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Runs and Patterns in a Sequence of Markov Dependent Bivariate Trials

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Abstract

In this paper we consider a sequence of Markov dependent bivariate trials whose each component results in an outcome success (0) and failure (1) *i.e.* we have a sequence $\left\{ \begin{pmatrix} X_n \\ Y_n \end{pmatrix}, n \geq 0 \right\}$ of $S = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ -valued Markov dependent bivariate trials. By using the method of conditional probability generating functions (*pgfs*), we derive the *pgf* of joint distribution of $\left(X_{n,k_0^1}^0, X_{n,k_1^1}^1; Y_{n,k_0^2}^0, Y_{n,k_1^2}^1 \right)$ where for $i = 0,1$, $X_{n,k_i^1}^i$ denotes the number of occurrences of *i*-runs of length k_i^1 in the first component and $Y_{n,k_i^2}^i$ denotes the number of occurrences of *i*-runs of length k_i^2 in the second component of Markov dependent bivariate trials. Further we consider two patterns Λ_1 and Λ_2 of lengths k_1 and k_2 respectively and obtain the *pgf* of joint distribution of $\left(X_{n,\Lambda_1}, Y_{n,\Lambda_2} \right)$ using method of conditional probability generating functions where $X_{n,\Lambda_1} \left(Y_{n,\Lambda_2} \right)$ denotes the number of occurrences of pattern $\Lambda_1 (\Lambda_2)$ of length $k_1 (k_2)$ in the first (second) *n* components of bivariate trials. An algorithm is developed to evaluate the exact probability distributions of the vector random variables from their derived probability generating functions. Further some waiting time distributions are studied using the joint distribution of runs.

Keywords: Markov Dependent Bivariate Trials, Conditional Probability Generating Function, Joint Distribution

1. Introduction

The distributions of several run statistics are used in various areas such as reliability theory, testing of statistical hypothesis, DNA sequencing, psychology [1], start up demonstration tests [2] etc. There are various counting schemes of runs. Some of the most popular counting schemes of runs are non-overlapping success runs of length k [3], overlapping success runs of length k [4], success runs of length at least k, ℓ- overlapping success runs of length k [5], success runs of exact length k [6].

The probability distribution of various run statistics associated with the above counting schemes have been studied extensively in the literature in different situations such as independent Bernoulli trials (BT), non-identical BT, Markov dependent BT (MBT), higher order MBT, binary sequence of order k, multi-state trials etc. But very little work is found on the distribution theory of run statistics in case of bivariate trials which has applications

in different areas such as start up demonstration tests with regard to simultaneous start ups of two equipment, reliability theory of two dimensional consecutive (k, r) out of $(k+1, n)$: F-Lattice system etc as specified by [7]. [7] have studied the distribution of sooner and later waiting time problems for runs in Markov dependent bivariate trials by giving system of linear equations of the conditional *pgfs* of the waiting times. The distribution of number of occurrences of runs in the two components of bivariate sequence of trials and their joint distributions are still unknown to the literature.

Consider a sequence $\{X_n, n \geq 0\}$ of S-valued trials where S is set of all possible outcomes of trials under study. The simple pattern Λ is composed of specified sequence of k states *i.e.* $\Lambda = a_1 a_2 \cdots a_k$ where $a_1, a_2, \cdots a_k \in S$. The number of occurrences of patterns can be counted according to the non-overlapping or overlapping counting scheme. The non-overlapping counting scheme starts recounting of the pattern immediately after the occurrence of the pattern while the overlapping

counting scheme of patterns allows an overlap of pre-specified fixed length in the successive occurrences of patterns.

Recently the study of distributions of different statistics based on patterns has become a focus area for many researchers due to its wide applicability area. Distribution of $W_{r,k,\Lambda}$, the waiting time for the rth occurrence of pattern Λ of length k in the sequence of multistate trials is studied by [1,8,9]. [10] considered the sequence $X_1, X_2, \cdots$ generated by Polya's urn scheme and study the waiting time distribution of $W_{r,k,\Lambda}$ for $r = 1$.

Joint distribution of number of occurrences of pattern Λ_1 of length k_1 and pattern Λ_2 of length k_2 in n Markov dependent multi-state trials is studied by [11]. [12] considered a sequence $\underline{X}_i, i = 1, 2, \cdots$ of m-dimensional i.i.d. Random column vectors whose entries are $\{0,1\}$-valued i.i.d. random variables and obtain the waiting time distribution of two dimensional patterns with general shape. The general method, which is an extension of method of conditional *pgf*s, is used to study these distributions by [12].

Even though the distribution of waiting time of the pattern of general shape in the sequence of multi-variate trials with i.i.d. components has been done, the joint distribution of number of occurrences of patterns $\Lambda_i, (i = 1, 2, \cdots, m)$ in the sequence of ith component of the m-variate trials $\underline{X}_1, \underline{X}_2, \cdots, \underline{X}_n$ is still unknown. Here we derive the *pgf* of joint distribution of number of occurrences of runs in both the components of the bivariate trials and generalize this study to the distribution of number of occurrences of patterns in both components of the bivariate trials.

In this paper we consider the sequence $\left\{ \begin{pmatrix} X_n \\ Y_n \end{pmatrix}, n \geq 0 \right\}$ of S-valued Markov dependent bivariate trials. In Section 2, we obtain the *pgf* of joint distribution of number of occurrences of i-runs of length k_i^1 in first components and i-runs of length k_i^2 in the second components of the bivariate trials $(i = 0, 1)$. We study this joint distribution of runs under the non-overlapping counting scheme of runs by using the method of conditional *pgfs*. Further in section 3, we study the joint distribution of number of occurrences of pattern Λ_1 of length k_1 in the first component and number of occurrences of pattern Λ_2 of length k_2 in the second component of bivariate trials. In Section 4, we develop an algorithm to evaluate the exact probability distributions of the random variables under study. As an application of the derived joint distributions, in Section 5, we obtain distributions of several waiting times associated with the runs and patterns in bivariate trials. In Section 6 we present some numerical work based on distribution of runs and patterns. Finally in Section 7, we discuss an application and

generalization of the studied distributions.

2. The Joint Distribution of Number of Occurrences of Runs

Let $S = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$. Consider the sequence $\left\{ \begin{pmatrix} X_n \\ Y_n \end{pmatrix}, n \geq 0 \right\}$ of S-valued Markov dependent bivariate trials with the transition probabilities,

$$P\left(\begin{pmatrix} X_i \\ Y_i \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \middle| \begin{pmatrix} X_{i-1} \\ Y_{i-1} \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} \right) = p_{uv,xy} \forall \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \in S$$

$i = 1, 2, \cdots, n$

and the initial probabilities

$$P\left(\begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \right) = \pi_{xy} \quad \text{for} \quad \begin{pmatrix} x \\ y \end{pmatrix} \in S.$$

Let $X_{n,k_i^1}^i \left(Y_{n,k_i^2}^i \right)$ be the number of i-runs ($i = 0, 1$) of length k_i^1 $\left(k_i^2 \right)$ in n trials associated with the first (second) component of bivariate trials (*i.e.* number of i-runs in $X_0, X_1, \cdots, X_n$ $(Y_0, Y_1, \cdots, Y_n)$). In this section we derive the joint distribution of

$$\left(X_{n,k_0}^0, X_{n,k_1^1}^1 ; Y_{n,k_0^2}^0, Y_{n,k_1^2}^1 \right).$$

Let $\phi_n (t_0, t_1; s_0, s_1)$ be the *pgf* of distribution of $\left(X_{n,k_0^1}^0, X_{n,k_1^1}^1 ; Y_{n,k_0^2}^0, Y_{n,k_1^2}^1 \right)$. Assume that for a non-negative integer $c \leq n$, we have observed until $(n - c)^{th}$ trial (*i.e.* we have observed $\begin{pmatrix} X_i \\ Y_i \end{pmatrix}, i = 0, 1, \cdots, n - c$). We define $\phi_c^{\left(i, m_i^1; j, m_j^2 \right)} = \phi_c^{\left(i, m_i^1; j, m_j^2 \right)} (t_0, t_1; s_0, s_1)$ as *pgf* of conditional distribution of number of i-runs of length k_i^1 in $X_{n-c+1}, \cdots, X_n$ and number of j-runs of length k_j^2 in $Y_{n-c+1}, \cdots, Y_n$ given that we have observed $\begin{pmatrix} X_0 \\ Y_0 \end{pmatrix}, \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}, \cdots, \begin{pmatrix} X_{n-c} \\ Y_{n-c} \end{pmatrix}$ and currently have i-run of length m_i^1 in first component and j-runs of length m_j^2 in the second component of bivariate trials is observed, $m_i^1 = 1, 2, \cdots, k_i^1$, $m_j^2 = 1, 2, \cdots, k_j^2$, $i, j = 0, 1$.

We define $a^c = 1 - a$ for $a = 0, 1$.

Now by assuming $\pi_{00} = 1$, we have,

$$\phi_n (t_0, t_1; s_0, s_1) = \phi_n^{(0,0;0,0)} (t_0, t_1; s_0, s_1) \tag{2.1}$$

Also we have,

$$\phi_0^{\left(i, m_i^1; j, m_j^2 \right)} (t_0, t_1; s_0, s_1) = 1 \text{ for } m_i^1 = 1, 2, ..., k_i^1,$$
$$m_j^2 = 1, 2, ..., k_j^2 \text{ and } i, j = 0, 1 \tag{2.2}$$

Conditioning on the first trial we have the following system of recurrent relation of conditional *pgfs* for

$c = 1, 2, \cdots, n$.

$$\phi_n^{(0,0;0,0)}\left(t_0, t_1; s_0, s_1\right)$$
$$= p_{00,00}\phi_{n-1}^{(0,1;0,1)} + p_{00,01}\phi_{n-1}^{(0,1;1,1)}$$
$$+ p_{00,10}\phi_{n-1}^{(1,1;0,1)} + p_{00,11}\phi_{n-1}^{(1,1;1,1)}$$

Conditioning on the next trial from each stage, we have the following system of recurrent relations of conditional *pgfs* for $c = 1, 2, \cdots, n$ and $i, j = 0, 1$.

$$\phi_c^{\left(i, m_i^1; j, m_j^2\right)}$$
$$= p_{ij,ij}\phi_{c-1}^{\left(i, m_i^1+1; j, m_j^2+1\right)} + p_{ij,i^c j}\phi_{c-1}^{\left(i^c, 1; j, m_j^2+1\right)}$$
$$+ p_{ij,ij^c}\phi_{c-1}^{\left(i, m_i^1+1; j^c, 1\right)} + p_{ij,i^c j^c}\phi_{c-1}^{\left(i^c, 1; j^c, 1\right)}$$

$\quad$ if $m_i^1 = 1, 2, \cdots, k_i^1 - 2$, $m_j^2 = 1, 2, \cdots, k_j^2 - 2$,

$$\phi_c^{\left(i, m_i^1; j, m_j^2\right)}$$
$$= p_{ij,ij}\phi_{c-1}^{\left(i, m_i^1+1; j, m_j^2+1\right)}t_i + p_{ij,i^c j}\phi_{c-1}^{\left(i^c, 1; j, m_j^2+1\right)}$$
$$+ p_{ij,ij^c}\phi_{c-1}^{\left(i, m_i^1+1; j^c, 1\right)}t_i + p_{ij,i^c j^c}\phi_{c-1}^{\left(i^c, 1; j^c, 1\right)}$$

$\quad$ if $m_i^1 = k_i^1 - 1$, $m_j^2 = 1, 2, \cdots, k_j^2 - 2$,

$$\phi_c^{\left(i, m_i^1; j, m_j^2\right)} = p_{ij,ij}\phi_{c-1}^{\left(i, 1; j, m_j^2+1\right)} + p_{ij,i^c j}\phi_{c-1}^{\left(i^c, 1; j, m_j^2+1\right)}$$
$$+ p_{ij,ij^c}\phi_{c-1}^{\left(i, 1; j^c, 1\right)} + p_{ij,i^c j^c}\phi_{c-1}^{\left(i^c, 1; j^c, 1\right)}$$

