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Hajós-Property for Direct Product of Groups

Khalid Amin

Department of Mathematics, University of Bahrain, Sakhir, Kingdom of Bahrain
Email: kameen@uob.edu.bh

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Abstract

We study decomposition of finite Abelian groups into subsets and show by examples a negative answer to the question of whether Hajós-property is inherited by direct product of groups which have Hajós-property.

Keywords

Abelian Groups, Hajós-Property, Factorization of Abelian Groups

1. Introduction

The general setting is as follows: Suppose we decompose a group G into direct product of subsets $A_1, A_2, \cdots, A_n$ of G in such a way that each element g in G has a *unique* representation of the form $g = a_1 a_2 \cdots a_n$, where $a_i \in A_i$. The question then asked is what we can say about the subsets $A_1, A_2, \cdots, A_n$.

The answer is rather difficult even if we do not impose many restrictions either on G or on the subsets. The most important special case has some connection with a group-theoretial formulation by G. Hajós [1] of a conjecture by H. Minkowski [2]; this is when G is a finite Abelian group and each of the subsets is of the form

$$A_i = \{e, g, g^2, \cdots, g^k\},$$

where $k < |g|$ is an integer; here e denotes the identity element of g and $|g|$ denotes order of the element g of G. Then a result due to Hajos states that one of the subsets A_i must be a subgroup of G. L. Rédei [3] generalizes this result to the case when the condition on the subsets A_i is that they contain a prime number of elements.

Another interesting question has also been asked by Hajos. It is concerned with the case in which G is an Abelian group and $n = 2$; the question then asked is as follows: Suppose G has a decomposition as $G = A_1 A_2$. Does it follow that one of the subsets A_1 or A_2 is a direct product of another subset and a proper subgroup of G?

The concept of Hajós factorization begin group-theoretical but now finds applications in diverse fields such as number theory, [4] coding theory [5] and even in music [6].

2. Preliminaries

Throughout this paper, G will denote a *finite* Abelian group, e the identity of G, and if $g \in G$, then $|g|$ will denote its order. We will also use $|A|$ to denote the number of elements of a subset A of G. A subset A of G of the form $A = \{e, g, g^2, \cdots, g^k\}$ is called a *cyclic* subset of G; here k is an integer with $k < |g|$. If

$$G = A_1 A_2 \cdots A_n$$

we say that we have a factorization of G. If in addition, each of the subsets A_i contains e, we say that we have a *normalized factorization* of G. A subset A of G is called *periodic* if there exists $g \in G - \{e\}$, such that $gA = A$. Such an element $g \in G$ if it exists is called a *period* for A. A group G is said to be of type $(p_1^{\alpha_1}, p_2^{\alpha_2}, \cdots, p_r^{\alpha_r})$, if it is a direct product of cyclic groups of orders $p_1^{\alpha_1}, p_2^{\alpha_2}, \cdots, p_r^{\alpha_r}$, (where of course $p_i's$ are primes and $\alpha_i's$ are non-negative integers).

3. Remarks

1) If $G = AB$ is a factorization of G, then for any $x, y \in G$, $xAyB = G$ is also a factorization of G. Similarly, with $G = A_1 A_2 \cdots A_n$. Thus, we may assume that all factorization we consider are normalized.

2) In the literature, a group G is said to be "good" if from each factorization $G = AB$, it follows that one the subsets A or B is periodic.

We extend the above definition as follows.

4. Definition

A group G has the Hajos-n-property or n-good if from any factorization

$$G = A_1 A_2 \cdots A_n$$

it follows that one of the subsets $A_1, A_2, \cdots, A_n$ is periodic. Otherwise it is n-bad. We will also say G is *totally-good* if it is n-good for all possible values of n.

The following results are known and will be used in this paper.

***Lemma* 1** [7]

If G is of type $(2^2, 2^2)$, then G is 2-good.

***Lemma* 2** [8]

A cyclic group G of order p^α, where $p > 3$ is prime is *totally-good*.

***Lemma* 3** [8]

If G is of type (p^α, p^β), where $1 \le \alpha \le \beta, \beta \ge 2$ and $p > 3$ is prime, then G is n-bad for all n, $2 \le n \le \alpha + \beta - 1$.

***Lemma* 4** [9]

If H is a proper subgroup of G, then there exists a non-periodic set N such that $G = HN$ is a factorization of G, except when H is a subgroup of index 2 in an elementary abelian 2-group.

***Lemma* 5** [7]

If A and B are non-periodic subsets o a group G and A is contained in a subgroup H of G such that $G = HB$ is a factorization of G, then AB is also non-periodic.

5. Results

Theorem 6

If G is of type $(2^2, 2^2)$, then G is *totally-good*.

