Advances in Pure Mathematics, 2011, 1, 193-200
doi:10.4236/apm.2011.14034 Published Online July 2011 (http://www.SciRP.org/journal/apm)

Inclusion and Argument Properties for Certain Subclasses of Analytic Functions Defined by Using on Extended Multiplier Transformations

Oh Sang Kwon
Department of Mathematics, Kyungsung University, Busan, Korea
E-mail: oskwon@ks.ac.kr
Received March 28, 2011; *revised April* 27, 2011; *accepted May* 5, 2011

Abstract

Making use of a multiplier transformation, which is defined by means of the Hadamard product (or convolution), we introduce some new subclasses of analytic functions and investigate their inclusion relationships and argument properties.

Keywords: Subordination, Starlike Functions, Convex Functions, Closed-to-Convex Functions, Multiplier Transformation, Multivalent Functions, Argument Principle

1. Introduction

Let A_p denote the class of functions *f* normalized by

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \quad (p \in \mathbb{N} := \{1,2,3,\cdots\}) \tag{1.1}$$

which are analytic and p-valent in the open unit disk

$$U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

If f and g are analytic in U, we say that f is subordinate to g, and write

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in U)$$

if there exists a Schwarz function $\omega(z)$, analytic in U with $\omega(0) = 0$ and $|\omega(z)| < 1$ in $z \in U$, such that $f(z) = g(\omega(z))$ for $z \in U$.

We denote by $S_p^*(\eta)$ and $C_p(\eta)$ the subclasses of A_p consisting of all analytic functions which are, respectively, p-valent starlike of order η ($0 \le \eta < p$) in U and p-valent convex of order η ($0 \le \eta < p$) in *U*.

Let M be the class of analytic functions φ with $\varphi(0) = 1$, which are convex and univalent in U and satisfy the following inequality:

$$\mathrm{Re}\{\varphi(z)\} > 0 \quad (z \in U)$$

Making use of the aforementioned principle of subordination between analytic functions, we define each of the following subclasses of A_p:

$$S_p^*(\eta;\varphi) := \mathrm{Re}\left\{ f : f \in A_p \text{ and } \frac{1}{p-\eta}\left(\frac{zf'(z)}{f(z)} - \eta\right) \prec \varphi(z) \right\} \tag{1.2}$$

$$(0 \le \eta < p; z \in U; \varphi \in M)$$

$$K_p(\eta;\varphi) := \mathrm{Re}\left\{ f : f \in A_p \text{ and } \frac{1}{p-\eta}\left(1 + \frac{zf''(z)}{f'(z)} - \eta\right) \prec \varphi(z) \right\}$$

$$(0 \le \eta < p; z \in U; \varphi \in M) \tag{1.3}$$

For $m \in \mathbb{N}_0 := \{0,1,2,\cdots\}$, we define the multiplier transformation $J^m(p,\lambda,l)$ of functions $f \in A_p$ by

$$C_p(\eta,\beta;\varphi,\psi) := \mathrm{Re}\left\{ f : f \in A_p \text{ and } \exists g \in S_p^*(\eta;\varphi)\, s.t. \frac{1}{p-\beta}\left(\frac{zf'(z)}{g(z)} - \beta\right) \prec \psi(z) \right\} \tag{1.4}$$

$$(0 \le \eta, \beta < p; z \in U; \phi, \psi \in M)$$

$$J^m(p,\lambda,l)f(z)=z^p+\sum_{k=1}^{\infty}\left(\frac{l+\lambda k}{l}\right)^m a_{k+p}z^{k+p} \quad (l>0;\lambda\ge 0;z\in U) \tag{1.5}$$

Put

$$\phi_{p,\lambda,l}^m(z)=z^p+\sum_{k=1}^{\infty}\left(\frac{l+\lambda k}{l}\right)^m z^{k+p} \quad (m\in\mathbb{N};l>0;\lambda\ge 0;z\in U) \tag{1.6}$$

The operators $\phi_{p,\lambda,l}^m$ and $\phi_{p,1,l}^m$, are the multiplier transformations introduced and studied earlier by Sarangi and Uralegaddi [16] and Uralegaddi and Somanatha ([1] and [2]), respectively. Corresponding to the function $\phi_{p,\lambda,l}^m(z)$ defined by (1.6), we introduce a function $\phi_{p,\lambda,l}^{m,\mu}(z)$ given by the Hadamard product (or convolution):

$$\phi_{p,\lambda,l}^m(z)*\phi_{p,\lambda,l}^{m,\mu}(z)=\frac{z^p}{(1-z)^{\mu+p}} \quad (\mu>-p)$$