$\quad$ if $m_i^1 = k_i^1$, $m_j^2 = 1, 2, \cdots, k_j^2 - 2$,

$$\phi_c^{\left(i, m_i^1; j, m_j^2\right)} = p_{ij,ij}\phi_{c-1}^{\left(i, m_i^1+1; j, m_j^2+1\right)}s_j + p_{ij,i^c j}\phi_{c-1}^{\left(i^c, 1; j, m_j^2+1\right)}s_j$$
$$+ p_{ij,ij^c}\phi_{c-1}^{\left(i, m_i^1+1; j^c, 1\right)} + p_{ij,i^c j^c}\phi_{c-1}^{\left(i^c, 1; j^c, 1\right)}$$

$\quad$ if $m_i^1 = 1, 2, \cdots, k_i^1 - 2$, $m_j^2 = k_j^2 - 1$,

$$\phi_c^{\left(i, m_i^1; j, m_j^2\right)} = p_{ij,ij}\phi_{c-1}^{\left(i, m_i^1+1; j, m_j^2+1\right)}t_i s_j + p_{ij,i^c j}\phi_{c-1}^{\left(i^c, 1; j, m_j^2+1\right)}s_j$$
$$+ p_{ij,ij^c}\phi_{c-1}^{\left(i, m_i^1+1; j^c, 1\right)}t_i + p_{ij,i^c j^c}\phi_{c-1}^{\left(i^c, 1; j^c, 1\right)}$$

$\quad$ if $m_i^1 = k_i^1 - 1$, $m_j^2 = k_j^2 - 1$,

$$\phi_c^{\left(i, m_i^1; j, m_j^2\right)} = p_{ij,ij}\phi_{c-1}^{\left(i, 1; j, m_j^2+1\right)}s_j + p_{ij,i^c j}\phi_{c-1}^{\left(i^c, 1; j, m_j^2+1\right)}s_j$$
$$+ p_{ij,ij^c}\phi_{c-1}^{\left(i, 1; j^c, 1\right)} + p_{ij,i^c j^c}\phi_{c-1}^{\left(i^c, 1; j^c, 1\right)}$$

$\quad$ if $m_i^1 = k_i^1$, $m_j^2 = k_j^2 - 1$,

$$\phi_c^{\left(i, m_i^1; j, m_j^2\right)} = p_{ij,ij}\phi_{c-1}^{\left(i, m_i^1+1; j, 1\right)} + p_{ij,i^c j}\phi_{c-1}^{\left(i^c, 1; j, 1\right)}$$
$$+ p_{ij,ij^c}\phi_{c-1}^{\left(i, m_i^1+1; j^c, 1\right)} + p_{ij,i^c j^c}\phi_{c-1}^{\left(i^c, 1; j^c, 1\right)}$$

$\quad$ if $m_i^1 = 1, 2, \cdots, k_i^1 - 2$, $m_j^2 = k_j^2$,

$$\phi_c^{\left(i, m_i^1; j, m_j^2\right)} = p_{ij,ij}\phi_{c-1}^{\left(i, m_i^1+1; j, 1\right)}t_i + p_{ij,i^c j}\phi_{c-1}^{\left(i^c, 1; j, 1\right)}$$
$$+ p_{ij,ij^c}\phi_{c-1}^{\left(i, m_i^1+1; j^c, 1\right)}t_i + p_{ij,i^c j^c}\phi_{c-1}^{\left(i^c, 1; j^c, 1\right)}$$

$\quad$ if $m_i^1 = k_i^1 - 1$, $m_j^2 = k_j^2$

$$\phi_c^{\left(i, m_i^1; j, m_j^2\right)} = p_{ij,ij}\phi_{c-1}^{\left(i, 1; j, 1\right)} + p_{ij,i^c j}\phi_{c-1}^{\left(i^c, 1; j, 1\right)}$$
$$+ p_{ij,ij^c}\phi_{c-1}^{\left(i, 1; j^c, 1\right)} + p_{ij,i^c j^c}\phi_{c-1}^{\left(i^c, 1; j^c, 1\right)}$$

$\quad$ if $m_i^1 = k_i^1$, $m_j^2 = k_j^2$

Thus we have $\left(k_0^1 + k_1^1\right)\left(k_0^2 + k_1^2\right)$ recurrent relations of conditional *pgfs* for $c = 1, 2, \cdots, n-1$ and these can be written as,

$$\underline{\phi}_c\left(t_0, t_1; s_0, s_1\right) =$$
$$\left(A + \sum_{i=0}^{1} B_i^1 t_i + \sum_{j=0}^{1} B_j^2 s_j + \sum_{i=0}^{1}\sum_{j=0}^{1} B_{ij}^{12} t_i s_j\right)\underline{\phi}_{c-1}\left(t_0, t_1; s_0, s_1\right)$$

$$(2.3)$$

where

$$\underline{\phi}_c = \left(\phi_c^{(0,0;0,0)} \quad \phi_c^{(0,1;0,1)} \quad \cdots \quad \phi_c^{(0,1;0,k_0^2)} \quad \phi_c^{(0,1;1,1)} \quad \cdots \right.$$
$$\left. \phi_c^{(0,1;1,k_1^2)} \quad \cdots \quad \phi_c^{(1,1;1,k_1^2)} \quad \cdots \phi_c^{(1,k_1^1;1,k_1^2)}\right)'$$

and $A + \sum_{i=0}^{1} B_i^1 t_i + \sum_{j=0}^{1} B_j^2 s_j + \sum_{i=0}^{1}\sum_{j=0}^{1} B_{ij}^{12} t_i s_j$ is a square matrix of size $\left(k_0^1 + k_1^1\right)\left(k_0^2 + k_1^2\right)$. Each row of this matrix corresponds to corresponding element of $\underline{\phi}_c\left(t_0, t_1; s_0, s_1\right)$ and its elements are the coefficients of elements of $\underline{\phi}_{c-1}\left(t_0, t_1; s_0, s_1\right)$.

For $c = 0$, we have,

$$\underline{\phi}_0\left(t_0, t_1; s_0, s_1\right) = \underline{1}$$

where $\underline{1}$ is the column vector with all the elements equal to 1.

Using (2.3) recurrently for $c = 1, 2, \cdots, n$, we have,

$$\underline{\phi}_n\left(t_0, t_1; s_0, s_1\right)$$
$$= \left(A + \sum_{i=0}^{1} B_i^1 t_i + \sum_{j=0}^{1} B_j^2 s_j + \sum_{i=0}^{1}\sum_{j=0}^{1} B_{ij}^{12} t_i s_j\right)^n \underline{\phi}_0\left(t_0, t_1; s_0, s_1\right)$$

$$(2.4)$$

Hence from (2.1) and (2.4) we get the *pgf* of distribution of $\left(X_{n,k_0}^0, X_{n,k_1}^1; Y_{n,k_0}^0, Y_{n,k_1}^1\right)$ as,

$$\phi_n\left(t_0, t_1; s_0, s_1\right)$$
$$= \underline{p}'\left(A + \sum_{i=0}^{1} B_i^1 t_i + \sum_{j=0}^{1} B_j^2 s_j + \sum_{i=0}^{1}\sum_{j=0}^{1} B_{ij}^{12} t_i s_j\right)^n \underline{1} \quad (2.5)$$

where p is the first column of identity matrix of order $\left(k_0^1 + k_1^1\right)\left(k_0^2 + k_1^2\right)$.

3. The Joint Distribution of Number of Occurrences of Patterns

Let $\Lambda_1 = a_1 a_2 \cdots a_{k_1}$ and $\Lambda_2 = b_1 b_2 \cdots b_{k_2}$ be two patterns of lengths k_1 and k_2 respectively where $a_i, b_j \in \{0,1\}$, $i = 1,2,\cdots,k_1; j = 1,2,\cdots,k_2$. We consider the non-overlapping counting scheme of patterns in which to count the number of occurrences of pattern Λ in a given sequence of n trials, one has to restart counting from scratch each time the pattern Λ occurs. Let X_{n,Λ_1} denotes the number of non-overlapping occurrences of patterns Λ_1 in the first component of n bivariate trials (*i.e.* in $X_1, X_2, \cdots, X_n$) and Y_{n,Λ_2} denotes the number of non-overlapping occurrences of patterns Λ_2 in the second component of n bivariate trials (*i.e.* in $Y_1, Y_2, \cdots, Y_n$). Consider the random vector $\left(X_{n,\Lambda_1}, Y_{n,\Lambda_2}\right)$. In this section we obtain the *pgf* of joint distribution of $\left(X_{n,\Lambda_1}, Y_{n,\Lambda_2}\right)$ using method of conditional *pgf*s.

Let *pgf* of joint distribution of $\left(X_{n,\Lambda_1}, Y_{n,\Lambda_2}\right)$ be $\phi_n(t_1, t_2) = \phi_n(\underline{t})$. Assume that for a non-negative integer $c \leq n$, we have observed until $(n-c)^{th}$ trial *i.e.* $\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}, \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}, \cdots, \begin{pmatrix} X_{n-c} \\ Y_{n-c} \end{pmatrix}$. For $c \leq n$, we define, the following conditional *pgf*s.

Let $\phi_c^{(i,j)}(\underline{t})$ be *pgf* of conditional distribution of number of occurrences of patterns Λ_1 in $X_{n-c+1}, \cdots, X_n$ and number of occurrences of pattern Λ_2 in $Y_{n-c+1}, \cdots, Y_n$ of bivariate trials given that currently (*i.e.* at $(n-c)^{th}$ trial) we have observed none of the sub-patterns of Λ_1 and Λ_2 and $\begin{pmatrix} X_{n-c} \\ Y_{n-c} \end{pmatrix} = \begin{pmatrix} i \\ j \end{pmatrix}$, where $\begin{pmatrix} i \\ j \end{pmatrix} \in S$.

Similarly let, $\phi_{c,\Lambda_1}^{(i,j)}(\underline{t})$ be *pgf* of conditional distribution of number of occurrences of patterns Λ_1 in $X_{n-c+1}, \cdots, X_n$ and patterns Λ_2 in $Y_{n-c+1}, \cdots, Y_n$ of bivariate trials given that at $(n-c)^{th}$ trial we have observed the sub-pattern of length i of Λ_1 in the first component and none of the sub pattern of Λ_2 is observed in the second component of bivariate trials and $\begin{pmatrix} X_{n-c} \\ Y_{n-c} \end{pmatrix} = \begin{pmatrix} a_i \\ j \end{pmatrix}$ where $i = 1,2,\cdots,k_1$ and $j = 0,1$.

Let $\phi_{c,\Lambda_2}^{(i,j)}(\underline{t})$ be *pgf* of conditional distribution of number of occurrences of patterns Λ_1 in $X_{n-c+1}, \cdots, X_n$ and patterns Λ_2 in $Y_{n-c+1}, \cdots, Y_n$ of bivariate trials given that at $(n-c)^{th}$ trial we have observed none of the sub-pattern of Λ_1 in the first component and a sub pattern of the length j of Λ_2 in the second component of bivariate trials is observed and $\begin{pmatrix} X_{n-c} \\ Y_{n-c} \end{pmatrix} = \begin{pmatrix} i \\ b_j \end{pmatrix}$ where $i = 0,1$ and $j = 1,2,\cdots,k_2$.

Also let $\phi_{c,\Lambda_1,\Lambda_2}^{(i,j)}(\underline{t})$ as *pgf* of conditional distribution of number of occurrences of pattern Λ_1 in $X_{n-c+1}, \cdots, X_n$ and pattern Λ_2 in $Y_{n-c+1}, \cdots, Y_n$ of bivariate trials given that at $(n-c)^{th}$ trial we have observed the sub-pattern of length i of Λ_1 in the first component and a sub pattern of the length j of Λ_2 in the second component of bivariate trials and $\begin{pmatrix} X_{n-c} \\ Y_{n-c} \end{pmatrix} = \begin{pmatrix} a_i \\ b_j \end{pmatrix}$ where $i = 1,2,\cdots,k_1$ and $j = 1,2,\cdots,k_2$.

