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Certain Algebraic Test for Analyzing Aperiodic Stability of Two-Dimensional Linear Discrete Systems

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Abstract

This paper addresses the new algebraic test to check the aperiodic stability of two dimensional linear time invariant discrete systems. Initially, the two dimensional characteristics equations are converted into equivalent one-dimensional equation. Further Fuller's idea is applied on the equivalent one-dimensional characteristics equation. Then using the co-efficient of the characteristics equation, the routh table is formed to ascertain the aperiodic stability of the given two-dimensional linear discrete system. The illustrations were presented to show the applicability of the proposed technique.

Keywords

Routh Table, Aperiodic Stability, Characteristics Equation, Two-Dimensional, Linear Discrete Systems

1. Introduction

Stability is the unique and basic property to be possessed by all kinds of systems. To investigate this property, various graphical and analytical methods are available. For a given absolutely stable linear time invariant discrete system represented by its characteristics equation $f(Z) = 0$, with all the roots having $z < 1$, the aperiodic stability can be obtained in the given stable system. If all the roots are simple and lie on the negative real axis, it represents aperiodically stable condition. Using fuller's concept routh table is formed to check the aperiodic stability of the system. Information about the aperiodic stability of a control system is of paramount importance for any design problem. This is generally used for the design of instrumentation systems, network analysis and au-

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tomatic controls. The existence of real and distinct roots in the negative real axis determines the aperiodic behavior of a linear system. The presence of any complex roots shows that the system is aperiodicallly stable. To analyze the aperiodic stability, a generalized method was investigated in the literature by szaraniec. A three-step transformation procedure is presented by fuller, which develops a polynomial whose number of right hand poles equals the number of complex roots present in the original polynomial.

In the Routh-Hurwitz stability criterion, the coefficient of the polynomial is arranged in two rows. When n is even, the S^n row is formed by coefficients of even order terms (*i.e.*, coefficients of even power of S) and S^{n-1} row is formed by coefficients of odd order terms (*i.e.*, coefficients of odd power of S). When n is odd, the S^n row is formed by coefficients of odd terms (*i.e.*, coefficients of odd power of S) and S^{n-1} row is formed by coefficients of even order terms (*i.e.*, coefficients of even powers of S). In our proposed scheme, the analysis of aperiodic stability of a given stable linear discrete system is presented with a help of fullers equation as follows it accounts all the coefficients of the equivalent one dimensional characteristics equations in the S^n row when n is either even or odd and S^{n-1} row is formed using the coefficients by differentiating the equivalent one dimensional characteristics equation. The other rows of routh array up to S^0 row can be formed by normal Routh-Hurwitz stability criterion. Illustrative examples show the applicability of the proposed scheme.

2. Literature Survey

The stability analysis is the most important test that should be considered in LTIDS analysis and design. Therefore stability determination is very important. Hence many researches had been conducted to ascertain, the stability in last few years. A new formulation of the critical constrains for stability limits matrix and its bi-alternative product was discussed by Jury in [1]. Bistritz had introduced in [2] the efficient stability test that involves real univariate polynomials and real arithmetic only. It also examines the doubling degree technique in the development of two dimensional stability test discussed by Bistritz in [3]. The possible root locations of two dimensional polynomials were proposed by Khargonekerin [4] had proved the critical constrains of obtained from the polynomial and also obtained the stability within the unit circle. A new method to compute the stability margin of two dimensional continuous system was provided by Mastorakis in [5] and illustrated and discussed. A discussion of stability test was obtained for two dimensional and sufficient conditions for asymptotic stability were easy to checked by Anderson *et al.* in [6] [7]. Results obtained in [8] proved that double bi-linear transformation does not preserve the stability in either direction. *i.e.* Continuous to discrete domain and vice versa proposed by Jury et al. The algorithm was proposed by Bose *et al.* in [9] explains the positive definiteness of arbitrary quadratic forms that is expressed in terms of inner wise number. The necessary and sufficient conditions are used to form the inner of square matrix discussed by Jury in [10]. A new algebraic procedure was solving the problem of stability in very low count of arithmetic operations was given by Bistritz and Ahmed in [11] [12]. A simplify algebraic equations were Presented by Jury in [13] it was analyzed for its consistency and stability. Goodman had focused in [14] the suitable difference between the one dimensional and two dimensional test cases were presented. Two dimensional recursive filtering were proposed by Huang in [15]. It derived and simplified version of stability theorem. The Hurwitz character of the system was determined by Bauer in [16] and Also proved a relationship between the Hurwitz character of the denominator polynomial of the two dimensional transfer function. Vimalsingh had proposed in [17] a new criterion for the general asymptotic stability of two dimensional discrete system discussed by the roesser model using saturation arithmetic was proposed. The implementation procedure for stability test for two dimensional digital filter was presented by katbab *et al.* in [18] it had reduced the computations, to determine a conservative coefficient space within the coefficient of a real two dimensional digital filter. Kamat *et al.* in [19] had presented the root distribution polynomial with real coefficients with respect to the unit circle is equivalent to one proposed with the same degree of complexity. The condition for aperiodicity was discussed by szaraniec in [20]. Fullers proposed methodology in [21] to check whether a control system was having the dead-beat condition (Aperiodic stability) and revealed that the characteristics equation with real coefficients and can transformed into characteristics equation with complex coefficients. Later, szaraniec methodology was analyzed and modified by jury in [22] and it explains the new theorem for the aperiodicity. Jury *et al.* in [23] proposed a simple stability test similar to routh table was being introduced for linear discrete systems. Bose *et al.* in [24] presented a procedure to determine whether or not a polynomial in several real variables is globally positive. A method for determining stability by use of a table form has been presented by jury in [25], the table had also been used in determining the roots distribution within the unit circle.

