



Solitons and Heat Transfer in Nonlinear Lattices with Cubic On-Site and Quartic Interaction Potentials

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Received 2 July 2014; revised 8 August 2014; accepted 18 September 2014

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Abstract

This paper deals with the transfer of soliton-like heat waves in nonlinear lattices with cubic on-site potential and quartic interparticle interaction potential. A model Hamiltonian was proposed using the second quantized operators and the same was averaged using a suitable wavefunction. The equations were derived numerically in the discrete form for the field amplitude. Moreover the resulting equations were analyzed analytically using the continuous approximation technique and the properties of heat transfer were examined theoretically.

Keywords

Soliton, Nonlinear Lattice, Inhomogeneity

Subject Areas: Dynamical System, Theoretical Physics

1. Introduction

The control of heat flow in electronic devices has drawn a great deal of attention in recent years since the rapid increase in temperature could deteriorate the device in process of time [1]-[8]. Attempts have been made by scientists to study the transfer of heat by considering simple linear lattice models fixed with heat baths [9]-[12]. Later on these linear models have been extended to the nonlinear models of one, two and three dimensions [13]-[20]. Various models like thermal rectifiers, diodes and transistors have been designed using the nonlinearity property which has been found useful in controlling heat flow [21]-[25]. It is worth noting that the propagation of heat in the form of solitons has been confirmed by Toda and the concept of soliton has been established as a mode of heat propagation in nonlinear lattices [26]. Moreover all the broad ranging reviews of mathematical techniques and physical applications which have been performed previously are based on classical theories. The classical

molecular dynamic simulations which can incorporate anharmonic interactions to all orders are restricted to high temperatures or at weak interactions [27]-[29]. Quantum heat transfer studies have been done by Lian Ao Wu *et al.* by developing a Born-Oppenheimer formalism and have been found that the vibrational or electronic energy dynamics could be described in a new way by the incorporation of quantum effects and nonlinearities [30]. However analytical studies on quantum heat conduction in nonlinear lattices have not been reported yet in the literature. This has prompted us to investigate theoretically the dynamics of heat transfer in nonlinear lattices using the principles of quantum mechanics.

The nonlinear lattices play a vital role in the construction of electronic devices and optical fiber communication systems. Hence we give special consideration to it. The volumetric rise in temperature must be evacuated very efficiently because the heat that is formed by joule effect in these devices will soon reach a level of high temperature and can cause possible failures of the systems. Studies based on temperature and heat effects have been studied by many scientists [31]-[35]. A better understanding of the dynamics of heat conduction may also lead to potentially interesting applications based on the possibility to control the heat flow. Since a sharp knowledge of the phenomena governing the heat transfer must be gained in order to achieve this task, we perform deep investigations on heat conduction in nonlinear lattices. The presence of impurities, defects or imperfections in fact gives rise to inhomogeneities and in turn influences the propagation of heat. Hence both homogeneous and inhomogeneous nonlinear lattices have been taken into account for our study.

Hence our overriding concern is with the role of nonlinearity in the conduction of heat in both homogeneous and inhomogeneous lattices which lead to the emergence of stable localized structures. We adopt a different approach namely the standard approach in quantum field theory in order to study the quantum mechanical effects. The main aim of this approach is to follow the method of coherent structures first, *i.e.* to calculate the Heisenberg equations of motion for the second quantized operators which are required to obey commutation relations. By proposing a model Hamiltonian using the second quantized operators of quantum field theory and averaging it using a suitable wave function, we derive the equations for the field amplitude. The resulting nonlinear equation in its natural form is very difficult to solve because of its high nonlinearity and discreteness. Hence we analyze it numerically and in addition we go for the continuum limit and using the perturbation techniques, we solve the resulting dynamics and analyze the nature of heat conduction in lattices.

Section 2 deals with the numerical analysis of heat conduction in the homogeneous and inhomogeneous nonlinear lattices and Section 3 describes the heat transfer in the continuum level and the results are concluded in Section 4.