Let,
$a_i^c = 1 - a_i$, $i = 1,2,\cdots,k_1$
and $b_j^c = 1 - b_j$, $j = 1,2,\cdots,k_2$.

For $c \leq n$, we assume that at $(n-c)^{th}$ trial, the sub-pattern of length i of Λ_1 is observed in the first component of bivariate trials. If we condition on the next trial as $\begin{pmatrix} X_{n-c+1} \\ Y_{n-c+1} \end{pmatrix} = \begin{pmatrix} a_{i+1}^c \\ y_{n-c+1} \end{pmatrix}$ *i.e.* the sub-pattern of length i of Λ_1 observed at $(n-c)^{th}$ trial breaks at $(n-c+1)^{st}$ trial then to check whether any sub-pattern of Λ_1 of length $r (r \leq i)$ has occurred, we define the indicator function ε_{ir}^1 as,

$$\varepsilon_{ir}^1 = I\left\{a_1 = a_{i-r+2}, a_2 = a_{i-r+3}, \ldots, a_{r-1} = a_i, a_r = a_{i+1}^c\right\}$$
$$i = 1,2,\cdots,k_1 - 1; r \leq i.$$

Similarly we have indicator function ε_{jr}^2 as,

$$\varepsilon_{jr}^2 = I\left\{b_1 = b_{j-r+2}, b_2 = b_{j-r+3}, \ldots, b_{r-1} = b_j, b_r = b_{j+1}^c\right\}$$
$$j = 1,2,\cdots,k_2 - 1; r \leq j.$$

Let

$$u_1^i = \max_{1 \leq r \leq i} \varepsilon_{ir}^1, \quad u_2^i = \max\left\{r \,\middle|\, \varepsilon_{ir}^1 = 1, r = 1,2,\cdots,i\right\},$$
$$i = 1,2,\cdots,k_1 - 1$$

$$v_1^j = \max_{1 \leq r \leq j} \varepsilon_{jr}^2 \text{ and } v_2^j = \max\left\{r \,\middle|\, \varepsilon_{jr}^2 = 1, r = 1,2,\cdots,j\right\}$$
$$j = 1,2,\cdots,k_2 - 1$$

Assuming $\pi_{00} = 1$, we have,

$$\phi_n(t_1, t_2) = \phi_n^{(0,0)}(t_1, t_2) \tag{3.1}$$

We also have,

$$\phi_0^{(i,j)}(\underline{t}) = 1 \quad \text{for } \forall \begin{pmatrix} i \\ j \end{pmatrix} \in S;$$

$$\phi_{0,\Lambda_1}^{(i,j)}\left(\underline{t}\right)=1 \qquad \text{for } i=1,2,\cdots,k_1 \text{ and } j=0,1;$$

$$\phi_{0,\Lambda_2}^{(i,j)}\left(\underline{t}\right)=1 \qquad \text{for } i=0,1 \text{ and } j=1,2,\cdots,k_2;$$

$$\phi_{0,\Lambda_1,\Lambda_2}^{(i,j)}\left(\underline{t}\right)=1 \quad \text{for } i=1,2,\cdots,k_1 \text{ and } j=1,2,\cdots,k_2 . \quad (3.2)$$

Now for each $1 \le c \le n$ (c denotes the number of trials remaining to observe) we condition on the next trial to obtain the recurrent relations of conditional *pgf*s as follows.

$$\phi_c^{(i,j)}\left(\underline{t}\right)=p_{ij,a_1b_1}\phi_{c-1,\Lambda_1,\Lambda_2}^{(1,1)}\left(\underline{t}\right)+p_{ij,a_1^cb_1}\phi_{c-1,\Lambda_2}^{(a_1^c,1)}\left(\underline{t}\right)$$
$$+p_{ij,a_1b_1^c}\phi_{c-1,\Lambda_1}^{(1,b_1^c)}\left(\underline{t}\right)+p_{ij,a_1^cb_1^c}\phi_{c-1}^{(a_1^c,b_1^c)}\left(\underline{t}\right)$$
$$\text{if } i,j=0,1 ,$$

$$\phi_{c,\Lambda_1}^{(i,j)}\left(\underline{t}\right)=p_{a_ij,a_{i+1}b_1}\phi_{c-1,\Lambda_1,\Lambda_2}^{(i+1,1)}\left(\underline{t}\right)$$
$$+p_{a_ij,a_{i+1}^cb_1}\left(\phi_{c-1,\Lambda_1,\Lambda_2}^{(u_2^i,1)}\left(\underline{t}\right)u_1^i+\phi_{c-1,\Lambda_2}^{(a_{i+1}^c,1)}\left(\underline{t}\right)\left(1-u_1^i\right)\right)$$
$$+p_{a_ij,a_{i+1}^cb_1^c}\left(\phi_{c-1,\Lambda_1}^{(u_2^i,b_1^c)}\left(\underline{t}\right)u_1^i+\phi_{c-1}^{(a_{i+1}^c,b_1^c)}\left(\underline{t}\right)\left(1-u_1^i\right)\right)$$
$$+p_{a_ij,a_{i+1}b_1^c}\phi_{c-1,\Lambda_1}^{(i+1,b_1^c)}\left(\underline{t}\right)$$
$$\text{if } i=1,2,\cdots,k_1-2; j=0,1$$

$$\phi_{c,\Lambda_1}^{(i,j)}\left(\underline{t}\right)=p_{a_ij,a_{i+1}b_1}\phi_{c-1,\Lambda_1,\Lambda_2}^{(i+1,1)}\left(\underline{t}\right)t_1$$
$$+p_{a_ij,a_{i+1}^cb_1}\left(\phi_{c-1,\Lambda_2}^{(u_2^i,1)}\left(\underline{t}\right)u_1^i+\phi_{c-1,\Lambda_2}^{(a_{i+1}^c,1)}\left(\underline{t}\right)\left(1-u_1^i\right)\right)$$
$$+p_{a_ij,a_{i+1}^cb_1^c}\left(\phi_{c-1,\Lambda_2}^{(u_2^i,b_1^c)}\left(\underline{t}\right)u_1^i+\phi_{c-1}^{(a_{i+1}^c,b_1^c)}\left(\underline{t}\right)\left(1-u_1^i\right)\right)$$
$$+p_{a_ij,a_{i+1}b_1^c}\phi_{c-1,\Lambda_1}^{(i+1,b_1^c)}\left(\underline{t}\right)t_1$$
$$\text{if } i=k_1-1; j=0,1$$

$$\phi_{c,\Lambda_1}^{(i,j)}\left(\underline{t}\right)=p_{a_ij,a_1b_1}\phi_{c-1,\Lambda_1,\Lambda_2}^{(1,1)}\left(\underline{t}\right)+p_{a_ij,a_1^cb_1}\phi_{c-1,\Lambda_2}^{(a_1^c,1)}\left(\underline{t}\right)$$
$$+p_{a_ij,a_1b_1^c}\phi_{c-1,\Lambda_1}^{(1,b_1^c)}\left(\underline{t}\right)+p_{a_ij,a_1^cb_1^c}\phi_{c-1}^{(a_1^c,b_1^c)}\left(\underline{t}\right)$$
$$\text{if } i=k_1, j=0,1 .$$

Similarly we have recurrent relations of the conditional *pgf*s $\phi_{c,\Lambda_2}^{(i,j)}\left(\underline{t}\right)$ for $i=0,1$ and $j=1,2,\cdots,k_2$ as follows.

$$\phi_{c,\Lambda_2}^{(i,j)}\left(\underline{t}\right)=p_{ib_j,a_1b_{j+1}}\phi_{c-1,\Lambda_1,\Lambda_2}^{(1,j+1)}\left(\underline{t}\right)$$
$$+p_{ib_j,a_1b_{j+1}^c}\left(\phi_{c-1,\Lambda_1,\Lambda_2}^{(1,v_2^j)}\left(\underline{t}\right)v_1^j+\phi_{c-1,\Lambda_1}^{(1,b_{j+1}^c)}\left(\underline{t}\right)\left(1-v_1^j\right)\right)$$
$$+p_{ib_j,a_1^cb_{j+1}^c}\left(\phi_{c-1,\Lambda_2}^{(a_1^c,v_2^j)}\left(\underline{t}\right)v_1^j+\phi_{c-1}^{(a_1^c,b_{j+1}^c)}\left(\underline{t}\right)\left(1-v_1^j\right)\right)$$
$$+p_{ib_j,a_1^cb_{j+1}}\phi_{c-1,\Lambda_2}^{(a_1^c,j+1)}\left(\underline{t}\right)$$
$$\text{if } j=1,2,\cdots,k_2-2;$$

$$\phi_{c,\Lambda_2}^{(i,j)}\left(\underline{t}\right)=p_{ib_j,a_1b_{j+1}}\phi_{c-1,\Lambda_1,\Lambda_2}^{(1,j+1)}\left(\underline{t}\right)t_2$$
$$+p_{ib_j,a_1b_{j+1}^c}\left(\phi_{c-1,\Lambda_1,\Lambda_2}^{(1,v_2^j)}\left(\underline{t}\right)v_1^j+\phi_{c-1,\Lambda_1}^{(1,b_{j+1}^c)}\left(\underline{t}\right)\left(1-v_1^j\right)\right)$$
$$+p_{ib_j,a_1^cb_{j+1}^c}\left(\phi_{c-1,\Lambda_2}^{(a_1^c,v_2^j)}\left(\underline{t}\right)v_1^j+\phi_{c-1}^{(a_1^c,b_{j+1}^c)}\left(\underline{t}\right)\left(1-v_1^j\right)\right)$$
$$+p_{ib_j,a_1^cb_{j+1}}\phi_{c-1,\Lambda_2}^{(a_1^c,j+1)}\left(\underline{t}\right)t_2$$
$$\text{if } j=k_2-1;$$

$$\phi_{c,\Lambda_2}^{(i,j)}\left(\underline{t}\right)=p_{ib_j,a_1b_1}\phi_{c-1,\Lambda_1,\Lambda_2}^{(1,1)}\left(\underline{t}\right)+p_{ib_j,a_1^cb_1}\phi_{c-1,\Lambda_2}^{(a_1^c,1)}\left(\underline{t}\right)$$
$$+p_{ib_j,a_1b_1^c}\phi_{c-1,\Lambda_1}^{(1,b_1^c)}\left(\underline{t}\right)+p_{ib_j,a_1^cb_1^c}\phi_{c-1}^{(a_1^c,b_1^c)}\left(\underline{t}\right)$$
$$\text{if } j=k_2$$

The recurrent relations of the conditional *pgf*s $\phi_{c,\Lambda_1,\Lambda_2}^{(i,j)}\left(\underline{t}\right)$ for $i=1,2,\cdots,k_1$ and $j=1,2,\cdots,k_2$ are as follows.

