Advances in Pure Mathematics, **2011, 1, 210-217**
doi:10.4236/apm.2011.14037 Published Online July 2011 (http://www.SciRP.org/journal/apm)

Normal Criteria and Shared Values by Differential Polynomials*

Jihong Wang[1], Qian Lu[2], Qilong Liao[3]
[1]*Department of Mathematics, Southwest University of Science and Technology, Mianyang, China*
[2]*Department of Mathematics, Southwest University of Science and Technology, Mianyang, China*
[3]*Department of Material Science and Engineer, Southwest University of Science and Technology, Mianyang, China*
E-mail: wangjihong@swust.edu.cn, luqiankuo1965@hotmail.com, liaoql@swust.edu.cn
Received April 5, 2011; *revised April* 28, 2011; *accepted May* 8, 2011

Abstract

For a family F of meromorphic functions on a domain *D*, it is discussed whether F is normal on *D* if for every pair functions $f(z)$, $g(z) \in F$, $f' - af^n$ and $g' - ag^n$ share value d on *D* when $n = 2,3$, where *a*, *b* are two complex numbers, $a \neq 0, \infty, b \neq \infty$. Finally, the following result is obtained:Let F be a family of meromorphic functions in *D*, all of whose poles have multiplicity at least 4 , all of whose zeros have multiplicity at least 2. Suppose that there exist two functions $a(z)$ not idendtically equal to zero, $d(z)$ analytic in *D*, such that for each pair of functions f and g in F, $f' - a(z)f^2$ and $g' - a(z)g^2$ share the function $d(z)$. If $a(z)$ has only a multiple zeros and $f(z) \neq \infty$ whenever $a(z) = 0$, then F is normal in *D*.

Keywords: Normal Family, Meromorphic Function, Shared Value, Differential Polynomial

1. Introduction and the Main Result

In 1959,Hayman[4] proved

Theorem 1.1. *Let f be meromorphic functions in C, n be a positive integer and a, b be two constant such that $n \geq 5$, $a \neq 0, \infty$ and $b \neq \infty$. If*

$$f' - af^n \neq b$$

then f is a constant.

Corresponding to Theorem 1.1 there is the following theorems which confirmed a Hayman's well-known conjecture about normal families in [5].

Theorem 1.2. *Let F be a meromorphic function family in D, n be a positive integer and a, b be two constant such that $a \neq 0, \infty$ and $b \neq \infty$. If $n \geq 3$ and for each function $f \in F$, $f' - af^n \neq b$, then F is normal in D.*

This result is due to S. Y. Li [8]($n \geq 5$), X. J. Li [9] ($n \geq 5$), X. C. Pang [10]($n = 4$), H. H. Chen and M. L. Fang [2]($n = 3$).

In 2001, M. L. Fang and W. J. Yuan [3] obtained

Theorem 1.3. *Let F be a meromorphic function family in D, a, b be two constants such that $a \neq 0, \infty$ and $b \neq \infty$. If, for each function $f \in F$, $f' - af^2 \neq b$ and the poles of $f(z)$ are of multiplic*ity 3 *at least, then F is normal in D.*

Let *D* be a domain in *C*, $f(z)$ be meromorphic on *D*,and. $a \in C$

$$E_f(a) = f^{-1}(a) \cap D = \{Z \in D : f(z) = a\}$$

Two functions *f* and *g* are said to share the value *a* if $E_f(a) = Eg(a)$. For a case $n \geq 4$ in Theorem 1.2, Q. C. Zhang [14] improved Theorem 1.2 by the idea of shared values and obtained the following result.

Theorem 1.4. *Let F be a family of meromorphic functions in D, n be a positive integer and a, b be two constant such that $n \geq 4$, $a \neq 0, \infty$ and $b \neq \infty$. If, for each pair of functions f and g in F, $f' - af^n$ and $g' - ag^n$ share the value b, then F is normal in D.*

In this paper, we shall discuss a condition on which *F* still is normal in *D* for the case $2 \leq n \leq 3$ and obtain the following result.

Theorem 1.5. *Let F be a family of meromorphic functions in D, all of whose poles have multiplicity* 2 *at least, and a, b be two constant such that $a \neq 0, \infty$ and $b \neq \infty$.*

*Supported by China Industrial Technology Development Program (B3120110001).

If, for each pair of functions f and g in F, $f' - af^3$ *and* $g' - ag^3$ *share the value b in D, then F is normal in D.*

We denote $f^{\#}(z) = \frac{|f'(z)|}{1+|f(z)|^2}$ for the spherical derivatives of $f(z)$ .The following example imply that the restriction of poles in Theorem 1.5 is necessary.

Example 1. [14] Let $D = \{z : |z| < 1\}$ and $F = \{f_n\}$, where

$$f_n(z) = \frac{1}{\sqrt{n}(z - 1/n)}, z \in D, n = 1, 2, \cdots$$

Then for each pair *m*, *n*, $f'_m - f_m^3$ and $f'_n - f_n^3$ share the value 0 in *D*. But *F* is not normal at $z = 0$ since $f_n^{\#}(1/\sqrt{n}) \to \infty$.

But we also have the following examples which imply that on the same as restriction of poles in Theorem 1.5 *F* is not normal in *D* if for each pair of functions f and g in *F*, $f' - af^2$ and $g' - ag^2$ share the value *b* on *D*.

Example 2. [3] Let $f_n(z) = nz\Big/\left(z\sqrt{n} - 1\right)^2$ for $n = 1, 2, \cdots$, and $\Delta = \{z : |z| < 1\}$ Clearly,

$$f'_n(z) + f_n^2 = n\left(z\sqrt{n} - 1\right)^{-4} \neq 0,$$

and $f_n(z)$ only a double pole and a simple zero. Since $f_n^{\#}(0) = n \to \infty$, as $n \to \infty$ from Marty's criterion we have that $\{f_n(z)\}$ is not normal in Δ In fact, in the-present paper we also obtain two results as follows.