2. Nonlinear Lattice

2.1. Homogeneous

To describe the nonlinear excitations which are ubiquitous in nature and play an important role in the description of a vast variety of natural phenomena, we consider a homogeneous nonlinear lattice consisting of N atoms of equal mass m and lattice constant L . The nonlinear lattice Hamiltonian is described by [36]-[38]

$$H = \sum_{n=1}^N \frac{p_n^2}{2m} + U(x_n) + \sum_{n=0}^N V(x_{n+1}, x_n) \quad (1)$$

where x_n and p_n are the displacements and momenta of the particles about their equilibrium positions. $U(x_n)$ is the on-site potential term. The interaction between nearest neighbors with quartic interparticle interaction potential strength is described by $V(x_{n+1}, x_n)$ which assumes the following form

$$U(x_n) = \frac{k'}{3}(x_n)^3 \quad (2)$$

$$V(x_{n+1}, x_n) = \left[\frac{k}{2}(x_{n+1} - x_n - L)^2 + \frac{\beta}{4}(x_{n+1} - x_n - L)^4 \right] \quad (3)$$

where k' , k and β represents the cubic on-site, squared interparticle interaction and quartic interparticle interaction potential strengths respectively. To bring out the nonlinearity, we now make a transition to a second quantized formalism and hence we bosonize the Hamiltonian (1) using the relations

$$p_n = i\sqrt{\frac{m\hbar\omega}{2}}(B_n^\dagger - B_n) \tag{4}$$

$$x_n = \rho(B_n + B_n^\dagger) \tag{5}$$

where ω is the angular frequency and $\rho = \sqrt{\frac{\hbar}{2m\omega}}$. The bosonic operators B_n and B_n^\dagger satisfy the usual commutation relations $[B_m, B_n^\dagger] = \delta_{m,n}$, $[B_m, B_n] = [B_m^\dagger, B_n^\dagger] = 0$. Using Equation (4) and Equation (5), the Hamiltonian (1) can be written as

$$H = \sum_{n=1}^N \frac{-\hbar\omega}{4}(B_n^\dagger - B_n)^2 + \frac{k'\rho^3}{3}(B_n + B_n^\dagger)^3 + \sum_{n=0}^N \frac{k}{2}[\delta_n(\rho^2\delta_n - L\tau) + L^2] + \sum_{n=0}^N \frac{\beta}{4}[(\rho\delta_n)^4 - 4L(\rho\delta_n)^3 + 6(\rho L\delta_n)^2 - 4L^3\rho\delta_n + L^4]. \tag{6}$$

where $\delta_n = B_{n+1} + B_{n+1}^* - (B_n + B_n^*)$ and $\tau = \sqrt{\frac{2\hbar}{m\omega}}$. In particular we are concerned with nonlinear excitations of atoms due to nonlinearity in the system in which a cluster of atoms may undergo a large excursion as compared to the rest of the atoms. Physically the quantum state of such large amplitude collective modes may be represented by coherent states. Hence we introduce the coherent state representation [39] of a harmonic oscillator given by

$$|\alpha\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right) \sum_{j=0}^{\infty} \frac{\alpha^j}{\sqrt{j!}} |j\rangle \tag{7}$$

We now construct the equation of motion for the boson operator using

$$i\hbar \frac{d}{dt} \langle B_n \rangle = \langle [B_n, H] \rangle \tag{8}$$

and

$$i\hbar \frac{d}{dt} \langle B_n^\dagger \rangle = \langle [B_n^\dagger, H] \rangle \tag{9}$$

For the bosonic operators, $B_n^\dagger |\alpha\rangle = a_n^* |\alpha\rangle$, $B_n |\alpha\rangle = a_n |\alpha\rangle$ with $\langle \alpha | \alpha \rangle = 1$, where a_n is the coherent amplitude of the operator B_n for the system in the state $|\alpha\rangle$. Now we write down the equation of motion using Equations (6)-(9) as

$$i\hbar \frac{da_n}{dt} = \frac{-\hbar\omega}{2}(a_n^* - a_n) + k'\rho^3 q_n^2 + k\rho^2(2q_n - r_n - s_n) + \frac{kL^2}{2} + \beta\rho^4(2q_n^3 - r_n^3 - s_n^3 - 3q_n^2 r_n + 3r_n^2 q_n + 3s_n^2 q_n - 3q_n^2 s_n) - 3L\beta\rho^3(r_n^2 - s_n^2 - 2q_n r_n + 2q_n s_n) + 3L^2\beta\rho^2(2q_n - r_n - s_n) + \frac{L^4\beta}{4}, \tag{10}$$

Equation (10) represents the nature of heat conduction in the homogeneous nonlinear lattice.