3. Proposed Method

The two dimensional discrete system is represented in transfer function [5] by Mastorakis (1998) as,

$$H(Z_1, Z_2) = \frac{A(Z_1, Z_2)}{B(Z_1, Z_2)} \tag{1}$$

where *A* and *B* are non-cancellable polynomials in Z_1 and Z_2. Method proposed in [5] is more complicated and requires more computations to check the aperiodic stability of the given system. In this paper to avoid more computations a simple algebraic procedure is presented

To convert two-dimensional characteristics equation in to one-dimensional characteristics equation:

In general the following form of Two-dimensional (2D) [11] equation is chosen:

$$B(Z_1, Z_2) = T_0(Z_1) Z_2^n + T_1(Z_1) Z_2^{n-1} + \cdots + T_n(Z_1) = 0 \tag{2}$$

The reciprocals of Z_1 and Z_2 are $\left(\frac{1}{Z_1}\right)$ and $\left(\frac{1}{Z_2}\right)$ respectively are utilized so that the Equation (2) is re-written as:

$$B\left(\frac{1}{Z_1}, \frac{1}{Z_2}\right) = T_0\left(\frac{1}{Z_1}\right)\left(\frac{1}{Z_2}\right)^n + T_1\left(\frac{1}{Z_1}\right)\left(\frac{1}{Z_2}\right)^{n-1} + \cdots + T_n\left(\frac{1}{Z_1}\right) = 0 \tag{3}$$

Again Equation (3) is represented as,

$$M(Z_1, Z_2)\big|_{Z_1 = Z_2 = Z} = f(Z) = 0 \tag{4}$$

This $f(Z) = 0$ is one dimensional equation and for aperiodic stability $|Z| < 1$.

Then $f(Z)$ can be analyzed by any algebraic method for checking aperiodic stability of the equivalent one-dimensional characteristics equation from two-dimensional characteristics equation.

Aperiodic Stability Test

If the characteristics roots of $f(Z) = 0$ lie in the sector region $0 \le Z < 1$ and all are simple (distinct) and all the roots are positive then the system becomes aperiodically stable and the characteristic equation of the system can be written as,

$$f(z) = (z - x_1)(z - x_2)\cdots(z - x_n) = 0 \tag{5}$$

where, x_i are distinct roots of $F(Z) = 0$.

In general, the Equation (5) can be arranged as

$$f(z) = z^n - a_{n-1} z^{n-1} + a_{n-2} z^{n-2} - \cdots + a_0 = 0 \tag{6}$$

The coefficients of $f(Z) = 0$ should alternate in sign. It is observed in Equation (6).

With $Z = -Z$, the coefficients of Equation (6) will become positive in the sector region, $-1 < Z \le 0$. To test the aperiodic stability of a given linear time-invariant continuous system represented in the form of its characteristics equation.

$$f(Z) = Z^n + a_{n-1} Z^{n-1} + a_{n-2} Z^{n-2} + \cdots + a_0 = 0 \tag{7}$$

Fuller (1955) formulated [20] a transformed equation of $F(s) = 0$ as,

$$F(s) = f(s)\big|_{s=s^2} + s \frac{\mathrm{d}f(s)}{\mathrm{d}s}\bigg|_{s=s^2} = 0 \tag{8}$$

Routh's test referred in Anderson *et al.* (1987) and Byrne (1975) is applied for the Equation (8). If the first column of Routh's table does not possess any sign change, then the system represented by the Equation (8) is aperiodically stable. Thus, extending Fuller's idea, the following transformed equation is written for the Equation (9) as,

$$F(Z) = f(Z)\Big|_{Z=Z^2} + Z\frac{\mathrm{d}f(Z)}{\mathrm{d}Z}\Bigg|_{z=Z^2} = 0 \tag{9}$$

The transformed Equation (9) can be handled by Routh's test verify the sufficiency condition for aperiodic stability. Thus, above proposed procedure is applied for the following illustrative examples.