Then, analogous to $J^m(p,\lambda,l)$, we have define a new multiplier transformation

$$I_\mu^m(p,\lambda,l):A_p\to A_p$$

as follows:

$$I_\mu^m(p,\lambda,l)f(z)=\phi_{p,\lambda,l}^{m,\mu}(z)*f(z) \tag{1.7}$$

We note that

$$I_{-p}^0(p,1,1)f(z)=f(z) \text{ and } I_2^1(1,1,2)f(z)=zf'(z)$$

It is easily verifed from the above definition of the operator $I_\mu^m(p,\lambda,l)$, that

$$z\left(I_\mu^m(p,\lambda,l)f(z)\right)'=(\mu+p)I_{\mu+1}^m(p,\lambda,l)f(z)-\mu I_\mu^m(p,\lambda,l)f(z) \tag{1.8}$$

and

$$\lambda z\left(I_\mu^m(p,\lambda,l)f(z)\right)'=lI_\mu^m(p,\lambda,l)f(z)-(\lambda p-l)I_\mu^{m+1}(p,\lambda,l)f(z) \tag{1.9}$$

The definition (1.6) of the multiplier transformation $\phi_{p,\lambda,l}^m$ is motivated essentially by the Choi-Saigo-Srivastava operator [3] for analytic functions, which includes a simpler integral operator studied earlier by Noor [7] and others (cf. [4-6]).

Next, by using the operator $I_\mu^m(p,\lambda,l)$ defined by (1.7), we introduce the following subclasses of analytic functions:

$$S_{p,\lambda,l}^{m,\mu}(\eta;\varphi)=\left\{f: f\in A_p \text{ and } I_\mu^m(p,\lambda,l)f(z)\in S_p^*(\eta;\varphi)\right\} \quad (\varphi\in M;\lambda,l,\mu>0;m\in\mathbb{Z};0\le\eta<1) \tag{1.10}$$

$$K_{p,\lambda,l}^{m,\mu}(\eta;\varphi)=\left\{f: f\in A_p \text{ and } I_\mu^m(p,\lambda,l)f(z)\in K_p(\eta;\varphi)\right\} \quad (\varphi\in M;\lambda,l,\mu>0;m\in\mathbb{N};0\le\eta<1) \tag{1.11}$$

and

$$C_{p,\lambda,l}^{m,\mu}(\eta,\beta;\varphi,\psi)=\left\{f: f\in A_p \text{ and } I_\mu^m(p,\lambda,l)f(z)\in C(\eta,\beta;\varphi,\psi)\right\} \quad (\varphi,\psi\in M;\lambda,l,\mu>0;m\in\mathbb{N};0\le\eta,\beta<1) \tag{1.12}$$

We also note that

$$f(z)\in K_{p,\lambda,l}^{m,\mu}(\eta;\varphi)\Leftrightarrow zf'(z)\in S_{p,\lambda,l}^{m,\mu}(\eta;\varphi) \tag{1.13}$$

In particular, we set

$$S_{p,\lambda,l}^{m,\mu}\left(\eta;\frac{1+Az}{1+Bz}\right)=S_{p,\lambda,l}^{m,\mu}(\eta;A,B) \quad (-1<B<A\le 1) \tag{1.14}$$

and

$$K_{p,\lambda,l}^{m,\mu}\left(\eta;\frac{1+Az}{1+Bz}\right)=K_{p,\lambda,l}^{m,\mu}(\eta;A,B) \quad (-1<B<A\le 1) \tag{1.15}$$

In the present paper, we investigate some inclusion relationships and argument properties associated with such multivalent functions in the class A_p as those belonging to the subclasses $S_{p,\lambda,l}^{m,\mu}(\eta;\varphi)$, $K_{p,\lambda,l}^{m,\mu}(\eta;\varphi)$ and $C_{p,\lambda,l}^{m,\mu}(\eta,\beta;\varphi,\psi)$ defined by (1.10), (1.11) and (1.12), respectively.

2. Inclusion Properties

Lemma 2.1: *Let* φ *be convex univalent in* U *with* $\varphi(0)=1$ *and* $\mathrm{Re}\{\beta\varphi(z)+\nu\}>0$ ($\beta,\nu\in\mathbb{C}$). *If* p *is analytic in* U *with* $p(0)=1$, *then*

$$p(z)+\frac{zp'(z)}{\beta\varphi(z)+\nu}\prec\varphi(z) \quad (z\in U)$$

implies that $p(z)<\varphi(z)$ ($z\in U$).

Theorem 2.2: *Let* $\varphi\in M$ *with*

$$\min_{z\in U}\left(\mathrm{Re}\{\varphi(z)\}\right)>\max\left(\frac{\eta+\mu}{\eta-p},\frac{\eta-p+\frac{l}{\lambda}}{\eta-p}\right)$$

then $S^{m,\mu+1}_{p,\lambda,l}(\eta;\varphi)\subset S^{m,\mu}_{p,\lambda,l}(\eta;\varphi)\subset S^{m+1,\mu}_{p,\lambda,l}(\eta;\varphi)$.

Proof. First of all, we show that $S^{m,\mu+1}_{p,\lambda,l}(\eta;\varphi)\subset S^{m,\mu}_{p,\lambda,l}(\eta;\varphi)$. Let $f\in S^{m,\mu+1}_{p,\lambda,l}(\eta;\varphi)$ and set

$$p(z)=\frac{1}{p-\eta}\left(\frac{z\left(I^m_\mu(p,\lambda,l)f(z)\right)'}{I^m_\mu(p,\lambda,l)f(z)}-\eta\right) \tag{2.1}$$

where the function $p(z)$ is analytic in U with $p(0)=1$.