$$\phi_{c,\Lambda_1,\Lambda_2}^{(i,j)}\left(\underline{t}\right)=p_{a_ib_j,a_{i+1}b_{j+1}}\phi_{c-1,\Lambda_1,\Lambda_2}^{(i+1,j+1)}\left(\underline{t}\right)$$
$$+p_{a_ib_j,a_{i+1}b_{j+1}^c}\left(\phi_{c-1,\Lambda_1,\Lambda_2}^{(i+1,v_2^j)}\left(\underline{t}\right)v_1^j+\phi_{c-1,\Lambda_1}^{(i+1,b_{j+1}^c)}\left(\underline{t}\right)\left(1-v_1^j\right)\right)$$
$$+p_{a_ib_j,a_{i+1}^cb_{j+1}}\left(\phi_{c-1,\Lambda_1,\Lambda_2}^{(u_2^i,j+1)}\left(\underline{t}\right)u_1^i+\phi_{c-1,\Lambda_1}^{(a_{i+1}^c,j+1)}\left(\underline{t}\right)\left(1-u_1^i\right)\right)$$
$$+p_{a_ib_j,a_{i+1}^cb_{j+1}^c}\left(\phi_{c-1,\Lambda_1,\Lambda_2}^{(u_2^i,v_2^j)}\left(\underline{t}\right)u_1^iv_1^j\right.$$
$$+\phi_{c-1,\Lambda_1}^{(u_2^i,b_{j+1}^c)}\left(\underline{t}\right)u_1^i\left(1-v_1^j\right)$$
$$+\phi_{c-1,\Lambda_2}^{(a_{i+1}^c,v_2^j)}\left(\underline{t}\right)\left(1-u_1^i\right)v_1^j$$
$$\left.+\phi_{c-1}^{(a_{i+1}^c,b_{j+1}^c)}\left(\underline{t}\right)\left(1-u_1^i\right)\left(1-v_1^j\right)\right)$$
$$\text{if } i=1,2,\cdots,k_1-2; j=1,2,\cdots,k_2-2$$

$$\phi_{c,\Lambda_1,\Lambda_2}^{(i,j)}\left(\underline{t}\right)=p_{a_ib_j,a_{i+1}b_{j+1}}\phi_{c-1,\Lambda_1,\Lambda_2}^{(i+1,j+1)}\left(\underline{t}\right)t_1$$
$$+p_{a_ib_j,a_{i+1}b_{j+1}^c}\left(\phi_{c-1,\Lambda_1,\Lambda_2}^{(i+1,v_2^j)}\left(\underline{t}\right)v_1^j+\phi_{c-1,\Lambda_1}^{(i+1,b_{j+1}^c)}\left(\underline{t}\right)\left(1-v_1^j\right)\right)t_1$$
$$+p_{a_ib_j,a_{i+1}^cb_{j+1}}\left(\phi_{c-1,\Lambda_1,\Lambda_2}^{(u_2^i,j+1)}\left(\underline{t}\right)u_1^i+\phi_{c-1,\Lambda_1}^{(a_{i+1}^c,j+1)}\left(\underline{t}\right)\left(1-u_1^i\right)\right)$$
$$+p_{a_ib_j,a_{i+1}^cb_{j+1}^c}\left(\phi_{c-1,\Lambda_1,\Lambda_2}^{(u_2^i,v_2^j)}\left(\underline{t}\right)u_1^iv_1^j\right.$$
$$+\phi_{c-1,\Lambda_1}^{(u_2^i,b_{j+1}^c)}\left(\underline{t}\right)u_1^i\left(1-v_1^j\right)$$
$$+\phi_{c-1,\Lambda_2}^{(a_{i+1}^c,v_2^j)}\left(\underline{t}\right)\left(1-u_1^i\right)v_1^j$$
$$\left.+\phi_{c-1}^{(a_{i+1}^c,b_{j+1}^c)}\left(\underline{t}\right)\left(1-u_1^i\right)\left(1-v_1^j\right)\right)$$
$$\text{if } i=k_1-1; j=1,2,\cdots,k_2-2$$

$$\phi_{c,\Lambda_1,\Lambda_2}^{(i,j)}\left(\underline{t}\right) = p_{a_ib_j,a_1b_{j+1}}\phi_{c-1,\Lambda_1,\Lambda_2}^{(1,j+1)}\left(\underline{t}\right)$$

$$+ p_{a_ib_j,a_1b_{j+1}^c}\left(\phi_{c-1,\Lambda_1,\Lambda_2}^{\left(1,v_2^j\right)}\left(\underline{t}\right)v_1^j + \phi_{c-1,\Lambda_1}^{\left(1,b_{j+1}^c\right)}\left(\underline{t}\right)\left(1-v_1^j\right)\right)$$

$$+ p_{a_ib_j,a_1^cb_{j+1}}\phi_{c-1,\Lambda_2}^{\left(a_1^c,j+1\right)}\left(\underline{t}\right)$$

$$+ p_{a_ib_j,a_1^cb_{j+1}^c}\left(\phi_{c-1,\Lambda_2}^{\left(a_1^c,v_2^j\right)}\left(\underline{t}\right)v_1^j + \phi_{c-1}^{\left(a_1^c,b_{j+1}^c\right)}\left(\underline{t}\right)\left(1-v_1^j\right)\right)$$

$$\text{if } i = k_1; j = 1,2,\cdots,k_2-2$$

$$\phi_{c,\Lambda_1,\Lambda_2}^{(i,j)}\left(\underline{t}\right) = p_{a_ib_j,a_{i+1}b_{j+1}}\phi_{c-1,\Lambda_1,\Lambda_2}^{(i+1,j+1)}\left(\underline{t}\right)t_2$$

$$+ p_{a_ib_j,a_{i+1}b_{j+1}^c}\left(\phi_{c-1,\Lambda_1,\Lambda_2}^{\left(i+1,v_2^j\right)}\left(\underline{t}\right)v_1^j + \phi_{c-1,\Lambda_1}^{\left(i+1,b_{j+1}^c\right)}\left(\underline{t}\right)\left(1-v_1^j\right)\right)$$

$$+ p_{a_ib_j,a_{i+1}^cb_{j+1}}\left(\phi_{c-1,\Lambda_1,\Lambda_2}^{\left(u_2^i,j+1\right)}\left(\underline{t}\right)u_1^i + \phi_{c-1,\Lambda_1}^{\left(a_{i+1}^c,j+1\right)}\left(\underline{t}\right)\left(1-u_1^i\right)\right)t_2$$

$$+ p_{a_ib_j,a_{i+1}^cb_{j+1}^c}\left(\phi_{c-1,\Lambda_1,\Lambda_2}^{\left(u_2^i,v_2^j\right)}\left(\underline{t}\right)u_1^iv_1^j\right.$$

$$\left. + \phi_{c-1,\Lambda_1}^{\left(u_2^i,b_{j+1}^c\right)}\left(\underline{t}\right)u_1^i\left(1-v_1^j\right) + \phi_{c-1,\Lambda_2}^{\left(a_{i+1}^c,v_2^j\right)}\left(\underline{t}\right)\left(1-u_1^i\right)v_1^j\right.$$

$$\left. + \phi_{c-1}^{\left(a_{i+1}^c,b_{j+1}^c\right)}\left(\underline{t}\right)\left(1-u_1^i\right)\left(1-v_1^j\right)\right)$$

$$\text{if } i = 1,2,\cdots,k_1-2; j = k_2-1$$

$$\phi_{c,\Lambda_1,\Lambda_2}^{(i,j)}\left(\underline{t}\right) = p_{a_ib_j,a_{i+1}b_{j+1}}\phi_{c-1,\Lambda_1,\Lambda_2}^{(i+1,j+1)}\left(\underline{t}\right)t_1t_2$$

$$+ p_{a_ib_j,a_{i+1}b_{j+1}^c}\left(\phi_{c-1,\Lambda_1,\Lambda_2}^{\left(i+1,v_2^j\right)}\left(\underline{t}\right)v_1^j + \phi_{c-1,\Lambda_1}^{\left(i+1,b_{j+1}^c\right)}\left(\underline{t}\right)\left(1-v_1^j\right)\right)t_1$$

$$+ p_{a_ib_j,a_{i+1}^cb_{j+1}}\left(\phi_{c-1,\Lambda_1,\Lambda_2}^{\left(u_2^i,j+1\right)}\left(\underline{t}\right)u_1^i + \phi_{c-1,\Lambda_1}^{\left(a_{i+1}^c,j+1\right)}\left(\underline{t}\right)\left(1-u_1^i\right)\right)t_2$$

$$+ p_{a_ib_j,a_{i+1}^cb_{j+1}^c}\left(\phi_{c-1,\Lambda_1,\Lambda_2}^{\left(u_2^i,v_2^j\right)}\left(\underline{t}\right)u_1^iv_1^j\right.$$

$$\left. + \phi_{c-1,\Lambda_1}^{\left(u_2^i,b_{j+1}^c\right)}\left(\underline{t}\right)u_1^i\left(1-v_1^j\right) + \phi_{c-1,\Lambda_2}^{\left(a_{i+1}^c,v_2^j\right)}\left(\underline{t}\right)\left(1-u_1^i\right)v_1^j\right.$$

$$\left. + \phi_{c-1}^{\left(a_{i+1}^c,b_{j+1}^c\right)}\left(\underline{t}\right)\left(1-u_1^i\right)\left(1-v_1^j\right)\right)$$

$$\text{if } i = k_1-1; j = k_2-1$$

$$\phi_{c,\Lambda_1,\Lambda_2}^{(i,j)}\left(\underline{t}\right) = p_{a_ib_j,a_1b_{j+1}}\phi_{c-1,\Lambda_1,\Lambda_2}^{(1,j+1)}\left(\underline{t}\right)t_2$$

$$+ p_{a_ib_j,a_1b_{j+1}^c}\left(\phi_{c-1,\Lambda_1,\Lambda_2}^{\left(1,v_2^j\right)}\left(\underline{t}\right)v_1^j + \phi_{c-1,\Lambda_1}^{\left(1,b_{j+1}^c\right)}\left(\underline{t}\right)\left(1-v_1^j\right)\right)$$

$$+ p_{a_ib_j,a_1^cb_{j+1}}\phi_{c-1,\Lambda_2}^{\left(a_1^c,j+1\right)}\left(\underline{t}\right)t_2$$

$$+ p_{a_ib_j,a_1^cb_{j+1}^c}\left(\phi_{c-1,\Lambda_2}^{\left(a_1^c,v_2^j\right)}\left(\underline{t}\right)v_1^j + \phi_{c-1}^{\left(a_1^c,b_{j+1}^c\right)}\left(\underline{t}\right)\left(1-v_1^j\right)\right)$$

$$\text{if } i = k_1; j = k_2-1$$

$$\phi_{c,\Lambda_1,\Lambda_2}^{(i,j)}\left(\underline{t}\right) = p_{a_ib_j,a_{i+i}b_1}\phi_{c-1,\Lambda_1,\Lambda_2}^{(i+1,1)}\left(\underline{t}\right)$$

$$+ p_{a_ib_j,a_{i+1}b_1^c}\left(\phi_{c-1,\Lambda_1,\Lambda_2}^{\left(u_2^i,1\right)}\left(\underline{t}\right)u_1^i + \phi_{c-1,\Lambda_2}^{\left(a_{i+1}^c,1\right)}\left(\underline{t}\right)\left(1-u_1^i\right)\right)$$

$$+ p_{a_ib_j,a_{i+1}b_1^c}\phi_{c-1,\Lambda_1}^{\left(i+1,b_i^c\right)}\left(\underline{t}\right)$$

$$+ p_{a_ib_j,a_{i+1}b_1^c}\left(\phi_{c-1,\Lambda_1}^{\left(u_2^i,b_1^c\right)}\left(\underline{t}\right)u_1^i + \phi_{c-1}^{\left(a_{i+1}^c,b_1^c\right)}\left(\underline{t}\right)\left(1-u_1^i\right)\right)$$

$$\text{if } i = 1,2,\cdots,k_1-2; j = k_2$$

$$\phi_{c,\Lambda_1,\Lambda_2}^{(i,j)}\left(\underline{t}\right) = p_{a_ib_j,a_{i+i}b_1}\phi_{c-1,\Lambda_1,\Lambda_2}^{(i+1,1)}\left(\underline{t}\right)t_1$$

$$+ p_{a_ib_j,a_{i+1}b_1^c}\left(\phi_{c-1,\Lambda_1,\Lambda_2}^{\left(u_2^i,1\right)}\left(\underline{t}\right)u_1^i + \phi_{c-1,\Lambda_2}^{\left(a_{i+1}^c,1\right)}\left(\underline{t}\right)\left(1-u_1^i\right)\right)$$

$$+ p_{a_ib_j,a_{i+1}b_i^c}\phi_{c-1,\Lambda_1}^{\left(i+1,b_i^c\right)}\left(\underline{t}\right)t_1$$