Theorem 1.6. *Let F be a family of meromorphic functions in D, all of whose poles have multiplicity* 4 *at least, all of whose zeros have multiplicity* 2 *at least, and a, b be two constant such that* $a \neq 0, \infty$ *and* $b \neq \infty$. *If, for each pair of functions f and g in F,* $f' - af^2$ *and* $g' - ag^2$ *share the value b in D, then F is normal in D.*

Theorem 1.7. *Let F be a family of meromorphic functions in D, all of whose poles have multiplicity at least* 4 , *all of whose zeros have multiplicity at least* 2. *Suppose that there exist two functions* $a(z)$ not idendtically equal to zero, $d(z)$ *analytic in D, such that for each pair of functions f and g in F,* $f' - a(z)f^2$ *and* $g' - a(z)g^2$ *share the function* $d(z)$ *in D. If* $a(z)$ *has only a multiple zeros and* $f(z) \neq \infty$ *whenever* $a(z) = 0$ *then F is normal in D.*

The following example shows that the condition $f(z) \neq \infty$ when $a(z) = 0$ in Theorem 1.7 is necessary.

Example 3. [7] Let $D = \{z : |z| < 1\}$ and $F = \{f_n\}$ where $f_n(z) = \frac{1}{nz^4}, z \in D, n = 1, 2, \cdots$. We take $a(z) = -4z^3$ and $d(z) \equiv 0$. Clearly, *F* fails to be normal at $z = 0$ However, all poles of $f_n(z)$ are of multiplicity 4, and for each pair *m*, *n*, $f'_m - a(z)f_m^2$ and $f'_n - a(z)f_n^2$ share analytic functions $d(z)$ in Δ.

2. Lemmas

To prove the above theorems, we need some lemma as follows:

Lemma 2.1. ([1,2]) *Let* $f(z)$ *be a meromorphic function in C, n be a positive integer and b be a non-zero constant. If* $f^n f' \neq b$, *then* f *is a constant. Moreover if f is a transcendental meromorphic function, then* $f^n f'(z)$ *assumes every finite non-zero value finitely often.*

Lemma 2.2. ([1]) *Let* $f(z)$ *be a transcendental meromorphic function with finite order in C. If* $f(z)$ *has only multiple zeros, then it's first derivative* f' *assumes every finite value except possibly zero infinitely often.*

Lemma 2.3. ([12]) *Let* $f(z)$ *be a non-polynomials rational function in* C. *If* $f(z)$ *has only zeros of multiplicity* 2 *at least, then* $f = \frac{(cz+d)^2}{az+b}$ *where a, b, c, d are four constants,* $a \neq 0, c \neq 0$.

Lemma 2.4. ([4]) *If* $f(z)$ *be a transcendental meromorphic function in C, then either* $f(z)$ *assumes every finite value infinitely often or every derivative* $f^{(l)}$ *assumes every finite value except possibly zero infinitely often. If* $f(z)$ *is a non-constant rational function and* $f(z) \neq a$, *a is a finite value, then* $f^{(l)}$ *assumes every finite value except possibly zero at least once.*

Lemma 2.5. ([11]) *Let* $f(z)$ *be a transcendental meromorphic function with finite order, all of whose zeroes are of multiplicity at least* $k+1$, *and let* $P(z)$ *be a polynomial,* $P(z)$ *is not idendtically equal to zero. Then* $f^{(k)}(z) - P(z)$ *has infinitely many zeros often.*

Lemma 2.6. ([6]) *Let* $f(z)$ *be a non-polynomial rational functions in* C, *all of whose zeroes are of multiplicity at least* 4. *Then* $f'(z) - z^r$ *has a zeros at least often.*

Lemma 2.7. ([13]) *Let F be a family of meromorphic functions on the unit disc* Δ, *all of whose zeroes have multiplicity p at least, all of whose poles have multiplicity q at least. Let* α *be a real number satisfying* $-p < \alpha < q$. *Then F is not normal at a point* $z_0 \in \Delta$ *if and only if there exist*

1) *points* $z_n \in \Delta$, $z_n \to z_0$;

2) *functions* $f_n \in F$; *and*

3) *positive numbers* $\rho_n \to 0$

such that

$$\rho_n^{\alpha} f_n\left(z_n+\rho_n\xi\right)=g_n\left(\xi\right)\to g\left(\xi\right)$$

spherically uniformly on each compact subset of C, where $g\left(\xi\right)$ is a non-constant meromorphic function satisfying the zeros of $g\left(\xi\right)$ are of multiplicities p at least and the poles of $g\left(\xi\right)$ are of multiplicities q at least. Moreover, the order of $g\left(\xi\right)$ is not greater than 2.

3. Proofs of Theorem 1.5.-1.7.

3.1. Proof of Theorem 1.5.

Suppose that there exists one point $z_0\in D$ such that F is not normal at point z_0. Without loss of generality we assume that $z_0=0$. By Lemma 2.7, there exist points, $z_n\in\Delta$, $z_n\to z_0$, functions $f_n\in F$ and positive numbers $\rho_n\to 0$ such that

$$g_j\left(\xi\right)=\rho_j^{\frac{1}{n-1}} f_j\left(z_j+\rho_j\xi\right)\Rightarrow g\left(\xi\right) \tag{3.1}$$

spherically uniformly on each compact subset of *C*, where $g\left(\xi\right)$ is a non-constant meromorphic function with order ≤ 2, all of whose poles are of multiplicities k at least.

From (3.1) we have

$$\begin{aligned}&\rho_j^{\frac{n}{n-1}}\left\{f_j'\left(z_j+\rho_j\xi\right)-af_j^n\left(z_j+\rho_j\xi\right)-b\right\}\\&=g_j'\left(\xi\right)-ag_j^n\left(\xi\right)-\rho_j^{\frac{n}{n-1}}b\Rightarrow g'\left(\xi\right)-ag^n\end{aligned} \tag{3.2}$$

By the same method as [14], from Lemma 2.1 it is not difficult to find that $g'-ag^n$ has just a unique zero $\xi=\xi_0$.

Set $g=1/\varphi$ again, if $n\ge 3$ then

$$g'-ag^n=-\left[\varphi'\varphi^{n-2}+a\right]/\varphi^n$$

thus $\left[\varphi'\varphi^{n-2}+a\right]/\varphi^n$ has just a unique zero $\xi=\xi_0$. Thus ξ_0 is a multiple pole of φ or else a zero of $\varphi'\varphi^{n-2}+a$.