2.2. Inhomogeneous

The effect of inhomogeneities in the nonlinear lattice during the propagation of heat can be analyzed by the inclusion of site-dependent inhomogeneities f_n and g_n in the Hamiltonian (1) and the inter particle interaction potential term $V(x_{n+1}, x_n)$ is given by

$$V(x_{n+1}, x_n) = \left[\frac{k}{2} f_n (x_{n+1} - x_n - L)^2 + \frac{\beta}{4} g_n (x_{n+1} - x_n - L)^4 \right] \tag{11}$$

Using the momentum and positon operators in the corresponding Hamiltonian we get

$$H = \sum_{n=1}^N \frac{-\hbar\omega}{4} (B_n^\dagger - B_n)^2 + \frac{k'}{3} \rho^3 (B_n + B_n^\dagger)^3 + \sum_{n=0}^N \frac{k}{2} [\rho f_n \delta_n - L\tau f_n \delta_n + L] + \sum_{n=0}^N \frac{\beta}{4} [g_n (\rho \delta_n)^4 - 4L\rho^3 g_n \delta_n^3 + 6L^2 \rho^2 g_n \delta_n^2 - 4L^3 \rho g_n \delta_n + L^4]. \tag{12}$$

The following equation is derived by the process of commutation method which is solved numerically

$$i\hbar \frac{da_n}{dt} = \frac{-\hbar\omega}{2} (a_n^* - a_n) + k' \rho^3 q_n^2 + \frac{k}{2} \left[\frac{\hbar}{m\omega} (f_{n-1} z_n + f_n y_n) + L\tau (f_n - f_{n-1}) + L^2 \right] + \frac{\beta}{4} \left[\frac{\hbar^2}{m^2 \omega^2} (g_{n-1} z_n^3 + g_n y_n^3) - 12L\rho^3 (g_{n-1} z_n^2 - g_n y_n^2) - 4L^3 \rho (g_{n-1} - g_n) + 12L^2 \rho^2 (g_{n-1} z_n + g_n y_n) + L^4 \right], \tag{13}$$

where $y_n = q_n - r_n$ and $z_n = q_n - s_n$. Equation (13) is a the discrete equation which describe the nature of heat conduction in the inhomogeneous nonlinear lattice. The periodic inhomogenities are prescribed values as $f_n = g_n = \tanh(n * h)$. An illustration of the picture displayed in **Figure 1** shows that the exponential decay gradually becomes less evident in process of time in the homogeneous nonlinear lattice. Also we observe that the thermal stability is considerably enhanced in the presence of inhomogenities. It is found that the amplitude decreases exponentially in the absence of inhomogenities and the effect of inhomogenities on the propagating solitons is to maintain its amplitude.

3. Continuum Level

3.1. Homogeneous Nonlinear Lattice

The direct and effective algebraic method for finding exact solutions of nonlinear equations is the tanh method [40]. The function a_n changes smoothly over one end of the lattice and hence we can replace $a_n(t)$ by $a(x,t)$ and proceed to the continuum limit. Using the Taylor series expansion

$$a_{n\pm 1} = 1 \pm ha_x + \frac{h^2 a_{xx}}{2!} \pm \frac{h^3 a_{xxx}}{3!} + \frac{h^4 a_{xxxx}}{4!} \pm \dots \tag{14}$$

where h is the lattice parameter. Using equation (14), equation (13) takes the form

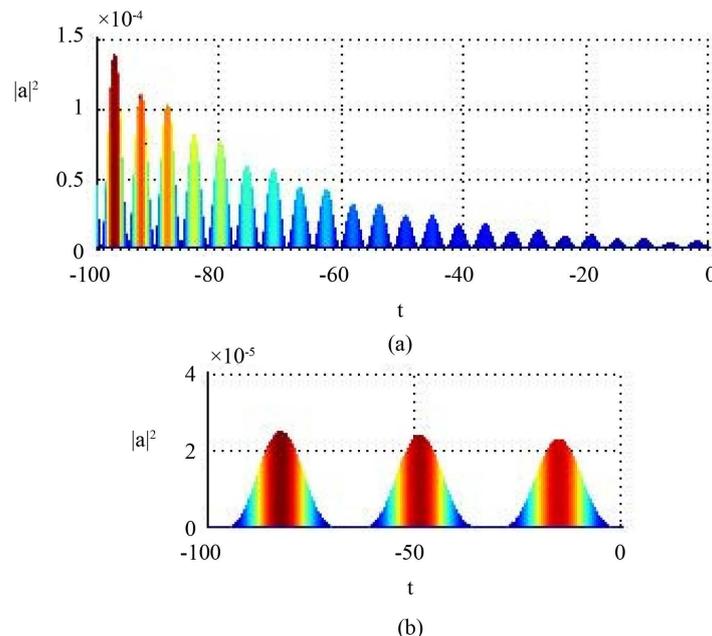


Figure 1. Snapshot of solitons in the (a) homogeneous nonlinear lattice (b) inhomogeneous nonlinear lattice.