4. Illustrations

Example 1: [15]

Consider a two dimensional characteristic equation,

$$f(z) = 1 - 0.75z_1 - 0.5z_2 + 0.3z_1z_2$$

Convert the two dimensional characteristic equation in to single dimensional characteristic equation,

$$1 - 0.75z_1 - 0.5z_2 + 0.3z_1z_2 = 0$$

Taking the inverse of variables z_1 and z_2

$$1 - \frac{0.75}{z_1} - \frac{0.5}{z_2} + \frac{0.3}{z_1z_2} = 0$$

$$z_1 = z_2 = x$$

$$1 - \frac{0.75}{x} - \frac{0.5}{x} + \frac{0.3}{x^2} = 0$$

The required one dimensional equation is,

$$x^2 - 1.25x + 0.3 = 0$$

$$f(x) = x^2 - 1.25x + 0.3 = 0$$

The coefficient of $f(x)$ has alternate in sign; then the necessary condition is satisfied.

Then Substitute $x = -x$ in $f(x)$

$$f(-x) = x^2 + 1.25x + 0.3$$

$$f'(-x) = 2x + 1.25$$

Applying Fuller's Concept $F(x) = f(-x) + f'(-x)$

$$F(x) = (x^2 + 1.25x + 0.3) + (2x + 1.25)$$

Routh Table:

From **Table 1**, all the elements in the first column are positive then the system is aperiodically stable.

Table 1. Routh table using fullers concept for Example 1.

1	−1.25	0.3
2	1.25	
0.625	0.3	
0.288		
0.3		

Output is verified using MATLAB

$$p = f(x);\ q = f(-x);\ r = f'(-x)$$

p = 1.0000 −1.2500 0.3000
q = 1.0000 1.2500 0.3000
r = 2.0000 1.2500

$$\text{sol mat} = \begin{bmatrix} 2.000 & 1.250 & 0 & 0 \\ 0.625 & 0.300 & 0 & 0 \\ 0.290 & 0 & 0 & 0 \\ 0.300 & 0 & 0 & 0 \\ 0.290 & 0 & 0 & 0 \end{bmatrix}$$

Example 2: [25]

Consider the following two dimensional characteristic equation

$$f(z) = 1 - 1.2z_2 + 0.3z_2^2 - 1.5z_1 + 1.8z_1z_2 - 0.75z_1z_2^2 + 0.6z_1^2 - 0.72z_1^2z_2 + 0.29z_1^2z_2^2 = 0$$

Convert the two dimensional characteristic equation in to single dimensional characteristic equation,

$$1 - 1.2z_2 + 0.3z_2^2 - 1.5z_1 + 1.8z_1z_2 - 0.75z_1z_2^2 + 0.6z_1^2 - 0.72z_1^2z_2 + 0.29z_1^2z_2^2 = 0$$

Taking inverse of variables z_1 and z_2,

$$1 - \frac{1.2}{z_2} + \frac{0.3}{z_2^2} - \frac{1.5}{z_1} + \frac{1.8}{z_1z_2} - \frac{0.75}{z_1z_2^2} + \frac{0.6}{z_1^2} - \frac{0.72}{z_1^2z_2} + \frac{0.29}{z_1^2z_2^2} = 0$$

$$z_1 = z_2 = x$$

$$1 - \frac{1.2}{x} + \frac{0.3}{x^2} - \frac{1.5}{x} + \frac{1.8}{x^2} - \frac{0.75}{x^3} + \frac{0.6}{x^2} - \frac{0.72}{x^3} + \frac{0.29}{x^4} = 0$$

The required one dimensional equation is,

$$x^4 - 2.7x^3 + 2.7x^2 - 1.47x + 0.29 = 0$$

$$f(x) = x^4 - 2.7x^3 + 2.7x^2 - 1.47x + 0.29$$

The coefficient of $f(x)$ has alternate sign; then the necessary condition is satisfied.
Then Substitute $x = -x$ in $f(x)$
The transformed equation is formed as,

$$f(-x) = x^4 + 2.7x^3 + 2.7x^2 + 1.47x + 0.29$$

$$f'(-x) = 4x^3 + 8.1x^2 + 5.4x + 1.47$$

Using Fuller's Concept, $F(x) = f(-x) + f'(-x)$

$$F(x) = f\left(x^4 + 2.7x^3 + 2.7x^2 + 1.47x + 0.29\right) + f'\left(4x^3 + 8.1x^2 + 5.4x + 1.47\right)$$

Routh Table:

From Table 2, there is two sign change in the first column therefore the given system is unstable.