Applying (2.1), we obtain

$$(\mu+p)\frac{I^m_{\mu+1}(p,\lambda,l)f(z)}{I^m_\mu(p,\lambda,l)f(z)}=(p-\eta)p(z)+\eta+\mu \tag{2.2}$$

By logarithmically differentiating both sides of (2.2) and multiplying the reseulting equation by z, we have

$$\begin{aligned}&\frac{1}{p-\eta}\left(\frac{z\left(I^m_\mu(p,\lambda,l)f(z)\right)'}{I^m_\mu(p,\lambda,l)f(z)}-\eta\right)\\&=p(z)+\frac{zp'(z)}{(p-\eta)p(z)+\eta+\mu}\quad(z\in U)\end{aligned} \tag{2.3}$$

Since $\operatorname{Re}\{(p-\eta)\varphi(z)+\eta+\mu\}>0$, by applying Lemma 2.1 to (2.3), it follows that $p(z)\prec\varphi(z)$ in U, that is, that $f(z)\in S^{m,\mu}_{p,\lambda,l}(\eta;\varphi)$.

To prove the second part of Theorem 2.1, let $f(z)\in S^{m,\mu}_{p,\lambda,l}(\eta;\varphi)$ and put

$$q(z)=\frac{1}{p-\eta}\left(\frac{z\left(I^m_{\mu+1}(p,\lambda,l)f(z)\right)'}{I^m_{\mu+1}(p,\lambda,l)f(z)}-\eta\right)$$

where the function $q(z)$ is analytic in U with $q(0)=1$.

In precisely the same manner, we can find the result that $q(z)\prec\varphi(z)$ in U, that is, that $f(z)\in S^{m+1,\mu}_{p,\lambda,l}(\eta;\varphi)$ under the hypothesis

$$\operatorname{Re}\left\{(p-\eta)\varphi(z)+\eta-p+\frac{l}{\lambda}\right\}>0$$

Theorem 2.3: *Let* $\varphi(z)\in M$ *with*

$$\min_{z\in U}\left(Re\{\varphi(z)\}\right)>\max\left(\frac{\eta+\mu}{\eta-p},\frac{\eta-p+\frac{l}{\lambda}}{\eta-p}\right)$$

then $K^{m,\mu+1}_{p,\lambda,l}(\eta;\varphi)\subset K^{m,\mu}_{p,\lambda,l}(\eta;\varphi)\subset K^{m+1,\mu}_{p,\lambda,l}(\eta;\varphi)$.

Proof. Applying (1.11) and Theorem 2.2, we observe that

$$\begin{aligned}&f(z)\in K^{m,\mu+1}_{p,\lambda,l}(\eta;\varphi)\Leftrightarrow zf'(z)\in S^{m,\mu+1}_{p,\lambda,l}(\eta;\varphi)\\&\Rightarrow zf'(z)\in S^{m,\mu}_{p,\lambda,l}(\eta;\varphi)\Leftrightarrow f(z)\in K^{m,\mu}_{p,\lambda,l}(\eta;\varphi)\end{aligned}$$

and

$$\begin{aligned}&f(z)\in K^{m,\mu}_{p,\lambda,l}(\eta;\varphi)\Leftrightarrow zf'(z)\in S^{m,\mu}_{p,\lambda,l}(\eta;\varphi)\\&\Rightarrow zf'(z)\in S^{m+1,\mu}_{p,\lambda,l}(\eta;\varphi)\Leftrightarrow f(z)\in K^{m+1,\mu}_{p,\lambda,l}(\eta;\varphi)\end{aligned}$$

which evidently prove Theorem 2.3.

By setting

$$\varphi(z)=\frac{1+Az}{1+Bz}\quad(-1<B<A\le 1;z\in U)$$

in Theorems 2.2 and 2.3, we deduce the following corollary.

Corollary 2.4: *Suppose that*

$$\frac{1-A}{1-B}>\max\left(\frac{\eta+\mu}{\eta-p},\frac{\eta-p+\frac{l}{\lambda}}{\eta-p}\right)$$

Then, for the function classes defined by (1.12) and (1.13),

$$S^{m,\mu+1}_{p,\lambda,l}(\eta;A,B)\subset S^{m,\mu}_{p,\lambda,l}(\eta;A,B)\subset S^{m+1,\mu}_{p,\lambda,l}(\eta;A,B)$$

and

$$K^{m,\mu+1}_{p,\lambda,l}(\eta;A,B)\subset K^{m,\mu}_{p,\lambda,l}(\eta;A,B)\subset K^{m+1,\mu}_{p,\lambda,l}(\eta;A,B)$$