$$+ p_{a_ib_j,a_{i+1}b_1^c}\left(\phi_{c-1,\Lambda_1}^{\left(u_2^i,b_1^c\right)}\left(\underline{t}\right)u_1^i + \phi_{c-1}^{\left(a_{i+1}^c,b_1^c\right)}\left(\underline{t}\right)\left(1-u_1^i\right)\right)$$

$$\text{if } i = k_1-1; j = k_2$$

$$\phi_{c,\Lambda_1,\Lambda_2}^{(i,j)}\left(\underline{t}\right) = p_{a_ib_j,a_1b_1}\phi_{c-1,\Lambda_1,\Lambda_2}^{(1,1)}\left(\underline{t}\right) + p_{a_ib_j,a_1^cb_1}\phi_{c-1,\Lambda_2}^{\left(a_1^c,1\right)}\left(\underline{t}\right)$$

$$+ p_{a_ib_j,a_1b_1^c}\phi_{c-1,\Lambda_1}^{\left(1,b_1^c\right)}\left(\underline{t}\right) + p_{a_ib_j,a_1^cb_1^c}\phi_{c-1}^{\left(a_1^c,b_1^c\right)}\left(\underline{t}\right)$$

$$\text{if } i = k_1; j = k_2$$

The above system of $(k_1+2)\times(k_2+2)$ recurrent relations of conditional *pgfs* $\phi_c^{(i,j)}\left(\underline{t}\right)$, $(i,j=0,1)$; $\phi_{c,\Lambda_1}^{(i,j)}\left(\underline{t}\right)$, $(i=0,1,\cdots,k_1; j=0,1)$; $\phi_{c,\Lambda_2}^{(i,j)}\left(\underline{t}\right)$, $(i=0,1; j=1,2,\cdots,k_2)$ and $\phi_{c,\Lambda_1,\Lambda_2}^{(i,j)}\left(\underline{t}\right)$ $(i=1,2,\cdots,k_1; j=1,2,\cdots,k_2)$ can be written as follows.

$$\underline{\phi}_c\left(\underline{t}\right)=\begin{cases}\left(A + B_1t_1 + B_2t_2 + B_{12}t_1t_2\right)\underline{\phi}_{c-1}\left(\underline{t}\right) & \text{if } c = 2,3,\cdots,n \\ \left(A + B_1t_1 + B_2t_2 + B_{12}t_1t_2\right)\underline{1} & \text{if } c = 1\end{cases}$$

$$(3.3)$$

where $\underline{1}$ is column vector with all its elements 1 and $\underline{\phi}_n\left(\underline{t}\right)$ is column vector with its elements as follows.

$$\left(\phi_c^{(0,0)} \ \phi_c^{(0,1)} \ \phi_c^{(1,0)} \ \phi_c^{(1,1)} \ \phi_{c,\Lambda_1}^{(1,0)}...\phi_{c,\Lambda_1}^{(k_1,1)} \ \phi_{c,\Lambda_2}^{(0,1)} \cdots\right.$$

$$\left.\phi_{c,\Lambda_2}^{(1,k_2)} \ \phi_{c,\Lambda_1,\Lambda_2}^{(1,1)} \ \phi_{c,\Lambda_1,\Lambda_2}^{(1,2)} \cdots\phi_{c,\Lambda_1,\Lambda_2}^{(k_1,k_2)}\right)$$

From (3.1) we get,

$$\phi_n\left(t_1,t_2\right) = \underline{p}'\underline{\phi}_n\left(t_1,t_2\right)$$

where $\underline{p}$ is first column of identity matrix of order $(k_1+2)\overline{\times}(k_2+2)$.

The recurrent use of (3.3) gives the *pgf* of $\left(X_{n,\Lambda_1}, Y_{n,\Lambda_2}\right)$ as follows.

$$\phi_n\left(\underline{t}\right) = \underline{p}'\left(A + B_1t_1 + B_2t_2 + B_{12}t_1t_2\right)^n\underline{1} \qquad (3.4)$$

4. Exact distribution of $\left(X_{n,k_0^1}^0, X_{n,k_1^1}^1; Y_{n,k_0^2}^0, Y_{n,k_1^2}^1 \right)$

We note that the *pgf* of $\left(X_{n,\Lambda_1}, Y_{n,\Lambda_2} \right)$ is a particular case of *pgf* of $\left(X_{n,k_0^1}^0, X_{n,k_1^1}^1; Y_{n,k_0^1}^0, Y_{n,k_1^1}^1 \right)$. Hence we develop an algorithm to obtain the exact probability distribution of $\left(X_{n,k_0^1}^0, X_{n,k_1^1}^1; Y_{n,k_0^2}^0, Y_{n,k_1^2}^1 \right)$ which can further beapplied to obtain the exact probability distribution of $\left(X_{n,\Lambda_1}, Y_{n,\Lambda_2} \right)$ from its *pgf*.

Observe that the *pgf* of joint distribution of $\left(X_{n,k_0^1}^0, X_{n,k_1^1}^1; Y_{n,k_0^2}^0, Y_{n,k_1^2}^1 \right)$ as obtained in (2.5) in general involves matrix polynomial $P_n \left(t_0, t_1; s_0, s_1 \right)$ in t_0, t_1, s_0 and s_1 of order n where

$$P_n \left(t_0, t_1; s_0, s_1 \right)$$
$$= \left(A + \sum_{i=0}^{1} B_i^1 t_i + \sum_{j=0}^{1} B_j^2 s_j + \sum_{i=0}^{1}\sum_{j=0}^{1} B_{ij}^{12} t_i s_j \right)^n.$$

Hence the joint probability distribution can be obtained by expanding the polynomial with respect to t_0, t_1, s_0 and s_1.

That is,

$$P\left(X_{n,k_0^1}^0 = x_0, X_{n,k_1^1}^1 = x_1; Y_{n,k_0^2}^0 = y_0, Y_{n,k_1^2}^1 = y_1 \right)$$
$$= \text{coefficient of } t_0^{x_0} t_1^{x_1} s_0^{y_0} s_1^{y_1} \text{ in } \phi_n \left(t_0, t_1; s_0, s_1 \right)$$

i.e.

$$P\left(\left(X_{n,k_0^1}^0, X_{n,k_1^1}^1 \right) = \underline{x}; \left(Y_{n,k_0^2}^0, Y_{n,k_1^2}^1 \right) = \underline{y} \right)$$
$$= \text{coefficient of } \underline{t}^{\underline{x}} \underline{s}^{\underline{y}} \text{ in } \phi_n \left(t_0, t_1; s_0, s_1 \right) \tag{4.1}$$

Exact formula to obtain the coefficient matrix is quiet tedious since multiplication of matrices is not commutative operation. But the interesting recurrent relations are found between these coefficient matrices. Let $C_n \left(x_0, x_1; y_0, y_1 \right) = C_n \left(\underline{x}; \underline{y} \right)$ be the coefficient matrix of $\underline{t}^{\underline{x}} \underline{s}^{\underline{y}}$ in the expansion of matrix polynomial $P_n \left(t_0, t_1; s_0, s_1 \right)$ and for $n \geq 1$, let $D_n = \left\{ \left(\underline{x}; \underline{y} \right) | x_i, y_i = 0, 1, \cdots, n \text{ for } i = 0, 1 \right\}$. The following Lemma gives the recurrent relations of the coefficient matrices of $C_n \left(\underline{x}; \underline{y} \right)$ with $C_{n-1} \left(\underline{x}; \underline{y} \right)$.

Lemma 4.1 Let $C_n \left(\underline{x}; \underline{y} \right)$ be the coefficient matrix of $\underline{t}^{\underline{x}} \underline{s}^{\underline{y}}$ in the expansion of the matrix polynomial $P_n \left(t_0, t_1; s_0, s_1 \right)$ Then $C_n \left(\underline{x}; \underline{y} \right)$ satisfies the following recurrent relation.

$$C_n \left(\underline{x}; \underline{y} \right) = C_{n-1} \left(\underline{x}; \underline{y} \right) A$$
$$+ \sum_{i=0}^{1} C_{n-1} \left(\underline{x} - \underline{e}_{i+1}; \underline{y} \right) B_i^1 I \left\{ \underline{x} - \underline{e}_{i+1} \geq \underline{0} \right\}$$
$$+ \sum_{j=0}^{1} C_{n-1} \left(\underline{x}; \underline{y} - \underline{e}_{j+1} \right) B_j^2 I \left\{ \underline{y} - \underline{e}_{j+1} \geq \underline{0} \right\}$$
$$+ \sum_{i=0}^{1}\sum_{j=0}^{1} C_{n-1} \left(\underline{x} - \underline{e}_{i+1}; \underline{y} - \underline{e}_{j+1} \right) B_{ij}^{12} I \left\{ \underline{x} - \underline{e}_{i+1} \geq \underline{0}; \underline{y} - \underline{e}_{j+1} \geq \underline{0} \right\}$$
$$\tag{4.2}$$

with $C_1 \left(\underline{0}; \underline{0} \right) = A$, $C_1 \left(\underline{e}_{i+1}; \underline{0} \right) = B_i^1$, $C_1 \left(\underline{0}; \underline{e}_{j+1} \right) = B_j^1$ and $C_1 \left(\underline{e}_{i+1}; \underline{e}_{j+1} \right) = B_{ij}^{12}$ for $i, j = 0, 1$. Here $\underline{e}_i$ is the ith row of the identity matrix of order 2.

Proof Obviously for $n = 1$, we have,

$$C_1 \left(\underline{x}; \underline{y} \right) = \begin{cases} A & \text{if } \underline{x} = \underline{0}; \underline{y} = \underline{0} \\ B_i^1 & \text{if } \underline{x} = \underline{e}_{i+1}; \underline{y} = \underline{0}, i = 0, 1 \\ B_j^2 & \text{if } \underline{x} = \underline{0}; \underline{y} = \underline{e}_{j+1}, j = 0, 1 \\ B_{ij}^2 & \text{if } \underline{x} = \underline{e}_{i+1}; \underline{y} = \underline{e}_{i+1}, i = 0, 1; j = 0, 1 \\ O & \text{otherwise} \end{cases}$$

where O is the null matrix of same order as that of A.

For $n = 2$, observe that $C_2 \left(\underline{x}, \underline{y} \right)$ satisfies (4.2).

Assume that Equation (4.2) is true for some $r \left(2 \leq r < n \right)$. Hence we have,

$$\left(A + \sum_{i=0}^{1} B_i^1 t_i + \sum_{j=0}^{1} B_j^2 s_j + \sum_{i=0}^{1}\sum_{j=0}^{1} B_{ij}^{12} t_i s_j \right)^r = \sum_{(\underline{x}, \underline{y}) \in D_r} C_r \left(\underline{x}, \underline{y} \right) \underline{t}^{\underline{x}} \underline{s}^{\underline{y}}$$

Then

$$\left(A + \sum_{i=0}^{1} B_i^1 t_i + \sum_{j=0}^{1} B_j^2 s_j + \sum_{i=0}^{1}\sum_{j=0}^{1} B_{ij}^{12} t_i s_j \right)^{r+1}$$
$$= \left(A + \sum_{i=0}^{1} B_i^1 t_i + \sum_{j=0}^{1} B_j^2 s_j + \sum_{i=0}^{1}\sum_{j=0}^{1} B_{ij}^{12} t_i s_j \right)^r$$
$$\cdot \left(A + \sum_{i=0}^{1} B_i^1 t_i + \sum_{j=0}^{1} B_j^2 s_j + \sum_{i=0}^{1}\sum_{j=0}^{1} B_{ij}^{12} t_i s_j \right)$$
$$= \left(\sum_{(\underline{x}, \underline{y}) \in D_r} C_r \left(\underline{x}, \underline{y} \right) \underline{t}^{\underline{x}} \underline{s}^{\underline{y}} \right)$$
$$\cdot \left(A + \sum_{i=0}^{1} B_i^1 t_i + \sum_{j=0}^{1} B_j^2 s_j + \sum_{i=0}^{1}\sum_{j=0}^{1} B_{ij}^{12} t_i s_j \right)$$
$$= \sum_{(\underline{x}, \underline{y}) \in D_{r+1}} \left\{ C_{r-1} \left(\underline{x}; \underline{y} \right) A + \sum_{i=0}^{1} C_{r-1} \left(\underline{x} - \underline{e}_{i+1}; \underline{y} \right) B_i^1 I \left\{ \underline{x} - \underline{e}_{i+1} \geq \underline{0} \right\} \right.$$
$$+ \sum_{j=0}^{1} C_{r-1} \left(\underline{x}; \underline{y} - \underline{e}_{j+1} \right) B_j^2 I \left\{ \underline{y} - \underline{e}_{j+1} \geq \underline{0} \right\}$$
$$+ \sum_{i=0}^{1}\sum_{j=0}^{1} C_{r-1} \left(\underline{x} - \underline{e}_{i+1}; \underline{y} - \underline{e}_{j+1} \right)$$
$$\left. \cdot B_{ij}^{12} I \left\{ \underline{x} - \underline{e}_{i+1} \geq \underline{0}; \underline{y} - \underline{e}_{j+1} \geq \underline{0} \right\} \right\} \underline{t}^{\underline{x}} \underline{s}^{\underline{y}}$$

Hence we get the required proof of the lemma.