If ξ_0 is a multiple pole of φ, since

$$\left[\varphi'\varphi^{n-2}+a\right]/\varphi^n$$

has only one zero ξ_0, then $\varphi'\varphi^{n-2}+a\ne 0$. By Lemma 2.1 again, φ is a constant which contradicts with *g* is not any constant.

So we have that φ has no multiple poles and $\varphi'\varphi'+a$ have only a unique zero. By Lemma 2.1, and Lemma 2.4, we have φ is not transcendental.

If φ is non-constant polynomial, then

$$\varphi'\varphi^{n-2}+a=A\left(\xi-\xi_0\right)^l .$$

Since all zeros of ψ are of multiplicity 2, then $l\ge 3$. Denoting ψ for $\varphi^{n-1}/\left(n-1\right)$, $\psi=\varphi^{n-1}/\left(n-1\right)$, we have $\psi'=A\left(\xi-\xi_0\right)^l-a$ and $\psi''=Al\left(\xi-\xi_0\right)^{l-1}$. Since all zeros of φ are of multiplicity $2\left(n-1\right)\ge 4$, then $\psi\left(\xi\right)\ne 0,\xi\ne\xi_0$.

If $\psi\left(\xi_0\right)=0$, then $\psi'\left(\xi_0\right)=0$ which contradicts with $\psi'\left(\xi_0\right)=-a\ne 0$. So ψ is a constant.

Next we prove that there exists no rational functions such as ψ. Noting that $\psi=\varphi^{n-1}/\left(n-1\right)$ and ψ has no multiple pole, we may set

$$\psi\left(\xi\right)=A\frac{\left(\xi-\xi_1\right)^{m_1}\left(\xi-\xi_2\right)^{m_2}\cdots\left(\xi-\xi_s\right)^{m_s}}{\left(\eta-\eta_1\right)^{n-1}\left(\eta-\eta_2\right)^{n-1}\cdots\left(\eta-\eta_t\right)^{n-1}}, \tag{3.3}$$

where A is a non-zero constant, $s\ge 1, t\ge 1$, $m_1,m_2,\cdots,m_s$ are s positive integers, $m_j\ge 2\left(n-1\right)$, $\left(j=1,2,\cdots,s\right)$. For a convenience of stating, we denote

$$m=m_1+m_2+\cdots+m_s, \tag{3.4}$$

then $m\ge 2\left(n-1\right)s$.

From (3.3), we have

$$\psi'\left(\xi\right)=A\frac{\left(\xi-\xi_1\right)^{m_1-1}\cdots\left(\xi-\xi_s\right)^{m_s-1}h\left(\xi\right)}{\left(\eta-\eta_1\right)^n\cdots\left(\eta-\eta_t\right)^n}=\frac{p_1\left(\xi\right)}{q_1\left(\xi\right)}, \tag{3.5}$$

where

$$\begin{aligned}h\left(\xi\right)&=\left[m-t\left(n-1\right)\right]\xi^{s+t-1}+a_{s+t-2}\xi^{s+t-2}+\cdots+a_0\\p_1\left(\xi\right)&=\left(\xi-\xi_1\right)^{m_1-1}\cdots\left(\xi-\xi_s\right)^{m_s-1}h\left(\xi\right)\\q_1\left(\xi\right)&=\left(\eta-\eta_1\right)^n\cdots\left(\eta-\eta_t\right)^n,\end{aligned} \tag{3.6}$$

are three polynomials. Since $\psi'\left(\xi\right)+a$ has only a unique zero ξ_0 then there exists a non-zero constant *B* such that

$$\psi'\left(\xi\right)+a=\frac{B\left(\xi-\xi_0\right)^l}{\left(\eta-\eta_1\right)^n\left(\eta-\eta_2\right)^n\cdots\left(\eta-\eta_t\right)^n}, \tag{3.7}$$

so

$$\psi''\left(\xi\right)=\frac{B\left(\xi-\xi_0\right)^{l-1}p_2\left(\xi\right)}{\left(\eta-\eta_1\right)^{n+1}\left(\eta-\eta_2\right)^{n+1}\cdots\left(\eta-\eta_t\right)^{n+1}}, \tag{3.8}$$

where $p_2\left(\xi\right)=\left(l-nt\right)\xi^t+b_{t-1}\xi^{t-1}+\cdots+b_0$ is a polynomial. From (3.5) we also have

$$\psi''\left(\xi\right)=A\frac{\left(\xi-\xi_1\right)^{m_1-2}\cdots\left(\xi-\xi_s\right)^{m_s-2}p_3\left(\xi\right)}{\left(\eta-\eta_1\right)^{n+1}\cdots\left(\eta-\eta_t\right)^{n+1}} \tag{3.9}$$

where $p_3\left(\xi\right)$ is a polynomial also.

We denote $\deg\left(p\right)$ for the degree of a polynomial $p\left(\xi\right)$, from (3.5) and (3.6) we may obtain

$$\begin{aligned}&\deg\left(h\right)\le s+t+1\\&\deg\left(p_1\right)\le m+t+1,\quad \deg\left(q_1\right)=nt\end{aligned} \tag{3.10}$$

From (3.8), (3.9) and (3.10) we may obtain

$$\deg(p_2) \le t, \tag{3.11}$$

$$\deg(p_3) \le 2t + 2s - 2. \tag{3.12}$$

Since $\psi'(\xi)+a$ has only a unique zero $\xi = \xi_0$ and

$$m_j - 2 \ge 1 \ (j = 1,2,\cdots,s),$$

then $\xi_0 \ne \xi_j$ $(j = 1,2,\cdots,s)$. From (3.8), (3.9) and (3.11) it follows that $\deg(p_3(\xi)) \ge l-1$ then

$$m - 2s \le \deg(p_2) \le t, \tag{3.13}$$

Since $m_j \ge 2(n-1)$, then $m \ge 2(n-1)s$, so by (3.13) we have $2s \le t$.