Theorem 2.5: *Let* $\varphi,\psi\in M$ *with*

$$\min_{z\in U}\left(Re\{\varphi(z)\}\right)>\max\left(\frac{\eta+\mu}{\eta-p},\frac{\eta-p+\frac{l}{\lambda}}{\eta-p}\right)$$

then

$$\begin{aligned}&C^{m,\mu+1}_{p,\lambda,l}(\eta,\beta;\varphi,\psi)\subset C^{m,\mu}_{p,\lambda,l}(\eta,\beta;\varphi,\psi)\\&\qquad\subset C^{m+1,\mu}_{p,\lambda,l}(\eta,\beta;\varphi,\psi)\end{aligned}$$

Proof. We begin by proving that $C^{m,\mu+1}_{p,\lambda,l}(\eta,\beta;\varphi,\psi)\subset C^{m,\mu}_{p,\lambda,l}(\eta,\beta;\varphi,\psi)$, which is the first inclusion relationship asserted by Theorem 2.5.

Let $f(z)\in C^{m,\mu+1}_{p,\lambda,l}(\eta,\beta;\varphi,\psi)$. Then there exists a function $k(z)\in S^*_p(\eta;\varphi)$ such that

$$\frac{1}{p-\beta}\left(\frac{z\left(I^m_{\mu+1}(p,\lambda,l)f(z)\right)'}{k(z)}-\beta\right)\prec\psi(z)\quad(z\in U)$$

Choose the function $g(z)$ such that

$$I^m_{\mu+1}(p,\lambda,l)g(z)=k(z)\in S^*_p(\eta;\varphi)$$

Then $g(z)\in S_{p,\lambda,l}^{m,\mu+1}(\eta;\varphi)\subset S_{p,\lambda,l}^{m,\mu}(\eta;\varphi)$, and

$$\frac{1}{p-\beta}\left(\frac{z\left(I_{\mu+1}^{m}(p,\lambda,l)f(z)\right)'}{I_{\mu+1}^{m}(p,\lambda,l)g(z)}-\beta\right)\prec\psi(z)\quad(z\in U)\tag{2.4}$$

Now let

$$p(z)=\left(\frac{z\left(I_{\mu}^{m}(p,\lambda,l)f(z)\right)'}{I_{\mu}^{m}(p,\lambda,l)g(z)}-\beta\right)\tag{2.5}$$

where the function $p(z)$ is analytic in U with $p(0)=1$.

Using (1.9), we find that

$$\frac{1}{p-\beta}\left(\frac{z\left(I_{\mu+1}^{m}(p,\lambda,l)f(z)\right)'}{I_{\mu+1}^{m}(p,\lambda,l)g(z)}-\beta\right)=\frac{1}{p-\beta}\cdot\left(\frac{z\left(z\left(I_{\mu}^{m}(p,\lambda,l)f(z)\right)'+\mu I_{\mu}^{m}(p,\lambda,l)f(z)\right)'}{z\left(I_{\mu}^{m}(p,\lambda,l)g(z)\right)'+\mu I_{\mu}^{m}(p,\lambda,l)g(z)}-\beta\right)$$

$$=\frac{1}{p-\beta}\cdot\left(\frac{(\mu+1)z\left(I_{\mu}^{m}(p,\lambda,l)f(z)\right)'+z^{2}\left(I_{\mu}^{m}(p,\lambda,l)f(z)\right)''}{z\left(I_{\mu}^{m}(p,\lambda,l)g(z)\right)'+\mu I_{\mu}^{m}(p,\lambda,l)g(z)}-\beta\right)$$

$$=\frac{1}{p-\beta}\cdot\left(\frac{(\mu+1)\dfrac{z\left(I_{\mu}^{m}(p,\lambda,l)f(z)\right)'}{I_{\mu}^{m}(p,\lambda,l)g(z)}+\dfrac{z^{2}\left(I_{\mu}^{m}(p,\lambda,l)f(z)\right)''}{I_{\mu}^{m}(p,\lambda,l)g(z)}}{\dfrac{z\left(I_{\mu}^{m}(p,\lambda,l)g(z)\right)'}{I_{\mu}^{m}(p,\lambda,l)g(z)}+\mu}-\beta\right)$$

Since $g(z)\in S_{p,\lambda,l}^{m,\mu}(\eta;\varphi)$, then we set

$$q(z)=\frac{1}{p-\eta}\left(\frac{z\left(I_{\mu}^{m}(p,\lambda,l)f(z)\right)'}{I_{\mu}^{m}(p,\lambda,l)g(z)}-\eta\right)\tag{2.6}$$

where $q(z)\prec\varphi(z)$ in U with the assumption that $\varphi\in M$. By (2.5),

$$\frac{z\left(I_{\mu}^{m}(p,\lambda,l)f(z)\right)'}{I_{\mu}^{m}(p,\lambda,l)g(z)}=(p-\beta)p(z)+\beta\tag{2.7}$$