Theorem 4.1 The exact probability distribution of $\left(X^0_{n,k^1_0}, X^1_{n,k^1_1}; Y^0_{n,k^2_0}, Y^1_{n,k^2_1}\right)$ is given by,

$$P\left((X^0_{n,k^1_0}, X^1_{n,k^1_1}) = \underline{x}; (Y^0_{n,k^2_0}, Y^1_{n,k^2_1}) = \underline{y}\right), \qquad (4.3)$$
$$= \underline{p}'C_n\left(\underline{x}, \underline{y}\right)\underline{1}$$

where $C_n\left(\underline{x}, \underline{y}\right)$ is the coefficient matrix of $\underline{t}^{\underline{x}}\underline{s}^{\underline{y}}$ in expansion of matrix polynomial $P_n\left(t_0, t_1; s_0, s_1\right)$ and it satisfies (4.2)

Proof The proof follows by applying the Lemma 4.1 to matrix polynomial $P_n\left(t_0, t_1; s_0, s_1\right)$ involved in the *pgf* of distribution of $\left(X^0_{n,k^1_0}, X^1_{n,k^1_1}; Y^0_{n,k^2_0}, Y^1_{n,k^2_1}\right)$ in (2.5).

Remark 4.1 The expected number of failure-runs of length k^1_0 in first components of n Markov dependent bivariate trials is given by,

$$E\left(X^0_{n,k^1_0}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\phi\left(t_0, 1; 1, 1\right)\big|_{t_0=1}.$$

On simplifying this expression, we have,

$$E\left(X^0_{n,k^1_0}\right) = \underline{p}'\sum_{j=1}^{n}\left(A + B^1_1 + B^2_0 + B^2_1 + B^{12}_{01} + B^{12}_{10} + B^{12}_{11}\right)^{j-1}$$
$$\cdot\left(B^1_0 + B^{12}_{01}\right)\underline{1}$$

$$(4.4)$$

5. Waiting Time Distributions Related to Runs and Patterns

The exact probability distribution of $\left(X^0_{n,k^1_0}, X^1_{n,k^1_1}; Y^0_{n,k^2_0}, Y^1_{n,k^2_1}\right)$ from its *pgf* given in (2.5) can be expressed as,

$$P\left((X^0_{n,k^1_0}, X^1_{n,k^1_1}) = \underline{x}; (Y^0_{n,k^2_0}, Y^1_{n,k^2_1}) = \underline{y}\right)$$
$$= \underline{p}'\Big\{C_{n-1}\left(\underline{x}; \underline{y}\right)A + \sum_{i=0}^{1}C_{n-1}\left(\underline{x}-\underline{e}_{i+1}; \underline{y}\right)B^1_i I\left\{\underline{x}-\underline{e}_{i+1}\geq \underline{0}\right\}$$
$$+\sum_{j=0}^{1}C_{n-1}\left(\underline{x}; \underline{y}-\underline{e}_{j+1}\right)B^2_j I\left\{\underline{y}-\underline{e}_{j+1}\geq \underline{0}\right\}$$
$$+\sum_{i=0}^{1}\sum_{j=0}^{1}C_{n-1}\left(\underline{x}-\underline{e}_{i+1}; \underline{y}-\underline{e}_{j+1}\right)$$
$$\cdot B^{12}_{ij}I\left\{\underline{x}-\underline{e}_{i+1}\geq \underline{0}; \underline{y}-\underline{e}_{j+1}\geq \underline{0}\right\}\Big\}\underline{1}$$

$$(5.1)$$

The components of the above expression can be interpreted as follows.

$$\underline{p}'\Big\{C_{n-1}\left(\underline{x}; \underline{y}\right)A\Big\}\underline{1}$$
$$= P\Big(\Big\{(X^0_{n,k^1_0}, X^1_{n,k^1_1}) = \underline{x}, (Y^0_{n,k^2_0}, Y^1_{n,k^2_1}) = \underline{y}\Big\};$$
$$\Big\{(X^0_{n-1,k^1_0}, X^1_{n-1,k^1_1}) = \underline{x}, (Y^0_{n-1,k^2_0}, Y^1_{n-1,k^2_1}) = \underline{y}\Big\}\Big)$$

For $i = 0, 1$,

$$\underline{p}'\Big\{C_{n-1}\left(\underline{x}-\underline{e}_{i+1}; \underline{y}\right)B^1_i\Big\}\underline{1}$$
$$= P\Big(\Big\{(X^0_{n,k^1_0}, X^1_{n,k^1_1}) = \underline{x}, (Y^0_{n,k^2_0}, Y^1_{n,k^2_1}) = \underline{y}\Big\};$$
$$\Big\{(X^0_{n-1,k^1_0}, X^1_{n-1,k^1_1}) = \underline{x}-\underline{e}_{i+1}, (Y^0_{n-1,k^2_0}, Y^1_{n-1,k^2_1}) = \underline{y}\Big\}\Big),$$

and

$$\underline{p}'\Big\{C_{n-1}\left(\underline{x}; \underline{y}-\underline{e}_{i+1}\right)B^2_i\Big\}\underline{1}$$
$$= P\Big(\Big\{(X^0_{n,k^1_0}, X^1_{n,k^1_1}) = \underline{x}, (Y^0_{n,k^2_0}, Y^1_{n,k^2_1}) = \underline{y}\Big\};$$
$$\Big\{(X^0_{n-1,k^1_0}, X^1_{n-1,k^1_1}) = \underline{x}, (Y^0_{n-1,k^2_0}, Y^1_{n-1,k^2_1}) = \underline{y}-\underline{e}_{i+1}\Big\}\Big).$$

Similarly for $i, j = 0, 1$,

$$\underline{p}'\Big\{C_{n-1}\left(\underline{x}-\underline{e}_{i+1}, \underline{y}-\underline{e}_{j+1}\right)B^{12}_{ij}\Big\}\underline{1}$$
$$= P\Big(\Big\{(X^0_{n,k^1_0}, X^1_{n,k^1_1}) = \underline{x}, (Y^0_{n,k^2_0}, Y^1_{n,k^2_1}) = \underline{y}\Big\};$$
$$\Big\{(X^0_{n-1,k^1_0}, X^1_{n-1,k^1_1}) = \underline{x}-\underline{e}_{i+1}, (Y^0_{n-1,k^2_0}, Y^1_{n-1,k^2_1}) = \underline{y}-\underline{e}_{j+1}\Big\}\Big)$$

$$(5.2)$$

so that

$$P\left((X^0_{n,k^1_0}, X^1_{n,k^1_1}) = \underline{x}; (Y^0_{n,k^2_0}, Y^1_{n,k^2_1}) = \underline{y}\right)$$
$$= P\Big(\Big\{(X^0_{n,k^1_0}, X^1_{n,k^1_1}) = \underline{x}, (Y^0_{n,k^2_0}, Y^1_{n,k^2_1}) = \underline{y}\Big\};$$
$$\Big\{(X^0_{n-1,k^1_0}, X^1_{n-1,k^1_1}) = \underline{x}, (Y^0_{n-1,k^2_0}, Y^1_{n-1,k^2_1}) = \underline{y}\Big\}\Big)$$
$$+\sum_{i=0}^{1}P\Big(\Big\{(X^0_{n,k^1_0}, X^1_{n,k^1_1}) = \underline{x}, (Y^0_{n,k^2_0}, Y^1_{n,k^2_1}) = \underline{y}\Big\};$$
$$\Big\{(X^0_{n-1,k^1_0}, X^1_{n-1,k^1_1}) = \underline{x}-\underline{e}_{i+1}, (Y^0_{n-1,k^2_0}, Y^1_{n-1,k^2_1}) = \underline{y}\Big\}\Big)$$
$$+\sum_{j=0}^{1}P\Big(\Big\{(X^0_{n,k^1_0}, X^1_{n,k^1_1}) = \underline{x}, (Y^0_{n,k^2_0}, Y^1_{n,k^2_1}) = \underline{y}\Big\};$$
$$\Big\{(X^0_{n-1,k^1_0}, X^1_{n-1,k^1_1}) = \underline{x}, (Y^0_{n-1,k^2_0}, Y^1_{n-1,k^2_1}) = \underline{y}-\underline{e}_{j+1}\Big\}$$
$$+\sum_{i=0}^{1}\sum_{j=0}^{1}P\Big(\Big\{(X^0_{n,k^1_0}, X^1_{n,k^1_1}) = \underline{x}, (Y^0_{n,k^2_0}, Y^1_{n,k^2_1}) = \underline{y}\Big\};$$
$$\Big\{(X^0_{n-1,k^1_0}, X^1_{n-1,k^1_1}) = \underline{x}-\underline{e}_{i+1}, (Y^0_{n-1,k^2_0}, Y^1_{n-1,k^2_1}) = \underline{y}-\underline{e}_{j+1}\Big\}\Big)$$

$$(5.3)$$

The above interpretation is useful for deriving different waiting time distributions.

Let F^i_0 be the event that failure-run of length k^i_0 occur for the first time in the ith component of the Markov dependent bivariate trials and F^i_1 be the event that success-run of length k^i_1 occur for the first time in the ith component of the Markov dependent bivariate trials $i = 1, 2$.

Let W_S be the waiting time for sooner occurring event between F^1_0, F^2_0 F^1_1, F^2_1. To obtain the distri-

bution of sooner waiting time we define the following random variables.

$W_{S,j}^i$: The waiting time for sooner occurring event between F_0^1, F_0^2 F_1^1, F_1^2 given that F_j^i is the sooner event (*i.e.* j -run of length k_j^i of ith component of bivariate trials is the sooner event) $i = 1, 2$ and $j = 0, 1$.

W_j^i : The waiting time for the first occurrence of j -run of length k_j^i in the ith component of the bivariate trials $\left\{ \begin{pmatrix} X_n \\ Y_n \end{pmatrix}, n \geq 0 \right\}$, $i = 1, 2$ and $j = 0, 1$.

W_{ij}^{12} : The waiting time until the occurrence of both events F_i^1 and F_j^2 simultaneously, $i, j = 0, 1$ (*i.e.* i - run of length k_i^1 in the first component and j -run of length k_j^2 in the second component of the bivariate trials occur simultaneously).

Then we have,

$$P(W_S = m) = \sum_{i=0}^{1} P(W_{S,i}^1 = m) + \sum_{j=0}^{1} P(W_{S,j}^2 = m)$$
$$+ \sum_{i=0}^{1} \sum_{j=0}^{1} P(W_{ij}^{12} = m) \qquad (5.4)$$

Let $\psi_S(t)$, $\psi_{S,j}^i(t)$, $\psi_j^i(t)$ and $\psi^{ij}(t)$ be the *pgf* of W_S, $W_{S,j}^i$, W_j^i and W_{kj}^{12}, $i = 1, 2$ and $k, j = 0, 1$ respectively. In the next sub-section we obtain the sooner waiting time distribution.