If $l \ge nt$, from (3.8), (3.9) and (3.12), we have

$$nt - 1 \le l - 1 \le \deg(p_3) \le 2t + 2s - 2$$

Then, $t \le 2s-1$. Combining with above inequality $2s \le t$, we bring about a contradiction.

If $l < nt$, then from (3.5) and (3.7) we have

$$\deg(p_1) = \deg(q_1)$$

that is $m - s + \deg(h) = nt$. If $m = t(n-1)$, then $\deg(h) \le s + t - 2$. So

$$\begin{aligned} m - t(n-1) &= s + nt - \deg(h) - t(n-1) \\ &= s + t - \deg(h) \\ &\ge s + t(s + t - 2) = 2 \end{aligned}$$

this is impossible. Thus, $m \ne t(n-1)$ and $\deg(h) = s + t - 1$.Therefore, $m = 1 + t(n-1)$. Again from (3.8) and (3.9), we have $m - 2s \le t$. Then $t \le 2s - 1$, this contradicts to $2s \le t$.

This completes the proof of Theorem 1.5.

3.2. Proof of Theorem 1.6.

For any points $z_0 \in D$, Without loss of generality, we set $z_0 = 0$. Suppose that *F* is not normal at $z_0 = 0$, then by Lemma 2.7, we have that there exist a subsequence $f_n \subset F$, points sequence $z_0 \in D$, and a positive numbers ρ_n, $\rho_n \to 0^+$, such that

$$g_n(\xi) = 1/\rho_n \quad f_n(z_n + \rho_n \xi) \to g(\xi), \tag{3.14}$$

spherically uniformly on each compact subset of *C*, where $g(\xi)$ is a non-constant meromorphic function with order ≤ 2, all of whose poles are ofmultiplicities at least 2, all of whose zeros are of multiplicities at least 4.

From (3.14) we have

$$\frac{1}{g_n^2(\xi)}\left(g_n'(\xi)+a\right) + \rho_n^2 d \to \frac{g'+a}{g^2(\xi)} \tag{3.15}$$

If $g'(\xi)+a \equiv 0$, then $g(\xi) = -a\xi + c_0$, this contradicts to which all zeros of $g(\xi)$ have multiplicity at least 4. If for any point $\xi \in C$, $g'(\xi)+a \ne 0$, then By Lemma 2.2, we have that $g(\xi)$ is not transcendental in *C*, so $g(\xi)$ is non-constant rational function in *C*. By Lemma 2.3 we also have that

$$g(\xi) = \frac{(c\xi + d)^3}{a\xi + b}$$

a contradictions. Therefore, $\left[g'(\xi)+a\right]/g^2(\xi)$ have a zeros. We may claim that $\left[g'(\xi)+a\right]/g^2(\xi)$ has a unique zero $\xi + \xi_0$. Otherwise, suppose that ξ_0, ξ_0^* are two distinguish zeros of

$$\left[g'(\xi)+a\right]/g^2(\xi)$$

then there exists a positive number $\delta > 0$ such that $N(\xi_0,\delta) \cap N(\xi_0^*,\delta) = \phi$ .On the other hand, by Hurwitz's Theorem we can find two point sequences $\xi_n \in N(\xi_0,\delta)$, $\xi_n^* \in N(\xi_0^*,\delta)$ Such that $\xi_n \to \xi_0, \xi_n^* \to \xi_0^*$, and

$$g_n^{-2}(\xi_n)\left[g_n'(\xi_n)+a\right] + \rho_n^2 d = 0$$

$$g_m^{-2}(\xi_m^*)\left[g_m'(\xi_m^*)+a\right] + \rho_m^2 d = 0$$

then, we have

$$f_n'(z_n + \rho_n \xi_n) - a f_n^2(z_n + \rho_n \xi_n) - d = 0,$$

$$f_m'(z_m + \rho_m \xi_m^*) - a f_m^2(z_m + \rho_m \xi_m^*) - d = 0.$$

From the hypothesis that for every pair functions f, g in *F*, $f'(z) - af^2$ and $g'(z) - ag^2$ share complex number d in *D*, we have

$$f_m'(z_n + \rho_n \xi_n) - a f_n^2(z_n + \rho_n \xi_n) - d = 0,$$

$$f_m'(z_n + \rho_n \xi_n^*) - a f_m^2(z_n + \rho_n \xi_n^*) - d = 0.$$

Fix *m*, let $n \to \infty$, then $f_m'(0) - a f_m^2(0) - d = 0$.

Since $f_m'(z) - a f_m^2(z) - d$ has no accumulation points, so for sufficiently large n we have

$$z_n + \rho_n \xi_n = 0,\ z_m + \rho_m \xi_m^* = 0$$

then

$$\xi_n = -\frac{z_n}{\rho_n}, \xi_n^* = -\frac{z_n}{\rho_n}$$

This contradicts to $N(\xi_0,\delta) \cap N(\xi_0^*,\delta) = \phi$. Thus, $\left[g'(\xi)+a\right]/g^2(\xi)$ has a unique zero $\xi = \xi_0$. Furthermore, we have that either $\xi = \xi_0$ is a multiple poles of $g(\xi)$ or $\xi = \xi_0$ is a unique zero of $g'(\xi)+a$. If $\xi = \xi_0$ is a multiple poles of $g(\xi)$, then $g'(\xi)+a \ne 0$, for any $\xi \in C$. By Lemma 2.2 and Lemma 2.3, we immediately deduce that $g(\xi)$ must be a constant in *C*, which contradicts to $g(\xi)$ is a non-constant meromorphic functions in C. Therefore, $g(\xi)$ has only a simple poles and $g'(\xi)+a$ has a unique $\xi=\xi_0$. But since $g(\xi)$ has only a multiple poles, so we have that $g(\xi)$ is entire in C and $g'(\xi)+a$ has a unique $\xi=\xi_0$. Also by Lemma 2.2, we have that $g(\xi)$ is a non-constant polynomials, all of whose zeros are of multiplicity at least 4. Setting

$$g(\xi)=A(\xi-\xi_1)^{m_1}(\xi-\xi_2)^{m_2}\cdots(\xi-\xi_s)^{m_s},$$

we have

$$g'(\xi)=A(\xi-\xi_1)^{m_1-1}(\xi-\xi_2)^{m_2-1}\cdots(\xi-\xi_2)^{m_s-1}h(\xi)$$

Where $h(\xi)=m\xi^{s-1}+a_0\xi^{s-2}+\cdots+a_{s-2}$, $A\neq 0$, $a_0,a_1,\cdots,a_{s-2}$ are some complex constants, $m_j\,(j=1,2,\cdots,s)$ are s positive integers, $m_j\geq 4$, and $m=\sum_{j=1}^{j=s}m_j$. Thus, we have

$$g'(\xi)+a=B(\xi-\xi_0)^l,$$

where $l\geq 3$. So we have that $g''(\xi)+a=Bl(\xi-\xi_0)^{l-1}$.

