence $\{x_n\}$ is bounded. So the sequences $\{f(x_n)\}$, $\{y_n\}$, Tx_n are also bounded. Therefore, we have:

$$\begin{aligned}\|y_n-y_{n-1}\| &= \|\beta_n x_n+(1-\beta_n)Tx_n-\beta_{n-1}x_{n-1}-(1-\beta_{n-1})Tx_{n-1}\| \\ &= \|\beta_n(x_n-x_{n-1})+(1-\beta_n)(Tx_n-Tx_{n-1})+(\beta_n-\beta_{n-1})(x_{n-1}-Tx_{n-1})\| \\ &\le \beta_n\|x_n-x_{n-1}\|+(1-\beta_n)\|Tx_n-Tx_{n-1}\|+|\beta_n-\beta_{n-1}|\|x_{n-1}-Tx_{n-1}\| \\ &\le \|x_n-x_{n-1}\|+|\beta_{n-1}-\beta_n|\|x_{n-1}-Tx_{n-1}\|\end{aligned}\tag{8}$$

$$\begin{aligned}\|x_{n+1}-x_n\| &= \left\|\alpha_n f(x_n)+(1-\alpha_n)PTy_n-\alpha_{n-1}f(x_{n-1})-(1-\alpha_{n-1})PTy_{n-1}\right\| \\ &= \left\|\alpha_n\big(f(x_n)-f(x_{n-1})\big)+(\alpha_n-\alpha_{n-1})\big(f(x_{n-1})-PTy_{n-1}\big)+(1-\alpha_n)\big(PTy_n-PTy_{n-1}\big)\right\| \\ &= \alpha_n\beta\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})-PTy_{n-1}\|+(1-\alpha_n)\|y_n-y_{n-1}\|\end{aligned}$$

by (8), we have

$$\begin{aligned}\|x_{n+1}-x_n\| &\le \alpha_n\beta\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})-PTy_{n-1}\| \\ &\quad+(1-\alpha_n)\big(\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\|x_{n-1}-Tx_{n-1}\|\big) \\ &\le \alpha_n\beta\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})-PTy_{n-1}\| \\ &\quad+(1-\alpha_n)\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\|x_{n-1}-Tx_{n-1}\|.\end{aligned}$$

Set $M_2=\max\left\{\|x_{n-1}-Tx_{n-1}\|,\|f(x_{n-1})-PTy_{n-1}\|\right\}$

$$\begin{aligned}\|x_{n+1}-x_n\| &\le (1-\alpha_n+\alpha_n\beta)\|x_n-x_{n-1}\|+\big(|\alpha_{n-1}-\alpha_n|+|\beta_{n-1}-\beta_n|\big)M_2 \\ &\le \big(1-\alpha_n(1-\beta)\big)\|x_n-x_{n-1}\|+\big(|\alpha_{n-1}-\alpha_n|+|\beta_{n-1}-\beta_n|\big)M_2.\end{aligned}$$

Set $a_{n+1}=\|x_{n+1}-x_n\|$, $t_n=1-\alpha_n(1-\beta)$, $b_n=0$, $c_n=\big(|\alpha_{n-1}-\alpha_n|+|\beta_{n-1}-\beta_n|\big)M_2$
by the lemma 2.2 we have

$$\lim_{n\to\infty}\|x_n-x_{n+1}\|=0.$$

Now we will proof $\|x_n-PTx_n\|\to 0$ as $n\to\infty$.

$$\begin{aligned}\|x_n-PTx_n\| &= \|x_n-x_{n+1}\|+\|x_{n+1}-PTx_n\| \\ &\le \|x_n-x_{n+1}\|+\|\alpha_n f(x_n)+(1-\alpha_n)PTy_n-PTx_n\| \\ &\le \|x_n-x_{n+1}\|+\alpha_n\|f(x_n)-PTx_n\|+(1-\alpha_n)\|PTy_n-PTx_n\| \\ &\le \|x_n-x_{n+1}\|+\alpha_n\|f(x_n)-PTx_n\|+(1-\alpha_n)\|y_n-x_n\|\end{aligned} \tag{9}$$

$$\|y_n-x_n\|=\|\beta_n x_n+(1-\beta_n)Tx_n-x_n\|=(1-\beta_n)\|x_n-Tx_n\|$$

as $n\to\infty$, $\alpha_n\to 0$, $\beta_n\to 1$ therefore

$$\lim_{n\to\infty}\|x_n-PTx_n\|=0. \tag{10}$$

Remark 3.3. From the lemma 3.6 we know that p is the unique solution in $F(T)$ to the following variational inequality:

$$\langle (I-F)p, j(p-u)\rangle\le 0 \quad \text{for all} \quad u\in F(T). \tag{11}$$