Differentiating both side of (2.7) with respect to z and multiplying by z, we obtain

$$\begin{aligned}&\frac{z\left(I_{\mu}^{m}(p,\lambda,l)f(z)\right)''}{\left(I_{\mu}^{m}(p,\lambda,l)f(z)\right)'}\\&=(p-\eta)q(z)+\eta-1+\frac{(p-\beta)zp'(z)}{(p-\beta)p(z)+\beta}\end{aligned}\tag{2.8}$$

Hence

$$\begin{aligned}&\frac{z^{2}\left(I_{\mu}^{m}(p,\lambda,l)f(z)\right)''}{I_{\mu}^{m}(p,\lambda,l)g(z)}\\&=(p-\beta)zp'(z)+\left((p-\eta)q(z)+\eta-1\right)\\&\quad\cdot\left((p-\beta)p(z)+\beta\right)\end{aligned}$$

Computing the above equations, we can obtain

$$\begin{aligned}&\frac{1}{p-\beta}\left(\frac{z\left(I_{\mu+1}^{m}(p,\lambda,l)f(z)\right)'}{I_{\mu+1}^{m}(p,\lambda,l)g(z)}-\beta\right)\\&=\frac{1}{p-\beta}\left(\frac{(\mu+1)\left((p-\beta)p(z)+\beta\right)+(p-\beta)zp'(z)+\left((p-\eta)q(z)+\eta-1\right)\left((p-\beta)p(z)+\beta\right)}{(p-\eta)q(z)+\eta+\mu}\right)\\&=p(z)+\frac{zp'(z)}{(p-\eta)q(z)+\eta+\mu}\end{aligned}$$

Since $\operatorname{Re}\{(p-\eta)\varphi(z)+\eta+\mu\}>0$, applying Lemma 2.1 with $w(z)=\dfrac{1}{(p-\eta)\varphi(z)+\eta+\mu}$, we can show that $p(z)\prec\psi(z)$ in U, so that $f(z)\in C_{p,\lambda,l}^{m,\mu}(\eta,\beta;\phi,\psi)$.

3. Argument Properties

Lemma 3.1: *Let φ be convex univalent in U and ω be analytic in U with $\operatorname{Re}\{\omega(z)\}\geq 0$. If $p(z)$ is analytic in U and $p(0)=\varphi(0)$, then*

$$p(z)+\omega(z)zp'(z)\prec\varphi(z)\quad (z\in U)$$

implies that $p(z)\prec\varphi(z)$ ($z\in U$).

Lemma 3.2: *Let p be analytic in U with $p(0)=1$ and $p(z)\neq 0$ for all $z\in U$. If there exist two points $z_1,z_2\in U$ such that*

$$-\frac{\pi}{2}\alpha_1=\arg\{p(z_1)\}<\arg\{p(z)\}<\arg\{p(z_2)\}=\frac{\pi}{2}\alpha_2 \tag{3.1}$$

for some α_1 and α_2 ($\alpha_1,\alpha_2>0$) and for all z ($|z|<|z_1|=|z_2|$).

$$\frac{z_1p'(z_1)}{p(z_1)}=-i\left(\frac{\alpha_1+\alpha_2}{2}\right)m \text{ and } \frac{z_2p'(z_2)}{p(z_2)}=i\left(\frac{\alpha_1+\alpha_2}{2}\right)m \tag{3.2}$$

where $m\geq\dfrac{1-|b|}{1+|b|}$ and $b=i\tan\dfrac{\pi}{4}\left(\dfrac{\alpha_2-\alpha_1}{\alpha_1+\alpha_2}\right)$.

Theorem 3.3: *Let $f\in A_p$. $0<\delta_1,\delta_2\leq 1$. $0<\gamma<p$. If*

$$-\frac{\pi}{2}\delta_1<\arg\left(\frac{z\left(I_{\mu+1}^m(p,\lambda,l)f(z)\right)'}{I_{\mu+1}^m(p,\lambda,l)g(z)}-\gamma\right)<\frac{\pi}{2}\delta_2$$

for some $g\in S_{p,\lambda,l}^{m,\mu+1}(\eta,p;A,B)$, then

$$-\frac{\pi}{2}\alpha_1<\arg\left(\frac{z\left(I_{\mu}^m(p,\lambda,l)f(z)\right)'}{I_{\mu}^m(p,\lambda,l)g(z)}-\gamma\right)<\frac{\pi}{2}\alpha_2$$

where α_1, α_2 are the solutions for the following equations:

$$\delta_1=\alpha_1+\frac{2}{\pi}\tan^{-1}\left(\frac{(\alpha_1+\alpha_2)(1-|b|)\cos\left(\frac{\pi}{2}t_1\right)}{2\left(\frac{(p-\eta)(1+A)}{1+B}+\eta+\mu\right)(1+|b|)+(\alpha_1+\alpha_2)(1-|b|)\sin\left(\frac{\pi}{2}t_1\right)}\right)$$

and

$$\delta_2=\alpha_2+\frac{2}{\pi}\tan^{-1}\left(\frac{(\alpha_1+\alpha_2)(1-|b|)\cos\left(\frac{\pi}{2}t_1\right)}{2\left(\frac{(p-\eta)(1+A)}{1+B}+\eta+\mu\right)(1+|b|)+(\alpha_1+\alpha_2)(1-|b|)\sin\left(\frac{\pi}{2}t_1\right)}\right)$$

b is given by (3.2), and

$$t_1=t_1(\lambda)=\frac{2}{\pi}\cos^{-1}\left(\frac{(p-\eta)(A-B)}{(p-\eta)(1-AB)+(\eta+\mu)(1-B^2)}\right) \tag{3.3}$$