If $g(\xi_0)=0$, then $g'(\xi_0)=g''(\xi_0)=g'''(\xi_0)=0$.

But $g'(\xi_0)=-a\neq 0$, a contradictions.

Therefore, F is normal at $z=0$.

3.3. Proof of Theorem 1.7.

For any $z\in D$, if $a(z)\neq 0$, we may give the complete proof of Theorem 1.7 by the same argument as Theorem 1.6, we emit the detail. In the sequel, we shall prove that F is normal at which $a(z)=0$. Set $a(z)=z^r b(z)$, where $b(z)$ is analytic at $z=0$, $b(0)=1$, r is a positive integer, $r\geq 2$.

$$F_1=\left\{F:F(z)=\frac{1}{z^r f(z)},f(z)\in F\right\}$$

For every function $F(z)$ in F_1, from the hypothesis in Theorem 1.7, we can see that all zeros of $F(z)$ are of order at least 4, all poles of $F(z)$ are of multiplicity at least 2.

Suppose that F_1 is not normal at $z=0$, then by Lemma 2.7, there exists a subsequence $F_n\subset F_1$, a point sequence $z_n,|z_n|<r<1$,and a positive number sequence ρ_n, $\rho_n\to 0^+$, such that

$$\begin{aligned}g_n(\xi)&=\rho_n^{-1}F_n(z_n+\rho_n\xi)\\&=\rho_n^{-1}(z_n+\rho_n\xi)^{-r}f_n^{-1}(z_n+\rho_n\xi)\quad(3.16)\\&\to g(\xi)\end{aligned}$$

spherically uniformly on compact subsets of C, where $g(\xi)$ is a non-constant meromorphic function on C, all of whose zeros are of multiplicity at least 4, and all of whose poles are multiple. Moreover, $g(\xi)$ has an order at most 2.

Now we distinguish two cases:

Case 1. $z_n/\rho_n\to\infty$. Without loss of a generalization, we assume that there exists a point z' such that $z_n\to z',|z'|\leq r\leq 1$, we have

$$\begin{aligned}&f_n'(z_n+\rho_n\xi)\\&=-\frac{g_n'(\xi)}{\rho_n^2(z_n+\rho_n\xi)^r g_n^2(\xi)}-\frac{r}{\rho_n(z_n+\rho_n\xi)^{r+1}g_n(\xi)}\\&=-\frac{1}{\rho_n^2(z_n+\rho_n)^r}\left\{\frac{g_n'}{g_n^2}+r\left(\frac{z_n}{\rho_n}+\xi\right)^{-1}\cdot\frac{1}{g_n(\xi)}\right\}\end{aligned}\quad(3.17)$$

For the sake of convenience, we denote S_1 for the set of all zeros of $g(\xi)$, S_2 for the set of all zeros of $g'(\xi)$, and S_3 for the set of all poles of $g(\xi)$.

Since $\lim_{n\to+\infty}\frac{g_n'(\xi)}{g_n^2(\xi)}=\frac{g'(\xi)}{g^2(\xi)}$, $\lim_{n\to+\infty}\frac{1}{g_n(\xi)}=\frac{1}{g(\xi)}$ uniformly on compact subsets of $C\setminus S_1$, and

$\lim_{n\to+\infty}\frac{r}{z_n/(\rho_n+\xi)}=0$ uniformly on compact subsets of C,

thus $\lim_{n\to\infty}f_n(z_n+\rho_n\xi)=\infty$, uniformly on compact subsets of $C\setminus(S_1\cup S_2\cup S_3)$. Thus, it is not difficult to see that

$$\begin{aligned}&\frac{f_n'(z_n+\rho_n\xi)-a(z_n+\rho_n\xi)f_n^2(z_n+\rho_n\xi)}{a(z_n+\rho_n\xi)f_n^2(z_n+\rho_n\xi)-d(z_n+\rho_n\xi)}\\&-\frac{d(z_n+\rho_n\xi)}{a(z_n+\rho_n\xi)f_n^2(z_n+\rho_n\xi)-d(z_n+\rho_n\xi)}\\&\to-\frac{g'(\xi)}{b(z')}-1\end{aligned}\quad(3.18)$$

uniformly on compact subsets of $C\setminus(S_1\cup S_2\cup S_3)$. If $-\frac{g'(\xi)}{b(z')}-1\neq 0$, then $g'(\xi)\neq -b(z')$, for any $\xi\in C\setminus(S_1\cup S_2\cup S_3)$. Thus, $g'(\xi)\neq -b(z')$ for any $\xi\in C$. By Lemma 2.5, we can see that $g(\xi)$ is not transcendental in C, but is a rational function. Also from Lemma 2.3, we deduce that $g(\xi)$ is constant, which contradicts to the fact that $g(\xi)$ is non-constant. On the other hand, it is easy to see that $g'(\xi)$ is not identically equal to $-b(z')$. Hence, $g'(\xi)+b(z')$ has one zeros at least in C. In fact, by the same as the arguments in Theorem 1.5 and Theorem 1.6, we deduce that $g(\xi)$ has a unique zero $\xi=\xi_0$. By Lemma 2.5, we can see that $g(\xi)$ is not transcendental in C, so $g(\xi)$ is non-constant rational function in C. For a non-constant polynomials $g(\xi)$, and noting that $g(\xi)$ has only a zero with multiplicity at least 4, we have

$$g'(\xi)+b(z')=B(\xi-\xi_0)^l, \quad l\ge 3$$

Thus, $g''(\xi)=Bl(\xi-\xi_0)^{l-1}$. Hence, $g(\xi)$ has a zero $\xi=\xi_0$ at most. If $\xi=\xi_0$ is a zero of $g(\xi)$, then $g'(\xi_0)=g''(\xi_0)=g'''(\xi_0)=0$. But $g'(\xi_0)=-b(z')\ne 0$, a contradiction.