Now, we can take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle f(p)-p, j(x_n-p)\rangle=\limsup_{k\to\infty}\langle f(p)-p, j(x_{n_k}-p)\rangle$$

we may assume that $x_{n_k}\to x^*$ by X is reflexive and $\{x_n\}$ is bounded. It follows from Lemma 2.3, Lemma 2.4, and (10), we have $x^*\in F(T)=F(PT)$, by (11) we have

$$\limsup_{n\to\infty}\langle f(p)-p, j(x_n-p)\rangle=\limsup_{k\to\infty}\langle f(p)-p, j(x_{n_k}-p)\rangle\le 0.$$

Theorem 3.8. Let X be a reflexive Banach space which admits a weakly sequentially continuous duality mapping J from X to X^*. Suppose C is a nonexpansive retract of X which is also a sunny nonexpansive retract of X, and $T: C \to X$ is a nonexpansive mapping satisfying the weakly inward condition and $F(T) \neq \phi$, let $f: C \to C$ be a fixed contractive mapping from C to C. And $\{x_n\}_{n=0}^{\infty}$ is the sequence by definition 2.4 (3). Let us assume there are two sequences $\{\alpha_n\}$, $\{\beta_n\}$ in $[0,1]$ satisfying the following conditions:

$$\sum_{n=1}^{\infty}\left(|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|\right) < \infty$$

then the sequence $\{x_n\}$ converges strongly to the unique solution p of the variational inequality:

$$p \in F(T) \text{ and } \left\langle (I - f)p, j(p-u)\right\rangle \leq 0 \text{ for all } u \in F(T).$$

Proof. Since C is closed, by lemma 3.5, $\{x_n\}$ is bounded, so $\{f(x_n)\}$, $\{y_n\}$, $\{Tx_n\}$ are also bounded. Let $\{x_t\}$ be the sequence defined by

$$x_t = P\left(tf(x_t) + (1-t)Tx_t\right)$$

by the lemma 3.6 as $t \to 0$ we have $\{x_t\}$ converges strongly to a fixed point p of T and p is also the unique solution in $F(T)$ to the following variational inequality

$$\left\langle (I - f)p, j(p-u)\right\rangle \leq 0 \text{ for all } u \in F(T)$$

using the remark 3.3, we have

$$\limsup_{n\to\infty}\left\langle f(p) - p, j(x_n - p)\right\rangle \leq 0$$

By the definition 2.4 (3), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n f(x_n) + (1-\alpha_n)PTy_n - p\|^2 \\
&\leq \|\alpha_n(f(x_n) - p) + (1-\alpha_n)(PTy_n - p)\|^2 \\
&\leq (1-\alpha_n)^2\|PTy_n - p\|^2 + 2\alpha_n\left\langle f(x_n) - f(p), j(x_{n+1} - p)\right\rangle \\
&\quad + 2\alpha_n\left\langle f(p) - p, j(x_{n+1} - p)\right\rangle \\
&\leq (1-\alpha_n)^2\|y_n - p\|^2 + 2\alpha_n\beta\|x_n - p\|\|x_{n+1} - p\| \\
&\quad + 2\alpha_n\left\langle f(p) - p, j(x_{n+1} - p)\right\rangle \\
&\leq (1-\alpha_n)^2\|y_n - p\|^2 + \alpha_n\beta\left(\|x_n - p\|^2 + \|x_{n+1} - p\|^2\right) \\
&\quad + 2\alpha_n\left\langle f(p) - p, j(x_{n+1} - p)\right\rangle
\end{aligned}$$