Proof. Let $p(z)=\dfrac{1}{p-\gamma}\left(\dfrac{z\left(I_{\mu}^m(p,\lambda,l)f(z)\right)'}{I_{\mu}^m(p,\lambda,l)g(z)}-\gamma\right)$.

Then $p(z)$ is analytic in U with $p(0)=1$. By using (1.9), we obtain

$$\begin{aligned}&\left((p-\gamma)p(z)+\gamma\right)\left(I_{\mu}^m(p,\lambda,l)g(z)\right)\\&=(\mu+p)I_{\mu+1}^m(p,\lambda,l)f(z)-\mu I_{\mu}^m(p,\lambda,l)f(z)\end{aligned} \tag{3.4}$$

Differentiating both sides of the above equation and multiplying the resulting equation by z, we find that

$$\begin{aligned}&(p-\gamma)p'(z)\left(I_{\mu}^m(p,\lambda,l)g(z)\right)\\&+\left((p-\gamma)p(z)+\gamma\right)\left(I_{\mu}^m(p,\lambda,l)g(z)\right)'\\&=(\mu+p)\left(I_{\mu+1}^m(p,\lambda,l)f(z)\right)'-\mu\left(I_{\mu}^m(p,\lambda,l)f(z)\right)'\end{aligned}$$

Since $g(z)\in S_{p,\lambda,l}^{m,\mu+1}(\eta,p;A,B)$, by Corollary 2.4, it follows that $g(z)\in S_{p,\lambda,l}^{m,\mu}(\eta,p;A,B)$.

Next we let $q(z)=\dfrac{1}{p-\eta}\left(\dfrac{z\left(I_{\mu}^m(p,\lambda,l)g(z)\right)'}{I_{\mu}^m(p,\lambda,l)g(z)}-\eta\right)$.

Then, using (1.9), we have

$$(\mu+p)\frac{I_{\mu}^{m}(p,\lambda,l)g(z)}{I_{\mu}^{m}(p,\lambda,l)g(z)}=(p-\eta)q(z)+\eta+\mu \quad (3.5)$$

From (3.4) and (3.5), we obtain

$$\frac{1}{p-\gamma}\left(\frac{z\left(I_{\mu+1}^{m}(p,\lambda,l)f(z)\right)'}{I_{\mu+1}^{m}(p,\lambda,l)g(z)}-\gamma\right)$$

$$=p(z)+\frac{zp'(z)}{(p-\eta)q(z)+\eta+\mu}$$

Furthermore, by using a known result, we have

$$\left|q(z)-\frac{1-AB}{1-B^{2}}\right|<\frac{A-B}{1-B^{2}} \quad (3.6)$$

Thus, from (3.6), we obtain

$$(p-\eta)q(z)+\eta+\mu=\rho\exp\left(\frac{i\pi}{2}\phi\right)$$

where, in terms of t_1 given by (3.3).

$$\frac{(p-\eta)(1-A)}{1-B}+\eta+\mu<\rho<\frac{(p-\eta)(1-A)}{1-B}+\eta+\mu$$
$$-t_1<\varphi<t_1$$

We note that p is analytic in U with $p(0)=1$. Let $\omega=h(z)$ be the function which maps U onto the angular domain

$$\left\{\omega:-\frac{\pi}{2}\delta_1<arg(\omega)<\frac{\pi}{2}\delta_2\right\} \quad \text{with} \quad h(0)=1$$

Applying Lemma 3.1 for this function h with

$$\omega(z)=\frac{1}{(p-\eta)q(z)+\eta+\mu}$$

we see that $\operatorname{Re}\{p(z)\}>0$ ($z\in U$), and hence $p(z)\neq 0$ ($z\in U$). By using Lemma 3.2, if there exist two points $z_1,z_2\in U$ such that the condition (3.1) is satisfied, then we obtain (3.2) under the constraint (3.2). And we obtain

$$\arg\left(p(z_1)+\frac{z_1p'(z_1)}{(p-\eta)q(z_1)+\eta+\mu}\right)$$

$$=-\frac{\pi}{2}\alpha_1+\arg\left(1-i\left(\frac{\alpha_1+\alpha_2}{2}m\left(\rho\exp\left(\frac{i\pi\phi}{2}\right)\right)^{-1}\right)\right)\leq-\frac{\pi}{2}\alpha_1-\tan^{-1}\left(\frac{(\alpha_1+\alpha_2)m\cos\left(\frac{\pi}{2}(1-\phi)\right)}{2\rho+(\alpha_1+\alpha_2)m\cos\left(\frac{\pi}{2}(1-\phi)\right)}\right)$$