$$i\hbar \frac{da}{dt} = -\frac{\hbar\omega}{2}(a^* - a) + k'\rho^3\omega^2 - \frac{4k\pi^2\hbar^3}{m\omega} \left(\frac{1}{2!}w_{xx} + \frac{\hbar^2}{4!}w_{xxxx} \right) + \frac{kL^2}{2} - \frac{12\hbar^6\beta\pi^4}{m^2\omega^2}w_{xx}(w_x^2 + 2a_x^*a_x) + 6Lh^3\beta\rho^3(w_x w_{xx}) - 6L^2h^2\beta\rho \left(\frac{1}{2}w_{xx} + \frac{1}{4!}w_{xxxx} \right) + \frac{L^4\beta}{4}, \tag{15}$$

where, $w = a + a^*$. Using $a = U + iV$, the Equation (15) is reduced to the following crucial step by substituting the imaginary part into the real one

$$AU_{tt} + CU^2 - EU_{xx} - FU_{xxxx} + G - HU_{xx} - IU_{xxxx} - JU_x^2U_{xx} + KU_xU_{xx} + P = 0 \tag{16}$$

where

$$A = \frac{\hbar}{\omega}, \quad C = \rho 4k', \quad E = \frac{4k\pi^2\hbar^3}{m\omega}, \quad F = \frac{k\hbar^2\hbar^3\pi^2}{3m\omega}, \quad G = \frac{kL^2}{2}, \tag{17}$$

$$H = 6L^2h^2\beta\rho, \quad I = \frac{L^2h^4\beta}{2}, \quad J = \frac{96\hbar^6\beta\pi^4}{m^2\omega^2}, \quad K = 24L^2h^3\beta\rho^3, \quad P = \frac{L^4\beta}{4}.$$

Equation (16) is a nonintegrable equation which can only be solved using perturbation techniques and for the present study we employ the tanh method which leads to a further systemization of the method and seek solutions in the travelling frame of reference [40]

$$\xi = \sum_{j=1}^N (c_j x_j) \tag{18}$$

where the components c_j of the wave vector x are constants. We seek polynomial solutions expressible in hyperbolic tangent $T = \tanh(\xi)$. Based on the identity $\cosh^2(\xi) - \sinh^2(\xi) = 1$,

$$\tanh'(\xi) = \operatorname{sech}^2(\xi) = 1 - \tanh^2(\xi) \tag{19}$$

$$\tanh''(\xi) = -2 \tanh(\xi) + 2 \tanh^3(\xi), \text{ etc.} \tag{20}$$

Therefore the first and consequently all higher order derivatives are polynomial in T . Thus repeatedly applying the chain rule

$$\frac{\partial \bullet}{\partial x_j} = \frac{\partial \xi}{\partial x_j} \frac{dT}{d\xi} \frac{d\bullet}{dT} = c_j (1 - T^2) \frac{d\bullet}{dT} \tag{21}$$

Equation (16) can be transformed into a coupled system of nonlinear ODEs

$$\Gamma(T, U(T), U'(T), \dots) = 0 \tag{22}$$

where $U_n(T)$ corresponds to $a_n(x)$. Using this technique, Equation (16) is reduced into a nonlinear ODE as follows

$$Ac_2^2[U'\phi_2 + U''\phi_1] + CU^2 - Ec_1^2[U'\phi_2 + U''\phi_1] - Fc_4^4[U'\phi_3 + U''\phi_4 + U'''\phi_5 + U''''\phi_6] - Hc_1^2[U'\phi_2 + U''\phi_1] - Ic_1^4[U'\phi_3 + U''\phi_4 + U'''\phi_5 + U''''\phi_6] - Jc_1^4[U'U'^2\phi_7 + U''U'^2\phi_6] + Kc_1^3[U'U''\phi_8 + U'^2(\phi_9)] + P + G = 0. \tag{23}$$

where

$$\begin{aligned} \phi_1 &= 1 - 2T^2 + T^4, & \phi_2 &= -2T + 2T^3, & \phi_3 &= 16T - 40T^3 + 24T^5, & \phi_4 &= -8 + 52T^2 - 80T^4 + 36T^6, \\ \phi_5 &= -12T + 36T^3 - 36T^5 + 12T^7, & \phi_6 &= 1 - 4T^2 + 6T^4 - 4T^6 + T^8, & \phi_7 &= -2T + 6T^3 - 6T^5 + 2T^7, \\ \phi_8 &= 1 - 3T^2 + 3T^4 - T^6, & \phi_9 &= -2T + 4T^3 - 2T^5. \end{aligned} \tag{24}$$