While

$$\begin{aligned}
\|y_n - p\| &= \|\beta_n x_n + (1-\beta_n)Tx_n - p\| \leq \beta_n\|x_n - p\| + (1-\beta_n)\|Tx_n - p\| \\
&\leq \beta_n\|x_n - p\| + (1-\beta_n)\|x_n - p\| = \|x_n - p\|
\end{aligned}$$

therefore,

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \frac{(1-\alpha_n)^2 + \alpha_n\beta}{1-\alpha_n\beta}\|x_n - p\|^2 + \frac{2\alpha_n}{1-\alpha_n\beta}\left\langle f(p) - p, j(x_{n+1} - p)\right\rangle \\
&\leq \left(1 - \frac{1-2\alpha_n\beta}{1-\alpha_n\beta}\right)\|x_n - p\|^2 + \frac{(1-\alpha_n)^2}{1-\alpha_n\beta}\|x_n - p\|^2 + \frac{2\alpha_n}{1-\alpha_n\beta}r_{n+1}
\end{aligned}$$

where $r_{n+1} = \max\left\{\left\langle f(p) - p, j(x_{n+1} - p)\right\rangle, 0\right\}$

Setting $a_n = \|x_n - p\|$, $t_n = 1 - \frac{1-2\alpha_n\beta}{1-\alpha_n\beta}$, $b_n = \frac{(1-\alpha_n)^2}{1-\alpha_n\beta}\|x_n - p\|^2$, $c_n = \frac{2\alpha_n}{1-\alpha_n\beta} r_{n+1}$ and applying Lemma 2.1, we conclude that $x_n \to p$.

Let us prove *p* is the unique fixed point of *T*.

We assume that p^* is another solution of (12) in $F(T)$, then $\langle f(p) - p, j(p^* - p)\rangle \le 0$ and $\langle f(p^*) - p^*, j(p - p^*)\rangle \le 0$, so we have $(1-\alpha)\|p - p^*\| \le 0$, which implies the equality $p = p^*$.

Remark 3.4. When $\beta_n = 1$ for all $n \in \mathrm{N}$. The second type viscosity approximation methods for nonexpansive nonself-mappings (see definition 2.4) become the following iteration sequence:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) PTx_n .$$

So the theorem 3.8 improves the theorem 4.3 theorem 4.4 of Song-Li [2].

4. Conclusion

In this paper, we studied two new viscosity approximation methods for nonexpansive nonself-mappings, which were defined by definition 2.4. And then we proved that the sequences $\{x_n\}$ which were defined by definition 2.4 converged strongly to the fixed point of *T*, which were the nonexpansive nonself mappings in Banach space.

References

[1] Song, Y.S. and Chen, R.D. (2006) Viscosity Approximation Methods for Nonexpansive Nonself-Mappings. *Journal of Mathematical Analysis and Applications*, **321**, 316-326.

[2] Song, Y.S. and Li, Q.C. (2007) Viscosity Approximation for Nonexpansive Nonself-Mappings in Reflexive Banach Space. *J. Sys. Sci. and Math. Scis*, 481-487. (In Chinese)

[3] Halpern, B. (1967) Fixed Points of Nonexpanding Maps. *Bulletin of the American Mathematical Society*, **73**, 957-961. http://dx.doi.org/10.1090/S0002-9904-1967-11864-0

[4] Kalinde, A.K. (1992) Fixed Points Ishikawa Iterations. *Journal of Mathematical Analysis Applications*, 600-606.

[5] Dotson, W.G. (1970) On the Mann Iterative Process. *Transactions of the American Mathematical Society*, **149**, 65-73.

[6] Morales, C.H. and Jung, J.S. (2000) Convergence of Paths for Pseudo-Contractive Mappings in Banach Space. *Proceedings of The American Mathematical Society*, **128**, 3411-3419. http://dx.doi.org/10.1090/S0002-9939-00-05573-8

[7] Liu, L.S. (1995) Ishikawa and Mann Iterative Processes with Errors for Nonlinear Strongly Accretive Mapping in Banach Spaces. *Journal of Mathematical Analysis and Applications*, **194**, 114-125. http://dx.doi.org/10.1006/jmaa.1995.1289