$$\leq-\frac{\pi}{2}\alpha_1-\tan^{-1}\left(\frac{(\alpha_1+\alpha_2)(1-|b|)\cos\left(\frac{\pi}{2}t_1\right)}{2\left(\frac{(p-\eta)(1+A)}{1+B}+\eta+\mu\right)(1+|b|)+(\alpha_1+\alpha_2)(1-|b|)\cos\left(\frac{\pi}{2}t_1\right)}\right)=-\frac{\pi}{2}\delta_1$$

and

$$\arg\left(p(z_2)+\frac{z_2p'(z_2)}{(p-\eta)q(z_2)+\eta+\mu}\right)\geq\frac{\pi}{2}\alpha_2+\tan^{-1}\left(\frac{(\alpha_1+\alpha_2)(1-|b|)\cos\left(\frac{\pi}{2}t_1\right)}{2\left(\frac{(p-\eta)(1+A)}{1+B}+\eta+\mu\right)(1+|b|)+(\alpha_1+\alpha_2)(1-|b|)\cos\left(\frac{\pi}{2}t_1\right)}\right)=\frac{\pi}{2}\delta_2$$

which would obviously contradict the assertion of Theorem 3.3. We thus complete the proof of Theorem 3.3.

If we let $\delta_1=\delta_2$ in Theorem 3.5, we easily obtain the following consequence.

Corollary 3.4: *Let* $f\in A_p$. $0<\delta\leq 1$. $0<\gamma<p$. *If*

$$\left|\arg\left(\frac{z\left(I_{\mu}^{m}(p,\lambda,l)f(z)\right)'}{I_{\mu}^{m}(p,\lambda,l)g(z)}-\gamma\right)\right|<\frac{\pi}{2}\delta$$

for some $g\in S_{p,\lambda,l}^{m,\mu+1}(\eta,p;A,B)$, then

$$\left|\arg\left(\frac{z\left(I_{\mu}^{m}(p,\lambda,l)f(z)\right)'}{I_{\mu}^{m}(p,\lambda,l)g(z)}-\gamma\right)\right|$$
$$<\frac{\pi}{2}\alpha$$

where α is the solutions for the following equation:

$$\delta = \alpha + \frac{2}{\pi}\tan^{-1}\left(\frac{\alpha(1-|b|)\cos\left(\frac{\pi}{2}t_1\right)}{\left(\frac{(p-\eta)(1+A)}{1+B}+\eta+\mu\right)(1+|b|)+\alpha(1-|b|)\cos\left(\frac{\pi}{2}t_1\right)}\right)$$

b is given by (3.2), and

$$-\frac{\pi}{2}\delta_1 < \arg\left(\frac{z\left(I_\mu^m(p,\lambda,l)f(z)\right)'}{I_\mu^m(p,\lambda,l)g(z)}-\gamma\right) < \frac{\pi}{2}\delta_2 \quad (3.7)$$

Theorem 3.5: *Let* $f \in A_p$. $0 < \delta_1, \delta_2 \le 1$. $0 < \gamma < p$. If

$$-\frac{\pi}{2}\delta_1 < \arg\left(\frac{z\left(I_\mu^m(p,\lambda,l)f(z)\right)'}{I_\mu^m(p,\lambda,l)g(z)}-\gamma\right) < \frac{\pi}{2}\delta_2$$

for some $g \in S_{p,\lambda,l}^{m,\mu}(\eta,p;A,B)$, then

$$-\frac{\pi}{2}\alpha_1 < \arg\left(\frac{z\left(I_\mu^{m+1}(p,\lambda,l)f(z)\right)'}{I_\mu^{m+1}(p,\lambda,l)g(z)}-\gamma\right) < \frac{\pi}{2}\alpha_2$$

where α_1, α_2 are the solutions for the following equations:

$$\delta_1 = \alpha_1 + \frac{2}{\pi}\tan^{-1}\left(\frac{(\alpha_1+\alpha_2)(1-|b|)\cos\left(\frac{\pi}{2}t_1\right)}{2\left(\frac{(p-\eta)(1+A)}{1+B}+\eta-p+\frac{l}{\lambda}\right)(1+|b|)+(\alpha_1+\alpha_2)(1-|b|)\cos\left(\frac{\pi}{2}t_1\right)}\right)$$

and

$$\delta_2 = \alpha_2 + \frac{2}{\pi}\tan^{-1}\left(\frac{(\alpha_1+\alpha_2)(1-|b|)\cos\left(\frac{\pi}{2}t_1\right)}{2\left(\frac{(p-\eta)(1-A)}{1-B}+\eta-p+\frac{l}{\lambda}\right)(1+|b|)+(\alpha_1+\alpha_2)(1-|b|)\cos\left(\frac{\pi}{2}t_1\right)}\right)$$

b is given by (3.2), and

$$t_1 = t_1(\lambda) = \frac{2}{\pi}\cos^{-1}\frac{(p-\eta)(A-B)}{(p-\eta)(1-AB)+\left(\eta-p+\frac{l}{\lambda}\right)(1-B^2)} \quad (3.8)$$

If we let $\delta_1 = \delta_2$ in Theorem 3.5, we easily obtain the following consequence.