The tanh method admits the use of the finite expansion

$$U(T) = \sum_{j=0}^M (a_j T^j) \tag{25}$$

By balancing the linear and nonlinear terms in T , the value of M is found to be equal to 1. Equating the coefficients of the powers of T , we get

$$G + P + Ca_0^2 = 0 \tag{26}$$

$$-2c_2^2 a_1 A + 2Ec_1^2 a_1 - 16Fc_1^4 a_1 + 2Hc_1^2 a_1 - 16Ic_1^4 a_1 + 2Jc_1^4 a_1^3 - 2Kc_1^3 a_1^2 + 2Ca_0 a_1 = 0 \tag{27}$$

$$Ca_1^2 = 0 \tag{28}$$

$$2Ac_2^2 a_1 - 2Ec_1^2 a_1 + 40Fc_1^4 a_1 - 2Hc_1^2 a_1 + 40Ic_1^4 a_1 - 6Jc_1^4 a_1^3 + 4Kc_1^3 a_1^2 = 0 \tag{29}$$

$$-24Fc_1^4 a_1 - 24Ic_1^4 a_1 + 6Jc_1^4 a_1^3 - 2Kc_1^3 a_1^2 = 0 \tag{30}$$

$$-2Jc_1^4 a_1^3 = 0 \tag{31}$$

Assuming $J = 0$ and $a_1 \neq 0$ from the equations we get

$$a_1 = \frac{-2Ac_2^2 + 2Ec_1^2 - 40Fc_1^4 + 2Hc_1^2 - 40Ic_1^4}{4Kc_1^3} \tag{32}$$

The remaining constants G,A and K are now easily found through simple algebra by applying the conditions. As a result we get,

$$G = -P \tag{33}$$

$$A = \frac{Ec_1^2 - 8Fc_1^4 + Hc_1^2 - 8Ic_1^4 - Ka_1 c_1^3}{c_2^2} \tag{34}$$

$$K = \frac{-12Fc_1 - 12Ic_1}{a_1} \tag{35}$$

By solving the equations we obtain the explicit solution

$$U = \frac{-2Ac_2^2 + 2Ec_1^2 - 40Fc_1^4 + 2Hc_1^2 - 40Ic_1^4}{4Kc_1^3} \tanh(c_1 x + c_2 t) \tag{36}$$

The propagation of solitons in the homogeneous nonlinear lattice, preserving their phase and shape is given by the solution (36).

3.2. Inhomogeneous Nonlinear Lattice

To note the propagation of solitons in the presence of inhomogenities we use the prescribed values as $f_n = g_n = \sin(n)$ and the application of continuous approximation process to Equation (13) generates,

$$\begin{aligned}
 & i\hbar \frac{da}{dt} \\
 & = \alpha_1 (a^* - a) + k' \rho^3 \omega^2 \\
 & + \alpha_2 \left(-2fe^2 w_{xx} - 2f_x e^2 w_x + f_x e^3 w_{xx} + e^3 f_{xx} w_x - \frac{1}{2} f_{xx} e^4 w_{xx} - \frac{1}{3} f_{xxx} e^4 w_x - \frac{1}{3} f_x e^4 w_{xxx} - \frac{1}{6} fe^4 w_{xxxx} \right) \\
 & + \alpha_3 \left(f_x e - \frac{1}{2} f_{xx} e^2 + \frac{1}{6} f_{xxx} e^3 - \frac{1}{24} f_{xxxx} e^4 \right) + \frac{kL^2}{2} + \alpha_4 \left(-3a_x^2 ge^4 w_{xx} - g_x e^4 w_x^3 - 6a_x ge^4 a_x^* w_{xx} - 3ge^4 a_x^{*2} w_{xx} \right) \\
 & + \alpha_5 \left(-g_x e^3 w_x^2 - 2a_{xx}^* ge^3 w_x - 2a_{xx} ge^3 w_x + a_{xx} g_x e^4 w_x + \frac{1}{2} g_{xx} e^4 w_x^2 + g_x e^4 a_{xx}^* w_x \right) \\
 & + \alpha_6 \left(-g_x e + \frac{1}{2} g_{xx} e^2 - \frac{1}{6} g_{xxx} e^3 + \frac{1}{24} g_{xxxx} e^4 \right) + \alpha_7 \left[\left(4wg - 2wg_x e + wg_{xx} e^2 - \frac{1}{3} g_{xxx} e^3 w + \frac{1}{12} g_{xxxx} e^4 w \right) \right. \\
 & - 2 \left(2gw - g_x ew + ge^2 w_{xx} + \frac{1}{2} wg_{xx} e^2 + g_x e^2 w_x - \frac{1}{2} g_x e^3 w_{xx} - \frac{1}{2} g_{xx} e^3 w_x - \frac{1}{6} wg_{xxx} e^3 + \frac{1}{12} ge^4 w_{xxxx} + \frac{1}{6} g_x e^4 w_{xxx} \right. \\
 & \left. \left. + \frac{1}{4} g_{xx} e^4 w_{xx} + \frac{1}{6} g_{xxx} e^4 w_x + \frac{1}{24} g_{xxxx} e^4 w_x \right) \right] + \frac{L^4 \beta}{4},
 \end{aligned} \tag{37}$$