5.1. Sooner Waiting Time Distribution

The probability that 0-run of length k_0^1 (*i.e.* event F_0^1) occurs for the first time at m^{th} trial given that none of the events F_0^2 F_1^1, F_1^2 has occurred until m^{th} trial $\left(i.e.\ P(W_{S,0}^1 = m) \right)$ can be written as follows.

$$P(W_{S,0}^1 = m)$$
$$= P\left(\left(X_{m,k_0^1}^0, X_{m,k_1^1}^1 ; Y_{m,k_0^2}^0, Y_{n,k_1^2}^1 \right) = (1, 0, 0, 0); \right.$$
$$\left. \left(X_{m-1,k_0^1}^0, X_{m-1,k_1^1}^1 ; Y_{m-1,k_0^2}^0, Y_{m-1,k_1^2}^1 \right) = (0, 0, 0, 0) \right)$$
$$= \underline{p}' \left\{ C_{m-1}(0, 0, 0, 0) \right\} B_0^1 \underline{1}$$
$$= \underline{p}' A^{m-1} B_0^1 \underline{1} \qquad m \geq k_0^1$$

Hence *pgf* of $W_{S,0}^1$ can be given as,

$$\psi_{S,0}^1(t) = t \underline{p}'(I - At)^{-1} B_0^1 \underline{1}.$$

Similarly the probability that 1-run of length k_1^1 (*i.e.* event F_1^1) occurs for the first time at m^{th} trial given that none of the events F_0^1 F_0^2, F_1^2 has occurred until m^{th} trial $\left(i.e.\ P(W_{S,1}^1 = m) \right)$ can be written as follows.

$$P(W_{S,1}^1 = m)$$
$$= P\left(\left(X_{m,k_0^1}^0, X_{m,k_1^1}^1 ; Y_{m,k_0^2}^0, Y_{m,k_1^2}^1 \right) = (0, 1, 0, 0); \right.$$
$$\left. \left(X_{m-1,k_0^1}^0, X_{m-1,k_1^1}^1 ; Y_{m-1,k_0^2}^0, Y_{m-1,k_1^2}^1 \right) = (0, 0, 0, 0) \right)$$
$$= \underline{p}' \left\{ C_m(0, 1, 0, 0) \right\} B_1^1 \underline{1}$$
$$= \underline{p}' A^{m-1} B_1^1 \underline{1} \qquad m \geq k_1^1$$

Hence *pgf* of $W_{S,1}^1$ is,

$$\psi_{S,1}^1(t) = t \underline{p}'(I - At)^{-1} B_1^1 \underline{1}.$$

Similarly $P(W_{S,0}^2 = m)$ and $P(W_{S,1}^2 = m)$ is given by,

$$P(W_{S,0}^2 = m) = \underline{p}' A^{m-1} B_0^2 \underline{1}$$
$$P(W_{S,1}^2 = m) = \underline{p}' A^{m-1} B_1^2 \underline{1}$$

and *pgf* of $W_{S,j}^2$ is,

$$\psi_{S,j}^2(t) = t \underline{p}'(I - At)^{-1} B_j^2 \underline{1}, j = 0, 1.$$

Now the probability that both events F_i^1 and F_j^2 occurs simultaneously for the first time at m^{th} trial $\left(i.e.\ P(W_{i,j}^{12} = m) \right)$ can be written as follows.

$$P(W_{00}^{12} = m) = P\left((X_{m,k_0^1}^0, X_{m,k_1^1}^1 ; Y_{m,k_0^2}^0, Y_{m,k_1^2}^1) = (1, 0, 1, 0); \right.$$
$$\left. \left(X_{m-1,k_0^1}^0, X_{m-1,k_1^1}^1 ; Y_{m-1,k_0^2}^0, Y_{m-1,k_1^2}^1 \right) = (0, 0, 0, 0) \right)$$
$$= \underline{p}' \left\{ C_m(0, 0, 0, 0) \right\} B_{00}^{12} \underline{1} = \underline{p}' A^{m-1} B_{00}^{12} \underline{1}$$

$$P(W_{01}^{12} = m) = P\left(\left(X_{m,k_0^1}^0, X_{m,k_1^1}^1 ; Y_{m,k_0^2}^0, Y_{m,k_1^2}^1 \right) = (1, 0, 0, 1); \right.$$
$$\left. \left(X_{m-1,k_0^1}^0, X_{m-1,k_1^1}^1 ; Y_{m-1,k_0^2}^0, Y_{m-1,k_1^2}^1 \right) = (0, 0, 0, 0) \right)$$
$$= \underline{p}' \left\{ C_m(0, 0, 0, 0) \right\} B_{01}^{12} \underline{1} = \underline{p}' A^{m-1} B_{01}^{12} \underline{1}$$

$$P(W_{10}^{12} = m) = P\left(\left(X_{m,k_0^1}^0, X_{m,k_1^1}^1 ; Y_{m,k_0^2}^0, Y_{m,k_1^2}^1 \right) = (0, 1, 1, 0); \right.$$
$$\left. \left(X_{m-1,k_0^1}^0, X_{m-1,k_1^1}^1 ; Y_{m-1,k_0^2}^0, Y_{m-1,k_1^2}^1 \right) = (0, 0, 0, 0) \right)$$
$$= \underline{p}' \left\{ C_m(0, 0, 0, 0) \right\} B_{10}^{12} \underline{1} = \underline{p}' A^{m-1} B_{10}^{12} \underline{1}$$

$$P(W_{11}^{12} = m) = P\left((X_{m,k_0^1}^0, X_{m,k_1^1}^1 ; Y_{m,k_0^2}^0, Y_{m,k_1^2}^1) = (0, 1, 0, 1); \right.$$
$$\left. \left(X_{m-1,k_0^1}^0, X_{m-1,k_1^1}^1 ; Y_{m-1,k_0^2}^0, Y_{m-1,k_1^2}^1 \right) = (0, 0, 0, 0) \right)$$
$$= \underline{p}' \left\{ C_m(0, 0, 0, 0) \right\} B_{11}^{12} \underline{1} = \underline{p}' A^{m-1} B_{11}^{12} \underline{1}$$

In general,

$$P(W_{i,j}^{12} = m) = \underline{p}' A^{m-1} B_{ij}^2 \underline{1}, \quad i, j = 0, 1$$

Also *pgf* of W_{ij}^{12} is,

$$\psi^{ij}(t) = t \underline{p}'(I - At)^{-1} B_{ij}^{12} \underline{1}$$

Hence from (5.4), the exact probability distribution of

random variable W_S is given by,

$$P(W_S = m) = \underline{p}' A^{m-1} \left(\sum_{i=1}^{2} \sum_{j=0}^{1} B_j^i + \sum_{i=0}^{1} \sum_{j=0}^{1} B_{ij}^{12} \right) \underline{1},$$

$$m \geq \min\left(k_0^1, k_1^1, k_0^2, k_1^2\right). \qquad (5.5)$$

The *pgf* of W_S is given by,

$$\psi_S(t) = t \, \underline{p}' (I - At)^{-1} \left(\sum_{i=1}^{2} \sum_{j=0}^{1} B_j^i + \sum_{i=0}^{1} \sum_{j=0}^{1} B_{ij}^{12} \right) \underline{1} \quad (5.6)$$

Here we note that it is quiet difficult to study the later waiting time distribution between F_0^1, F_0^2 F_1^1, F_1^2, particularly when one is interested in studying the waiting time distribution for the later occurring event between more than two events. For the waiting time distribution of later occurring event between any two runs in the components of the bivariate trials or for the two patterns in the two components of bivariate trials the theory can be developed in general. For this we refer [13] who derive the waiting time distribution of later occurring event between the two events F_0 and F_1 where F_i $(i=0,1)$ is the event that i-run of length k_i occurs for the first time in the sequence of higher order Markov dependent BT. [13] use the joint distribution of number of occurrences of 1-runs (*i.e.* success-runs) of length k_1 and the number of occurrences of 0-runs (*i.e.* failure runs) of length k_0.

5.2. Waiting Time Distribution for Runs

Let $W_{r_i,i}^j$ be the waiting time for the r_i^{th} occurrence of i-run of length k_i^j for the jth component of bivariate trials, $i=0,1$; $j=1,2$. We obtain the distribution of $W_{r_i,i}^j$ using distribution of $X_{n,k_i^j}^j$ number of occurrences of i-run of length k_i^j for the jth component of n bivariate trials $i=0,1$; $j=1,2$. The *pgf* $\phi_n(t_0, t_1; s_0, s_1)$ of joint distribution of $\left(X_{n,k_0^1}^0, X_{n,k_1^1}^1; X_{n,k_0^2}^0, X_{n,k_1^2}^1\right)$ as obtained in (2.5) is as follows.

$$\phi_n(t_0, t_1; s_0, s_1)$$

$$= \underline{p}' \left(A + \sum_{i=0}^{1} B_i^1 t_i + \sum_{j=0}^{1} B_j^2 s_j + \sum_{i=0}^{1} \sum_{j=0}^{1} B_{ij}^{12} t_i s_j \right)^n \underline{1}$$

In particular *pgf* of marginal distribution of $X_{n,k_0^1}^1$ is obtained by setting $t_1 = s_0 = s_1 = 1$ in the *pgf* $\phi_n(t_0, t_1; s_0, s_1)$ of $\left(X_{n,k_0^1,0}^0, X_{n,k_1^1,1}^1; Y_{n,k_0^2,0}^0, Y_{n,k_1^2,1}^1\right)$ and is as follows.

$$\phi_n(t_0, 1; 1, 1)$$

$$= \underline{p}' \left(A + B_0^1 t_0 + B_1^1 + \sum_{j=0}^{1} B_j^2 + B_{00}^{12} t_0 + B_{01}^{12} t_0 + B_{10}^{12} + B_{11}^{12} \right)^n \underline{1}$$

$$\qquad (5.7)$$

Let *pgf* of $X_{n,k_i^j}^j$ be $\psi_{n,i}^j\left(t_i^j\right)$ for $i=0,1$; $j=1,2$. Then *pgf* of $X_{n,k_0^1}^1$ in (5.7) can be expressed in a simplified form as follows.

$$\psi_{n,0}^1\left(t_0^1\right) = \phi_n\left(t_0 = t_0^1, 1; 1, 1\right) = \underline{p}'\left(A_0^1 + D_0^1 t_0^1\right)^n \underline{1}$$

where $A_0^1 = A + B_1^1 + B_0^2 + B_1^2 + B_{10}^{12} + B_{11}^{12}$ and $D_0^1 = B_0^1 + B_{00}^{12} + B_{01}^{12}$.