Corollary 3.6: *Let* $f \in A_p$. $0 < \delta \le 1$. $0 < \gamma < p$. *If*

$$\left|\arg\left(\frac{z\left(I_\mu^m(p,\lambda,l)f(z)\right)'}{I_\mu^m(p,\lambda,l)g(z)}-\gamma\right)\right| < \frac{\pi}{2}\delta$$

for some $g \in S_{p,\lambda,l}^{m,\mu}(\eta,p;A,B)$, then

$$\left|\arg\left(\frac{z\left(I_\mu^{m+1}(p,\lambda,l)f(z)\right)'}{I_\mu^m(p,\lambda,l)g(z)}-\gamma\right)\right| < \frac{\pi}{2}\alpha$$

where α is the solutions for the following equation:

$$\delta = \alpha + \frac{2}{\pi}\tan^{-1}\left(\frac{\alpha(1-|b|)\cos\left(\frac{\pi}{2}t_1\right)}{\left(\frac{(p-\eta)(1+A)}{1+B}+\eta-p+\frac{l}{\lambda}\right)(1+|b|)+\alpha(1-|b|)\cos\left(\frac{\pi}{2}t_1\right)}\right)$$

b is given by (1.17), and

$$t_1 = t_1(\lambda)$$

$$= \frac{2}{\pi}\cos^{-1}\left(\frac{(p-\eta)(A-B)}{(p-\eta)(1-AB)+\left(\eta-p+\frac{l}{\lambda}\right)(1-B^2)}\right) \quad (3.9)$$

4. Acknowledgements

The research was supported by Kyungsung University Research Grants in 2011.

5. References

[1] B. A. Uralegaddi and C. Somanatha, "Certain Differential Operators for Meromorphic Functions," *Houston Journal of Mathematics*, Vol. 17, 1991, pp. 279-284.

[2] B. A. Uralegaddi and C. Somanatha, "New Critetia for Meromorphic Starlike Functions," *Bulletin of the Australian Mathematical Society*, Vol. 43, No. 1, 1991, pp. 137-140. doi:10.1017/S0004972700028859

[3] J. H. Choi, M. Saigo and H. M. Srivastava, "Some Inclusion Properties of a Certain Family of Integral Operators," *Journal of Mathematical Analysis and Applications*, Vol. 276, No. 1, 2002, pp. 432-445. doi:10.1016/S0022-247X(02)00500-0

[4] J.-L. Liu and K. I. Noor, "Some Properties of Noor Integral Operator," *Journal of Natural Geometry*, Vol. 21, 2002, pp. 81-90.

[5] J.-L. Liu, "The Noor Integral and Strongly Starlike Functions," *Journal of Mathematical Analysis and Applications*, Vol. 261, No. 2, 2001, pp. 441-447. doi:10.1006/jmaa.2001.7489

[6] K. I. Noor and M. A. Noor, "On Integral Operators," *Journal of Mathematical Analysis and Applications*, Vol. 238, No. 2, 1999, pp. 341-352. doi:10.1006/jmaa.1999.6501

[7] K. I. Noor, "On New Classes of Integral Operators," *Journal of Natural Geometry*, Vol. 16, 1999, pp. 71-80.

[8] K. S. Padmanabhan and R. Parvatham, "On Analytic Functions and Differential Subordination," *Bulletin Mathématique de la Société des Sciences*, *Mathématiques de Roumanie*, Vol. 31, 1987, pp. 237-248.

[9] M. Nunokawa, S. Owa, H. Saitoh, N. E. Cho and N. Ta-kahashi, "Some Properties of Analytic Functions at Extremal Points for Arguments," preprint, 2003.

[10] P. Eenigenburg, S. S. Miller, P. T. Mocanu and M. O. Reade, "On a Briot-Bouquet Differential Subordination," *General Inequalities*, Vol. 3, 1983, pp. 339-348.

[11] R. J. Libera and M. S. Robertson, "Meromorphic Close-to-Convex Functions," *Michigan Mathematical Journal*, Vol. 8, No. 2, 1961, pp. 167-176. doi:10.1307/mmj/1028998568

[12] S. K. Bajpai, "A Note on a Class of Meromorphic Univalent Functions," *Revue Roumaine de Mathématiques Pures et Appliquées*, Vol. 22, 1997, pp. 295-297.

[13] S. M. Sarangi and S. B. Uralegaddi, "Certain Differential Operators for Meromorphic Functions," *Bulletin of the Calcutta Mathematical Society*, Vol. 88, 1996, pp. 333-336

[14] S. S. Miller and P. T. Mocanu, "Differential Subordinations and Univalent Functions," *Michigan Mathematical Journal*, Vol. 28, No. 2, 1981, pp. 157-171.