where

$$\alpha_1 = \frac{-\hbar\omega}{2}, \quad \alpha_2 = \frac{k\rho^2}{2}, \quad \alpha_3 = \frac{kL}{2}\tau, \quad \alpha_4 = \beta\rho^4, \quad (38)$$

$$\alpha_5 = -3L\beta\rho^3, \quad \alpha_6 = -L^3\beta\rho, \quad \alpha_7 = \frac{3L^2\beta}{2}\rho, \quad (39)$$

and e is the lattice parameter. Equation (37) can be consequently reduced to the following form by using $a = U + iV$ and substituting the imaginary part into the real part, we get

$$K_1U_{tt} + K_2U + s_1U_x + s_2U_{xx} - s_3U_{xxx} - s_4U_{xxxx} + K_{11}U_xU_{xx} - K_{12}U_{xxx}U_x - K_9U_x^2U_{xx} + K_{10}U_x^2 - K_8U_x^3 + K_7 + K_{17} = 0. \quad (40)$$

where

$$K_1 = \frac{\hbar}{\omega}, \quad K_2 = \frac{k'\hbar}{m\omega}, \quad K_3 = \frac{ke^4f_x\hbar}{6m\omega}, \quad (41)$$

$$K_4 = -\frac{ke^2f_x\hbar}{m\omega} + \frac{ke^3f_{xx}\hbar}{2m\omega} - \frac{ke^4f_{xxx}\hbar}{6m\omega} \quad (42)$$

$$K_5 = -\frac{ke^2f\hbar}{m\omega} + \frac{ke^3f_x\hbar}{2m\omega} - \frac{ke^4f_{xx}\hbar}{4m\omega}, \quad K_6 = \frac{ke^4f\hbar}{12m\omega} \quad (43)$$

$$K_7 = \frac{kLef_x\hbar}{4m\omega}\tau - \frac{kLe^2f_{xx}\hbar}{8m\omega}\tau + \frac{kLe^3f_{xxx}\hbar}{24m\omega}\tau - \frac{kLe^4f_{xxxx}\hbar}{96m\omega}\tau + \frac{kL^3\hbar}{4m\omega}\tau, \quad K_8 = \frac{2e^4g_x\beta\hbar^2}{m^2\omega^2}, \quad K_9 = \frac{6e^4g\beta\hbar^2}{m^2\omega^2} \quad (44)$$

$$K_{10} = 12Le^3g_x\beta\rho^3 - 6e^4Lg_{xx}\beta\rho^3, \quad K_{11} = 24Le^3g\beta\rho^3 \quad (45)$$

$$K_{12} = 12Le^4g_x\beta\rho^3, \quad K_{13} = L^2e^4g_x\beta\rho \quad (46)$$

$$K_{14} = -6e^2L^2g_x\beta\rho + 3e^3L^2g_{xx}\beta\rho - e^4L^2g_{xxx}\beta\rho \quad (47)$$

$$K_{15} = -6L^2e^2g\beta\rho + 3L^2e^3g_x\beta\rho - \frac{3L^2e^4g_{xx}\beta}{2}\rho, \quad K_{16} = \frac{L^2e^4g\beta}{2}\rho \quad (48)$$

$$K_{17} = L^3eg_x\beta\rho - \frac{L^3e^2g_{xx}\beta}{2}\rho + \frac{L^3e^3g_{xxx}\beta}{6}\rho - \frac{L^3e^4g_{xxxx}\beta}{24}\rho + \frac{L^4\beta}{4} \quad (49)$$

$$s_1 = K_4 + K_{14}, \quad s_2 = K_5 + K_{15}, \quad s_3 = K_3 + K_{13}, \quad s_4 = K_6 + K_{16} \quad (50)$$