In the sequel, we denote $\deg(p)$ for the degree of a polynomial $p(\xi)$. If $g(\xi)$ is non polynomials rational functions, then we set

$$g(\xi)=A\frac{(\xi-\xi_1)^{m_1}(\xi-\xi_2)^{m_2}\cdots(\xi-\xi_s)^{m_s}}{(\xi-\eta_1)^{n_1}(\xi-\eta_2)^{n_2}\cdots(\xi-\eta_t)^{n_t}}, \tag{3.19}$$

Where $m_j\ge 4$, $j=1,2,\cdots,s$; $n_j\ge 2$, $j=1,2,\cdots,t$.

$$m=\sum_{j=1}^{s}m_j\ge 4s,\ q=\sum_{k=1}^{t}n_k\ge 2t \tag{3.20}$$

Then,

$$g'(\xi)=\frac{p_1(\xi)}{q_1(\xi)}=\frac{A(\xi-\xi_1)^{m_1-1}\cdots(\xi-\xi_s)^{m_s-1}h(\xi)}{(\xi-\eta_1)^{n_1+1}\cdots(\xi-\eta_t)^{n_t+1}} \tag{3.21}$$

where

$$h(\xi)=(m-q)\xi^{s+t-1}+a_0\xi^{s+t-2}+\cdots+a_{s+t-2},$$
$$\deg(h)\le s+t-1$$

$$p_1(\xi)=A(\xi-\xi_1)^{m_1-1}(\xi-\xi_2)^{m_2-1}\cdots(\xi-\xi_s)^{m_s-1}h(\xi)$$
$$q_1(\xi)=(\xi-\eta_1)^{n_1+1}(\xi-\eta_2)^{n_2+1}\cdots(\xi-\eta_t)^{n_t+1}$$

Since $g'(\xi)+b(z')$ has a unique zero $\xi=\xi_0$, so we set

$$g'(\xi)+b(z')=\frac{B(\xi-\xi_0)^l}{(\xi-\eta_1)^{n_1+1}\cdots(\xi-\eta_t)^{n_t+1}} \tag{3.22}$$

where *B* is a nonzero constant. Then from (3.22), we have

$$g''(\xi)=\frac{B(\xi-\xi_0)^{l-1}p_2(\xi)}{(\xi-\eta_1)^{n_1+2}\cdots(\xi-\eta_t)^{n_t+2}} \tag{3.23}$$

where $p_2(\xi)=(l-q-t)\xi^t+b_0\xi^{t-1}+\cdots+b_{t-1}$ is a polynomial, $\deg(p_2)\le t$.

From (3.21), it follow that

$$g''(\xi)=\frac{A(\xi-\xi_1)^{m_1-2}\cdots(\xi-\xi_s)^{m_s-2}p_3(\xi)}{(\xi-\eta_1)^{n_1+2}\cdots(\xi-\eta_t)^{n_t+2}} \tag{3.24}$$

where

$$p_3(\xi)=(m-q)(m-q+1)$$
$$\cdot\xi^{2s+2t-2}+c_0\xi^{2s+2t-3}+\cdots+c_{2s+2t-3}$$

is also a polynomial, $\deg(p_3)\le 2s+2t-2$.

We distinguish five cases to derivative a contradiction:

Subcase 1.1. $m=q$. Then from (3.21), we have $l=q+t$. So,

$$\deg(p_2)=t-i_2, 1\le i_2\le t,$$

$$\deg(h)=s+t-1-h_0,\ 1\le h_0\le s+t-1$$

and

$$\deg(p_3)=2s+2t-2-i_3, 1\le t_3\le 2s+2t-2$$

From (3.23) and (3.24), we have $i_2=i_3+1$. So also from (3.23) and (3.24), we also have $l-1\le\deg(p_3)$. Thus, we have $l\le 2s+2t-1-i_3=2s+2t-i_2$.

Since $l=q+t$ and $q\ge 2t$, then we have $t\le 2s-i_2$. On the other hand, from (3.23) and (3.24), we also have $m-2s\le\deg(p_2)$. Since $m\ge 4s$, we have $2s\le t-i_2$. This is impossible.

Subcase 1.2. $m=q-1$. Then $l=q+t$,

$$\deg(p_2)=t-i_2,\ 1\le i_2\le t,\ \deg(h)=s+t-1$$

and

$$\deg(p_3)=2s+2t-2-i_3,\quad 1\le t_3\le 2s+2t-2$$

Similarly to Subcase (1.1), from (3.23) and (3.24), we also have that $i_2=i_3+1$.

Also from (3.23) and (3.24), we have $l-1\le\deg(p_3)$, then, we have $t\le 2s+1-i_2$ On the other hand, similarly to the argument of Subcase (1.1), from (3.23) and (3.24), we also have $m-2s\le\deg(p_2)=t-i_2$, then $2s\le t-1-i_2$. This also is impossible.

Subcase 1.3. $m\le q-2$. Then we still have $l=q+t\ge 3t, \deg(p_2)=t-i_2, 1\le i_2\le t, \deg(h)=s+t-1$, and $\deg(p_3)=2s+2t-2$. Therefor, $l\le 2s+2t-2$, so $t\le 2s-2$. Similarly, we have $m-2s\le 2s+t-i_2$, then $2s\le t-i_2$. This is a contradiction.

Subcase 1.4. $m=q+1$. Then$l\le q+t$, $\deg(h)=s+t-1$, $\deg(p_3)=2s+2t-2$, and $\deg(p_2)=t-i_2, 0\le i_2\le t$. From (3.23) and (3.24), we have $m\le 2s+t-i_2$. Thus, $2s\le t-i_2$ and $t\le 2s-1-i_2$. This is impossible.