The exact probability distribution of $X_{n,k_0^1}^1$ can be obtained from Lemma 4.1 as follows.

$$P\left(X_{n,k_0^1}^1 = r\right) = \underline{p}' C_n(r) \underline{1} \qquad (5.8)$$

where $C_n(r)$ is the coefficient matrix of $\left(t_0^1\right)^r$ in the matrix polynomial $\left(A_0^1 + D_0^1 t_0^1\right)^n$ and in general for $2 \leq m \leq n$ $C_m(r)$ satisfies the recurrent relation $C_m(r) = C_{m-1}(r) A_0^1 + C_{m-1}(r-1) D_0^1$

with $C_1(i) = \begin{cases} A_0^1 & \text{if } i=0 \\ D_0^1 & \text{if } i=1 \\ O & \text{otherwise} \end{cases}$

where O is the null matrix of order same as matrix A_0^1 and D_0^1. The probability in (5.8) can be written as follows.

$$P\left(X_{n,k_0^1}^1 = r\right) = \underline{p}'\left\{C_{n-1}(r) A_0^1 + C_{n-1}(r-1) D_0^1\right\} \underline{1}$$

The above components of $P\left(X_{n,k_0^1}^1 = r\right)$ can be interpreted as follows.

$$\underline{p}'\left\{C_{n-1}(r) A_0^1\right\} \underline{1} = P\left(X_{n,k_0^1}^1 = r; X_{n-1,k_0^1}^1 = r\right)$$

$$\underline{p}'\left\{C_{n-1}(r) D_0^1\right\} \underline{1} = P\left(X_{n,k_0^1}^1 = r; X_{n-1,k_0^1}^1 = r-1\right) \quad (5.9)$$

Now we have,

$$P\left(W_{r_0,0}^1 = m\right) = P\left(X_{m,k_0^1}^1 = r_0; X_{m-1,k_0^1}^1 = r_0 - 1\right)$$

Using the *pgf* of $X_{n,k_0^1}^1$ and interpretations in (5.9) we have,

$$P\left(W_{r_0,0}^1 = m\right) = \underline{p}'\left\{C_{m-1}(r_0 - 1) D_0^1\right\} \underline{1}$$

Particularly when $r_0 = 1$, we have,

$$P\left(W_{1,0}^1 = m\right) = \underline{p}'\left(A^{m-1}\right) D_0^1 \underline{1}$$

The *pgf* of $W_{1,0}^1$ is given by,

$$\xi(s) = s \, \underline{p}'\left(I - A_0^1 s\right)^{-1} D_0^1 \underline{1}$$

and formula for exact probability is

$$P\left(W_{1,0}^1 = m\right) = \underline{p}'\left(A_0^1\right)^{m-1} D_0^1 \underline{1}, \quad m \geq k_0^1.$$

Similarly waiting time distributions of $W_{r_i,i}^j$, $i=0,1$ and $j=1,2$ can be obtained.

6. Numerical Study

In this section we present the numerical study based on the joint distribution of patterns in the sequence of bivariate trials. We consider the sequence

$$\left\{ \begin{pmatrix} X_n \\ Y_n \end{pmatrix}, n \geq 0 \right\} \text{ of } \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \text{-valued Markov}$$

dependent bivariate trials with $\pi_{00} = 1$ with transition probabilities as follows.

$p_{00,00} = 0.2$, $p_{00,01} = 0.3$, $p_{00,10} = 0.4$, $p_{00,11} = 0.1$,

$p_{01,00} = 0.3$, $p_{01,01} = 0.1$, $p_{01,10} = 0.5$, $p_{01,11} = 0.1$,

$p_{10,00} = 0.1$, $p_{10,01} = 0.4$, $p_{10,10} = 0.3$, $p_{10,11} = 0.2$,

$p_{11,00} = 0.2$, $p_{11,01} = 0.2$, $p_{11,10} = 0.2$, $p_{11,11} = 0.4$

Let $\Lambda_1 = 01$ and $\Lambda_2 = 110$. The joint distribution of $\left(X_{n,\Lambda_1}, Y_{n,\Lambda_2} \right)$ is evaluated numerically for $n = 10$ using the algorithm given in Theorem 4.1. The evaluated joint *pmf* of $\left(X_{n,\Lambda_1}, Y_{n,\Lambda_2} \right)$ is described in the following **Table 1**.

7. Application and Extension

[14] introduced the two-dimensional engineering system consisting of a grid of $m \times n$ components arranged in m-rows and n-columns. The system and its components can be either in working or failed state. The system fails if and only if a grid of size $r \times s$ components fails. Particularly, for $m = r + 1$, we can formulate the states of the components in this system as a sequence of n independent bivariate trials. We assume that the ith component in a column is in state 1 if it is in failed state and in state 0 otherwise. For $i = 1, 2, \cdots, n$ we define,

$$X_i = \begin{cases} 1 \text{ if first r components in } i^{th} \text{ column are in failed state} \\ 0 \text{ otherwise} \end{cases}$$

and

$$Y_i = \begin{cases} 1 \text{ if last r components in } i^{th} \text{ column are in failed state} \\ 0 \text{ otherwise} \end{cases}$$

The reliability of consecutive-(r, s)-out-of-$(r+1, n)$: F-Lattice system can now be obtained simply by using the joint distribution of $\left(X_{n,\Lambda_1}, Y_{n,\Lambda_2} \right)$ as,
P(consecutive-(r, s)-out-of-$(r+1, n)$: F-Lattice system works) $= P\left(X_{n,\Lambda_1} = 0, Y_{n,\Lambda_2} = 0 \right)$,
where Λ_1 is 1-run of length s in the first component and Λ_2 is 1-run of length s in the second component of the bivariate trials.

Extending the concept of bivariate trials to multivariate trials, the joint distributions of number of occurrences of patterns Λ_i of length k_i $(i = 1, 2, \cdots, m)$ in the ith component of m-variate trials can be used to get the reliability of general two-dimensional consecutive-$r \times s$-out-of-(m, n): F-Lattice system.

Table 1. Distribution of $\left(X_{10,\Lambda_1}, Y_{10,\Lambda_2} \right)$.

X_{10,Λ_1}	Y_{10,Λ_2}				Sum
	0	1	2	3	
0	0.00257	0.00666	0.00179	3.5E–05	0.01106
1	0.04765	0.08133	0.02006	0.00044	0.14949
2	0.16288	0.21551	0.04793	0.00113	0.42744
3	0.14927	0.15937	0.03184	0.00075	0.34122
4	0.03465	0.0294	0.00497	9.9E–05	0.06912
5	0.00101	0.00058	7.3E–05	1.2E–06	0.00167
Sum	0.39803	0.49285	0.10666	0.00245	1.00000

The *pgf* $\phi_n(t_1, t_2, \cdots, t_m) = \phi_n(\underline{t})$ of joint distribution of $X_{n,\Lambda_i}^i, i = 1, 2, \cdots, m$, the number of occurrences of patterns Λ_i of length k_i in the ith component of m-variate trials obtained in general by using the method of conditional *pgfs* is of form,

$$\phi_n(\underline{t}) = \underline{p}' \left(A + \sum_{i=1}^{n} B_i t_i + \sum_{\substack{i,j=1 \\ i<j}}^{m} B_{ij} t_i t_j + \ldots + B_{12\ldots k} t_1 t_2 \ldots t_k \right)^n \underline{1}$$

(7.1)

Extending the Lemma 4.1 for the *pgf* in (7.1) exact joint probability distribution of $X_{n,\Lambda_i}^i, i = 1, 2, \cdots, m$ can be obtained.

The above study of runs and patterns can be extended in another direction by generalizing the sequence of Markov dependent bivariate trials to the sequence of Markov dependent multivariate trials. In the following, we discuss briefly the method of deriving the joint distribution of $\left(Y_{n,\Lambda_1^2,k_1}, Y_{n,\Lambda_2^2,k_2}, \cdots, Y_{n,\Lambda_r^2,k_r} \right)$ where Y_{n,Λ_i^2,k_i}, $(i = 1, 2, \cdots, r)$ denotes the number of occurrences of two dimensional patterns of rectangular shape Λ_i^2 $(i = 1, 2, \cdots, r)$ in the sequence of Markov dependent multivariate trials.

Let $S_m^2 = \left\{ \underline{X} = (x_1, x_2, \cdots, x_m)' \, | \, x_i = 0, 1; i = 1, 2, \cdots, m \right\}$

so that $\#\left(S_m^2 \right) = 2^m$. Consider the sequence of m-variate S_m^2-valued Markov dependent trials $\{ \underline{X}_i, i \geq 0 \}$. Let the transition probabilities of these Markov dependent trials $\{ \underline{X}_i, i \geq 0 \}$ be,

$$P\left(\underline{X}_r = \underline{y} \, | \, \underline{X}_{r-1} = \underline{x} \right) = P_{\underline{x}\underline{y}}, \quad \underline{x}, \underline{y} \in S_m^2, \quad r \geq 1.$$

and initial probabilities be $P\left(\underline{X}_0 = \underline{x} \right) = \pi_{\underline{x}} \, \forall \, \underline{x} \in S_m^2$.

Consider the two dimensional pattern of rectangular shape, $\Lambda_i^2 = \underline{a}_{i1} \underline{a}_{i2} \cdots \underline{a}_{ik_i}$ $(i = 1, 2, \cdots, r)$ where $\underline{a}_{i1}, \underline{a}_{i2}, \cdots, \underline{a}_{ik_i} \in S_m^2$. Let Y_{n,Λ_i^2,k_i}, $(i = 1, 2, \cdots, r)$ be the number of occurrences of patterns Λ_i^2 in n trials

$\underline{X}_1, \underline{X}_2, \cdots, \underline{X}_n$. We obtain the joint distribution of $\left(Y_{n,\Lambda_1^2,k_1}, Y_{n,\Lambda_2^2,k_2}, \cdots, Y_{n,\Lambda_r^2,k_r} \right)$ using the method for deriving joint distribution of number of occurrences of patterns Λ_i $(i = 1, 2, \cdots, r)$ in the sequence of multi-state trials as done by [11]. For this consider the following transformation of sequence of m-variate sequence of S_m^2-valued Markov dependent trials $\{\underline{X}_i, i \geq 0\}$ to the univariate sequence of Markov dependent trials $\{Z_i, i \geq 0\}$.

Define the function $g : S_m^2 \to S_m^{10}$ such that

$$g(\underline{x}) = \sum_{i=1}^{m} 2^{i-1} x_i \quad \text{where} \quad S_m^{10} = \left\{ g(\underline{x}) \middle| \underline{x} \in S_m^2 \right\}. \text{ Hence}$$

$S_m^{10} = \{0, 1, 2, \cdots, 2^m - 1\}$. Each $\underline{x}$ in S_m^2 can be treated as a m-digit binary number. The function $g(\underline{x})$ converts this m-digit binary number into a unique equivalent decimal number in S_m^{10}.

Now corresponding to the m-variate sequence of S_m^2-valued Markov dependent trials $\{\underline{X}_i, i \geq 0\}$, we have the univariate sequence of S_m^{10}-valued Markov dependent trials $\{Z_i, i \geq 0\}$ where $Z_i = g(\underline{X}_i)$. The transition probabilities for the sequence of trials $\{\underline{X}_i, i \geq 0\}$ and for the converted sequence of trials $\{Z_i, i \geq 0\}$ are related as follows.

$$P\left(\underline{X}_r = \underline{y} \middle| \underline{X}_{r-1} = \underline{x} \right) = P_{\underline{x}\underline{y}}$$
$$= P\left(Z_r = g(\underline{y}) \middle| Z_{r-1} = g(\underline{x}) \right) = p_{ij}$$
$$\underline{x}, \underline{y} \in S_m^2, \ r \geq 1$$

where $g(\underline{x}) = i$ and $g(\underline{y}) = j$. Convert the pattern $\Lambda_i^2 = \underline{a}_{i1} \underline{a}_{i2} \cdots \underline{a}_{ik_i}$ $(i = 1, 2, \cdots, r)$ into a pattern $\Lambda_i^{10} = b_{i1} b_{i2} \cdots b_{ik_i}$ where $b_{ij} = g(\underline{a}_{ij})$ for $j = 1, 2, \cdots, k_i$. Now the original problem of studying the joint distribution of $\left(Y_{n,\Lambda_1^2,k_1}, Y_{n,\Lambda_2^2,k_2}, \cdots, Y_{n,\Lambda_r^2,k_r} \right)$ in the sequence of S_m^2-valued Markov dependent trials $\underline{X}_1, \underline{X}_2, \cdots, \underline{X}_n$ reduces to the problem of studying distribution of $\left(Y_{n,\Lambda_1^{10},k_1}, Y_{n,\Lambda_2^{10},k_2}, \cdots, Y_{n,\Lambda_r^{10},k_r} \right)$ for the sequence of S_m^{10}-valued Markov dependent trials $Z_1, Z_2, \cdots, Z_n$. [15] obtain the joint distribution of number of occurrences of i-runs $(i = 0, 1, 2, \cdots, m)$ of length k_i, while [11] obtain the joint distribution of number of occurrences of patterns in the sequence of $\{0, 1, \cdots, m\}$-valued Markov dependent trials using the method of conditional *pgfs*. The related waiting time distributions can also be studied following the same process as in Section 5 in the case of Markov dependent multivariate trials.

7. References

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