Table 2. Routh table using fullers concept for Example 2.

1	2.7	2.7	1.47	0.29
4	8.1	5.4	1.47	
0.68	1.35	1.10	0.29	
0.1	−1.11	−0.25		
8.84	2.77	0.29		
−1.14	−0.25			
0.833	0.29			
0.15				
0.29				

Output is verified using MATLAB

$$p = f(x);\, q = f(-x);\, r = f'(-x)$$

p = 1.0000 −2.7000 2.7000 −1.4700 0.2900
q = 1.0000 2.7000 2.7000 1.4700 0.2900
r = 4.0000 8.1000 5.4000 1.4700

$$\text{sol\_mat} = \begin{bmatrix} 1.000 & 2.700 & 2.700 & 1.470 & 0.290 & 0 \\ 4.000 & 8.100 & 5.400 & 1.470 & 0 & 0 \\ 0.675 & 1.350 & 1.102 & 0.290 & 0 & 0 \\ 0.100 & -1.133 & -0.248 & 0 & 0 & 0 \\ 0.900 & 2.780 & 0.290 & 0 & 0 & 0 \\ -1.164 & -0.252 & 0 & 0 & 0 & 0 \\ 0.833 & 0.290 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example 3: [15]

Consider a two dimensional characteristic equation,

$$(1-0.95z_1)-(0.95-0.5z_1)z_2 = 0$$

Convert the two dimensional characteristic equation in to single dimensional characteristic equation,

$$1-0.95z_1-0.95z_2+0.5z_1z_2 = 0$$

Taking inverse of variables z_1 and z_2,

$$1-\frac{0.95}{z_1}-\frac{0.95}{z_2}+\frac{0.5}{z_1z_2} = 0$$

$$z_1 = z_2 = x$$

$$1-\frac{0.95}{x}-\frac{0.95}{x}+\frac{0.5}{x_2} = 0$$

The required one dimensional equation is,

$$x^2-1.90x+0.5 = 0$$

$$f(x) = x^2-1.90x+0.5$$

The coefficient of $f(x)$ has alternate sign; then the necessary condition is satisfied.

Then Substitute $x=-x$ in $f(x)$

The transformed equation is formed as,

$$f(-x) = x^2+1.90x+0.5$$

$$f'(-x) = 2x+1.90$$

Using Fuller's concept $F(x) = f(-x)+f'(-x)$

$$F(x) = (x^2+1.90x+0.5)+(2x+1.90)$$

Routh Table:

From **Table 3**, all the elements in the first column of routh table are positive then the system is aperiodically stable.

Output is verified using MATLAB

$$p = f(x);\, q = f(-x);\, r = f'(-x)$$

Table 3. Routh table using fullers concept for Example 3.

1	1.90	0.5
2	1.90	
0.95	0.5	
0.848		
0.5		

$p = 1.0000 \quad -1.9000 \quad 0.5000$
$q = 1.0000 \quad 1.9000 \quad 0.5000$
$r = 2.0000 \quad 1.9000$

$$\text{sol\_mat} = \begin{bmatrix} 1.000 & 1.900 & 0.500 & 0 \\ 2.000 & 1.900 & 0 & 0 \\ 0.950 & 0.500 & 0 & 0 \\ 0.8474 & 0 & 0 & 0 \\ 0.500 & 0 & 0 & 0 \end{bmatrix}$$

5. Conclusion

The main contribution of this paper is to ascertain the aperiodic stability of two dimensional linear time invariant discrete systems with the help of Routh table [24] by applying fuller's concept [21] to form the S^{n-1} row in the Routh table which reduces the order of the characteristics equation and hence reduces the number of computations. The result of the analysis is verified by using MATLAB. In the view of conceptual simplicity of these tests, it would appear that development of dedicated implementation procedures were simple and worthy subject of further investigation. The contribution made in this paper can be extended to characteristics polynomial containing complex coefficients.

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