Subcase 1.5. $m\ge q+2$. Then $l>q+t$, $\deg(h)=s+t-1, \deg(p_3)=2s+2t-2$, and $\deg(p_2)=t$. From (3.23) and (3.24), we have $l-1\le\deg(p_3)=2s+2t-1$ and $m-2s\le\deg(p_2)=t$. So, we have that $t\le 2s-1$ and $2s\le t$. This is a contradiction.

Case 2. Suppose that there exists a complex number $\alpha\in C$ and a subsequence of sequence $\{z_n\rho_n^{-1}\}$, still noting it $z_n\rho_n^{-1}$, such that $z_n\rho_n^{-1}\to\alpha$. We have a converges

$$H_n(\xi)=\rho_n^{-1}F_n(\rho_n\xi)=\rho_n^{-1}F_n(z_n+\rho_n(\xi-z_n/\rho_n))\to g(\xi-\alpha)=\hat{g}(\xi) \tag{3.25}$$

spherically uniform on compact subsets of *C*. Clearly, all zeros of $\hat{g}(\xi)$ are of multiplicity at least 4, all poles of $\hat{g}(\xi)$ are of multiplicity at least 2. For each $\xi_0\neq 0$, it is easy to see that there exists a neighborhood $N(\xi_0,\delta)$ of ξ_0, such that $\xi^r H_n(\xi)\Rightarrow\xi^r\hat{g}(\xi)$, the convergence being spherically uniform on $N(\xi_0,\delta)$. For $\xi_0=0$, since ξ_0 is the pole of $g(\xi)$, then there exists $\delta>0$, such that $1/\hat{g}(\xi)$ is analytic on $D_{2\delta}=\{\xi:|\xi|<2\delta\}$, $1/H_n(\xi)$ are analytic on $D_{2\delta}=\{\xi:|\xi|<2\delta\}$ for sufficiently large n. Since

$$1/H_n(\xi)=\rho_n\xi^r f_n(\rho_n\xi)$$

then $\xi_0=0$ is a zero of $1/\hat{g}(\xi)$ has order at least r, we can deduce that $1/(\xi^r H_n(\xi))$ converges uniformly to $1/(\xi^r\hat{g}(\xi))$ on

$$D_{\delta/2}=\{\xi:|\xi|<\delta/2\}$$

Hence, we have

$$G_n(\xi)=\frac{1}{\rho_n^{r+1}f_n(\rho_n\xi)}=\xi^r H_n(\xi)\to\xi^r\hat{g}(\xi) \tag{3.26}$$

spherically uniform on compact subsets of *C*. It follows that $G(0)\neq 0$ from $f(\xi)\neq\infty$ whenever $a(\xi)=0$ for $\xi\in D$, hence all of zeros of $G(\xi)$ have order at least 4, all of poles of $G(\xi)$ have order at least 2. Noting that

$$\begin{aligned}&\left[G_n'(\xi)+b(\rho_n\xi)\xi^r\right]G_n^{-2}(\xi)+\rho_n^{r+2}d(\rho_n\xi)\\&=\rho_n^{r+2}\left\{-f_n(\rho_n\xi)+a(\rho_n\xi)f_n^2(\rho_n\xi)+d(\rho_n\xi)\right\}\\&\to\left[G'(\xi)+\xi^r\right]G^{-2}(\xi)\end{aligned} \tag{3.27}$$

If $\left[G'(\xi)+\xi^r\right]G^{-2}(\xi)\equiv 0$, then $G'(\xi)+\xi^r\equiv 0$, so

$$G'(\xi)=-\xi^r,\ G(\xi)=-\frac{\xi^{r+1}}{r+1}+C_0$$

for any $\xi\in C$. Since $G(0)\neq 0$, then $C_0\neq 0$. Also since $G(\xi)$ has the zeros of multiplicity at least 4, then $G(\xi)\neq 0$, this is a contradiction. Therefore,

$$\left[G'(\xi)+\xi^r\right]G^{-2}(\xi)$$

is not identically equal to zero.

If $\left[G'(\xi)+\xi^r\right]/G^{-2}(\xi)$ for any $\xi\in C$, then $G(\xi)$ has no multiple poles and $G'(\xi)+\xi^r\neq 0$. Note that $G(\xi)$ has only multiple poles, so $G(\xi)$ is entire on *C*. Also by Lemma 2.5, we have that $G(\xi)$ is not transcendental in C, and then $G(\xi)$ is a polynomial. Thus, $G'(\xi)=-\xi^r+C_0$, where $C_0\neq 0$. We have $G''(\xi)=-r\xi^{r-1}$, then from $G(0)\neq 0$ and a multiplicities of every zeros of $G(\xi)$ it follows that $G(\xi)\neq 0$ for any $\xi\in C$, this is impossible. Hence, $\left[G'(\xi)+\xi^r\right]/G^{-2}(\xi)$ has some zeros. In fact, by the same argument as the Case 1, we may deduce that $\left[G'(\xi)+\xi^r\right]/G^{2}(\xi)$ has a unique zero $\xi=\xi_0$. Thus, we have that either $\xi=\xi_0$ is multiple poles of $G(\xi)$ or $\xi=\xi_0$ is a unique zero of $G(\xi)+\xi^r$.

Similarly, if $\xi=\xi_0$ is multiple poles of $G(\xi)$, from that $\left[G'(\xi)+\xi^r\right]/G^{2}(\xi)$ has a unique zero $\xi=\xi_0$ it follows that $G'(\xi)\neq-\xi^r$ for any $\xi\in C$. By Lemma 2.5, we have that $G(\xi)$ is not transcendental. Again by Lemma 2.6, we have that $G(\xi)$ is a constant, which is a contradiction. Hence, $G(\xi)$ has no multiple pole and $G'(\xi)+\xi^r$ has a unique zero $\xi=\xi_0$. Thus, $G(\xi)$ is entire on *C* and $G'(\xi)+\xi^r$ has a unique zero $\xi=\xi_0$. By Lemma 2.5, we have that $G(\xi)$ must be a polynomial. Setting

$$g(\xi)=A(\xi-\xi_1)^{m_1}(\xi-\xi_2)^{m_2}\cdots(\xi-\xi_s)^{m_s}, \tag{3.28}$$

where, $m_1,m_2,\cdots,m_s$ are *s* positive integers, $m_j\geq 4$ $j=1,2,\cdots,s$, $m=\sum_{j=1}^{s}m_j$

$$G'(\xi)+\xi^r=B(\xi-\xi_0)^l, \tag{3.29}$$

where l is a positive integer, $l\geq 3$, we have

$$G''(\xi)+r\xi^{r-1}=Bl(\xi-\xi_0)^{l-1}, \tag{3.30}$$

$$G^{(3)}(\xi)+r(r-1)\xi^{r-1}=Bl(l-1)(\xi-\xi_0)^{l-2}. \tag{3.31}$$

For $G(0)\neq 0$, we have $\xi_0\neq 0$ and $\xi_j\neq 0$. From (3.29) it follows that $\xi_j\neq\xi_0$, $j=1,2,\cdots,s$.