Proof.

Let $G = A_1 A_2 \cdots A_n$ be a factorization of G.

Now, the possible values for n are 1, 2, 3 and 4.

The case $n = 1$ is trivial.

The case $n = 2$ follows from **Lemma 1**.

The case $n = 4$ follows from Rédei's theorem.

So, we only need details the case $n = 3$. So now, $G = A_1 A_2 A_3$.

We may assume $|A_1| = |A_2| = 2$. Now, $G = A_1(A_2 A_3)$ is also a factorization of *G*. Hence by **Lemma 1**, either A_1 is or $A_2 A_3$ is periodic. If A_1 is periodic, we are done. So assume $A_2 A_3$ is periodic, say with period $g \neq e$. We may assume $|g| = 2$.

Let $A_2 = \{e, x\}$ and $A_3 = \{e, y\}$. Then $A_2 A_3 = \{e, x, y, xy\}$. If $g = x$, then A_2 is a subgroup and hence periodic, while if $g = y$, then A_3 is a subgroup and hence periodic. Suppose $g = xy$, then we must have either 1) $x^2 y = x$ and $xy^2 = y$ both of which give $xy = e$, which is impossible; or 2) $x^2 y = y$ and $xy^2 = x$ both of which imply that both A_1 and A_2 are subgroups of *G*. This ends the proof. □

Theorem 7

If *G* is of type $(2^2, 2^2, 2)$, then *G* is 3-bad.

Proof.

Let $G = \langle x \rangle \times \langle y \rangle \times \langle z \rangle$, where $|x| = |y| = 4$ and $|z| = 2$.

Let $A_1 = \{e, x\}$, $A_2 = \{e, y\}$ and $A_3 = \{e, x^2, xy^2, x^3y^2, z, y^2z, x^2yz, x^2y^3z\}$.

Then $G = A_1 A_2 A_3$ is a factorization of *G* and none of the subsets A_1, A_2 or A_3 is periodic. This ends the proof. □

Theorem 8

Let *H* be a *proper* subgroup of a group *G*. If *H* is *n*-bad, then *G* is both *n* and $(n+1)$-bad.

Proof.

Since *H* is *n*-bad, there is a factorization $H = A_1 A_2 \cdots A_n$ of *H*, where none of the subsets $A_1, A_2, \cdots, A_n$ is periodic. Now, by **Lemma 5**, there is a factorization $G = HA_{n+1}$ of *G*, with A_{n+1} nonperiodic. Hence,

$$G = A_1 A_2 \cdots A_n A_{n+1}$$

is a factorization *G* with none none of the subsets $A_1, A_2, \cdots, A_n, A_{n+1}$ periodic. Thus, *G* is $(n+1)$-bad.

Also, $G = A_1 A_2 \cdots A_{n-1}(A_n A_{n+1})$ is a factorization *G* with none of the subsets periodic. Here, the non-periodicity of the factor $(A_n A_{n+1})$ follows from **Lemma 5**. This ends the proof. □

Theorem 9

If *G* is of type $(2^{\alpha_1}, 2^{\alpha_2}, \cdots, 2^{\alpha_r})$, where $r \geq 3$, $\alpha_1, \alpha_2 \geq 2$, then *G* is both 3 and 4-bad.

Proof.

G has a subgroup *H* of type $(2^2, 2^2, 2)$ which is 3-bad by **Theorem 7**.

So, the result follows from **Theorem 8**. This ends the proof.

Finally, we show by example what we aimed to show.

6. Example 1

Let G_1 be of type $(2^2, 2^2)$. Then by **Lemma 1**, G_1 is 2-good. Now, consider the group $G = G_1 \times G_1$ and note that *G* is of type $(2^2, 2^2, 2^2, 2^2)$ Observe that *G* has a subgroup *H* of type $(2^2, 2^2, 2)$ which is 3-bad by **Theorem 9**. Now, by **Lemma 4**, *G* has a factorization $G = HN$, where *N* is nonperiodic. Hence, *G* has a factorization $G = A_1 A_2 A_3 N$, where none of the factor is periodic. Thus *G* is 4-bad. This ends the proof. □

7. Example 2

Let G_1 be of type (p^α) and G_2 be of type (p^β), where α and β are positive integers and $p > 3$ is prime. Then by **Lemma 2**, G_1 is *m*-good for all *m*, $1 \leq m \leq \alpha$, and G_2 is *n*-good for all *m*, $1 \leq n \leq \beta$. Consider the group $G = G_1 \times G_2$. Then by **Lemma 3**, G_1 is $(m+n)$-bad for all $m+n$, $2 \leq m+n \leq \alpha+\beta$. This ends the proof. □

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