The transformation of Equation (40) to a nonlinear ODE is possible by the repeated application of chain rule (21) which is given by

$$\begin{aligned} &K_1c_2^2[U'\phi_2 + U''\phi_1] + K_2 - K_3c_1^3[U'\phi_{10} + U''\phi_{11} + U'''\phi_8] + K_4c_1U'\phi_{12} + K_5c_1^2[U'\phi_2 + U''\phi_1] \\ &- K_6c_1^4[U'\phi_3 + U''\phi_4 + U'''\phi_5 + U''''\phi_6] - K_8c_1^3[U'^3\phi_3] - K_9c_1^4[U'^2U'\phi_7 + U'^2U''\phi_6] + K_{10}c_1^2U'^2\phi_1 \\ &+ K_{11}c_1^3[U'U''\phi_8 + U'^2\phi_9] - K_{12}c_1^4[U''U'(-6T + 18T^3 - 18T^5 + 6T^7) + U'U''\phi_6 + U'^2(-2 + 10T^2 - 14T^4 + 6T^6)] \quad (51) \\ &- K_{13}c_1^3[U'\phi_{10} + U''\phi_{11} + U'''\phi_8] + K_{14}c_1U'\phi_{12} + K_{15}c_1^2[U'\phi_2 + U''\phi_1] - K_{16}c_1^4[U'\phi_3 + U''\phi_4 + U'''\phi_5 + U''''\phi_6] \\ &+ K_7 + K_{17} = 0. \end{aligned}$$

where

$$\phi_{10} = -2 + 8T^2 - 6T^4, \quad \phi_{11} = -6T + 12T^3 - 6T^5, \quad \phi_{12} = 1 - T^2 \quad (52)$$

Balancing the linear and nonlinear terms we get $M = 1$. Equating the coefficients of the highest powers of T we get

$$K_7 + K_{17} - K_8c_1^3a_1^3 + 2K_{12}c_1^4a_1^2 + K_{10}c_1^2a_1^2 + K_4c_1a_1 + K_{14}c_1a_1 + 2K_3c_1^3a_1 + 2K_{13}c_1^3a_1 + K_2a_0^2 = 0 \quad (53)$$

$$-16K_6c_1^4a_1 - 2K_5c_1^2a_1 - 16K_{16}c_1^4a_1 - 2K_{15}c_1^2a_1 + 2K_9c_1^4a_1^3 - 2K_{11}c_1^3a_1^2 - 2K_1c_1^2a_1 + 2K_2a_0a_1 = 0 \tag{54}$$

$$-K_4c_1a_1 - K_{14}c_1a_1 - 8K_3c_1^3a_1 - 8K_{13}c_1^3a_1 + 3K_8c_1^3a_1^3 - 10K_{12}c_1^4a_1^2 - 2K_{10}c_1^2a_1^2 + K_2a_1^2 = 0 \tag{55}$$

$$2K_1c_1^2a_1 + 40K_6c_1^4a_1 + 2K_5c_1^2a_1 + 40K_{16}c_1^4a_1 + 2K_{15}c_1^2a_1 - 6K_9c_1^4a_1^3 + 4K_{11}c_1^3a_1^2 = 0 \tag{56}$$

$$6K_3c_1^3a_1 + 6K_{13}c_1^3a_1 - 3K_8c_1^3a_1^3 + 14K_{12}c_1^4a_1^2 + K_{10}c_1^2a_1^2 = 0 \tag{57}$$

$$-24K_6c_1^4a_1 - 24K_{16}c_1^4a_1 + 6K_9c_1^4a_1^3 - 2K_{11}c_1^3a_1^2 = 0 \tag{58}$$

$$K_8c_1^3a_1^3 - 6K_{12}c_1^4a_1^2 = 0 \tag{59}$$

$$-2K_9c_1^4a_1^3 = 0 \tag{60}$$

Assuming $K_9 = 0$ and $a_1 \neq 0$ from the above equations,

$$a_0 = \frac{16c_1^4K_6K_8 + 2c_1^2K_5K_8 + 16c_1^4K_{16}K_8}{2K_2K_8} + \frac{2c_1^2K_{15}K_8 + 12c_1^4K_{11}K_{12} + 2K_1c_1^2K_8}{2K_2K_8} \tag{61}$$

$$a_1 = \frac{6K_{12}c_1}{K_8} \tag{62}$$

and adhering to the conditions we get

$$K_7 = -K_2a_0^2 - K_{17} + K_8c_1^3a_1^3 - 2K_{12}c_1^4a_1^2 - K_{10}c_1^2a_1^2 - K_4c_1a_1 - K_{14}c_1a_1 - 2K_3c_1^3a_1 - 2K_{13}c_1^3a_1 \tag{63}$$