From (3.29), (3.30)and (3.31), for $j=1,2,\cdots,s$, we have

$$\xi_j^r=B(\xi_j-\xi_0)^l \tag{3.32}$$

$$r\xi_j^{r-1}=Bl(\xi_j-\xi_0)^{l-1} \tag{3.33}$$

$$r(r-1)\xi_j^{r-2}=Bl(l-1)(\xi_j-\xi_0)^{l-2} \tag{3.34}$$

From (3.32) and (3.33), we have

$$(r-l)\xi_j=r\xi_0, j=1,2,\cdots,s \tag{3.35}$$

If $l=r$, then $\xi_0=0$, this is impossible. Therefore, we have $l\neq r$, and so

$$\xi_1 = \xi_2 = \cdots = \xi_s = \frac{r}{r-l}\xi_0$$

From (3.33) and (3.34), we also have ,

$$\xi_1 = \xi_2 = \cdots = \xi_s = \frac{r-1}{r-l}\xi_0$$

then $r\xi_0 = (r-1)\xi_0$. Thus, we have $\xi_0 = 0$, a contradiction.

Finally, we prove that F is normal at the origin. For any function sequence $\{f_n(z)\}$ in F, since F_1 is normal at $z=0$, then there exist a positive number $\delta < 1/2$ and subsequence $\{F_{n_k}\}$ of $\{F_n\}$ such that F_{n_k} converges uniformly to a meromorphic function $h(z)$ or ∞ on $N(0,2\delta)$. Noting $F_n(0)=\infty$, we deduce that there exists a positive number $M>0$ such that $\left|F_{n_k}(z)\right| \geq M$ for any $z \in N(0,\delta)$. Again noting that $f_{n_k}(0) \neq \infty$ we have that $f_{n_k}(z) \neq \infty$ for all $z \in N(0,\delta)$, that is, $f_{n_k}(z)$ is analytic in $N(0,\delta)$. Therefore, for all n_k, we have

$$\left|f_{n_k}(z)\right| = \left|\frac{1}{z^r F_{n_k}(z)}\right| \leq \frac{1}{M}\frac{2^r}{\delta^r}, |z| < \frac{\delta}{2}$$

By Montel's Theorem, $\{f_{n_k}(z)\}$ is normal at $z=0$, and thus F is normal at $z=0$. The complete proof of Theorem 1.7 is given.

4. Acknowledgements

The authors are grateful to the referee for a number of helpful suggestions to improve the paper.

5. References

[1] W. Bergweiler and A. Eremenko, "On the Singularities of the Inverse to a Meromorphic Function of Finite Order," *Revista Matemática Iberoamericana*, Vol. 11, No. 2, 1995, pp. 355-373.

[2] H. H. Chen and M. L. Fang, "On the Value Distribution of $f^n f'$," *Science in China Series A*, Vol. 38, No. 7, 1995, pp. 789-798.

[3] M. L. Fang and W. J. Yuan, "On the Normality for Families of Meromorphic Functions," *Indian Journal of Mathematics*, Vol. 43, 2001, pp. 341-350.

[4] W. K. Hayman, "Picard Values of Meromorphic Functions and the Its Derivatives," *Annals of Mathematics*, Vol. 70, 1959, pp. 9-42. doi:10.2307/1969890

[5] W. K. Hayman, "Meromorphic Functions," Clarendon, Oxford, 1964.

[6] X. J. Huang and Y. X. Gu, "Normal Families of Meromorphic Functions with Multiple Zeros and Poles," *Journal of Mathematical Analysis and Applications*, Vol. 295, No. 2, 2004, pp. 611-619. doi:10.1016/j.jmaa.2004.03.041

[7] X. J. Huang and Y. X. Gu, "Normal Families of Meromorphic Functions," *Results in Mathematics*, Vol. 49, 2006, pp. 279-288. doi:10.1007/s00025-006-0224-2

[8] S. Y. Li, "On Normal Criterion of Meromorphic Functions," *Journal of Fujian Normal University*, Vol. 25 1984, pp. 156-158.

[9] X. J. Li, "Proof of Hayman's Conjecture on Normal Families," *Science in China Series A*, Vol. 28, 1985, pp. 596-603.

[10] X. C. Pang, "On Normal Criterion of Meromorphic Functions," *Science in China Series A*, Vol. 33, No. 5, 1990, pp. 521-527.

[11] X. C. Pang, D. G. Yang and L. Zalcman, "Normal Families of Meromorphic Functions Omitting a Function ii," *Computational Methods and Function Theory*, Vol. 2, No. 1, 2002, pp. 257-265.

[12] Y. F. Wang and M. L. Fang, "Picard Values and Normal Families of Meromorphic Functions with Multiple Zeros," *Acta Mathematica Sinica, Chinese Series*, Vol. 41, No. 4, 1998, pp. 743-748.

[13] L. Zalcman, "Normal Families: New Perspectives," *Bulletin (New Series) of the American Mathematical Society*, Vol. 35, No. 3, 1998, pp. 215-230.

[14] Q. C. Zhang, "Normal Families of Meromorphic Functions Concerning Sharing Values," *Journal of Mathematical Analysis and Applications*, Vol. 338, No. 1, 2008, pp. 545-551. doi:10.1016/j.jmaa.2007.05.032