$$K_4 = \frac{K_2a_1^2 - 2c_1^2a_1^2K_{10} - c_1K_{14}a_1 - 8c_1^3K_3a_1 - 8c_1^3K_{13}a_1 + 3c_1^3K_8a_1^3 - 10c_1^4K_{12}a_1^2}{c_1a_1} \tag{64}$$

$$K_1 = \frac{-40K_6c_1^4 - 2K_5c_1^2 - 40K_{16}c_1^4 - 2K_{15}c_1^2 - 4K_{11}c_1^3a_1}{2c_1^2} \tag{65}$$

$$K_3 = \frac{-6K_{13}c_1 + 3K_8c_1a_1^2 - 14K_{12}c_1^2a_1 - K_{10}a_1}{6c_1} \tag{66}$$

$$K_6 = \frac{-24K_{16}c_1 - 2K_{11}a_1}{24c_1} \tag{67}$$

The following solution is obtained by solving the Equations from (53) to (60),

$$U = \frac{16c_1^4K_8K_6 + 2c_1^2K_5K_8 + 16c_1^4K_{16}K_8}{2K_2K_8} + \frac{2c_1^2K_{15}K_8 + 12c_1^4K_{11}K_{12} + 2c_1^2K_1K_8}{2K_2K_8} + \frac{6K_{12}c_1}{K_8} \tanh(c_1x + c_2t) \tag{68}$$

Equation (68) describes the nature of soliton propagation in the inhomogeneous nonlinear lattice. In this context it is noted that the effect of inhomogeneities in the form of tangent hyperbolic functions on soliton propagation induces a phase shift. Thus it is worth mentioning that the soliton on interaction with the inhomogeneities which arises due to defects or imperfections acquires a phase shift. Hence the only change the wave experiences in the presence of inhomogeneities is a shift in their relative phases and is shown in **Figure 2**.

4. Conclusions

To conclude, we summarize the main results. We have investigated the heat transport properties in various models like homogeneous and inhomogeneous nonlinear lattices with cubic on-site potential and quartic inter-particle interaction potential, by considering a model Hamiltonian using the second quantized operators of quantum field theory. We derived the equations for the field amplitude and after averaging the Hamiltonian using a suitable wave function, we analyzed it numerically and analytically.

From the numerical results of the homogeneous nonlinear lattice model, it is found that the soliton propagation has ultimately resulted in the dissipation of heat in an exponential fashion. On the other hand, the presence of inhomogeneities almost maintains the amplitude of the soliton, propagating in the inhomogeneous

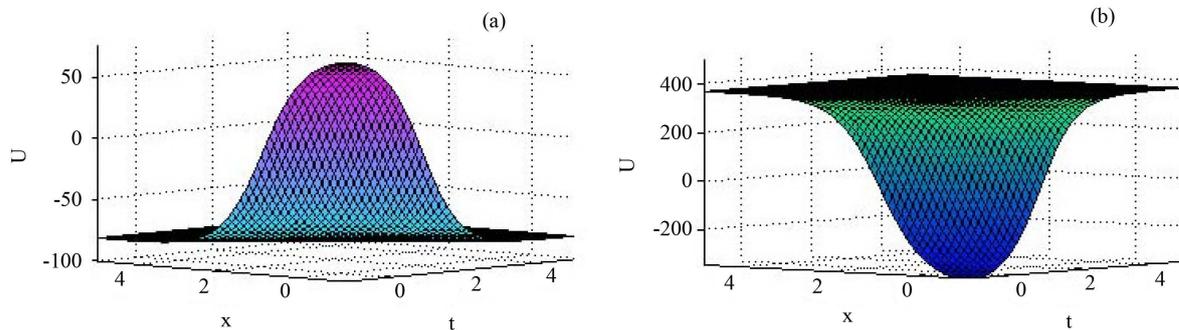


Figure 2. Snapshot of the solitary pulses in the (a) homogeneous (b) inhomogeneous nonlinear lattice in the continuum level.

nonlinear lattice. Hence this lattice is found to support the amplitude of heat without any energy loss and it could be used to design the heat transferring devices to transfer heat without energy loss. The realisation of soliton-like heat dissipation on the nonlinear lattice has clearly demonstrated that the results of the fundamental soliton theory can be successfully exploited in very important practical applications. From the continuum results it is found that the inhomogeneous structure accounts well for the shifting of the phase of the solitons propagating in the inhomogeneous nonlinear lattice. This suggests that the simplicity of this concept could be used in heat transferring devices which transfer heat using phase shift mechanism for instance.

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