

Asymptotic Behavior of Stochastic Strongly Damped Wave Equation with Multiplicative Noise

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Abstract

In this paper we study the asymptotic dynamics of the stochastic strongly damped wave equation with multiplicative noise under homogeneous Dirichlet boundary condition. We investigate the existence of a compact random attractor for the random dynamical system associated with the equation.

Keywords

Stochastic Strongly Damped Wave Equation, Random Dynamical System, Random Attractor

1. Introduction

Consider the following stochastic strongly damped wave equation with multiplicative noise:

$$u_{tt} - \alpha \Delta u_t + u_t + f(u) - \Delta u = g + cu \circ \frac{dW}{dt} \quad \text{in } U \times [0, +\infty) \quad (1.1)$$

with the homogeneous Dirichlet boundary condition

$$u(x, t) = 0 \quad \text{on } \partial U \times [0, +\infty), \quad (1.2)$$

and the initial value conditions

$$u(x, 0) = u_0(x) \in H_0^1(U), \quad u_t(x, 0) = u_1(x) \in L^2(U), \quad (1.3)$$

where Δ is the Laplacian with respect to the variable $x \in U$, $U \subset \mathbb{R}^n$ is a bounded open set with a smooth

boundary ∂U ; $u = u(x, t)$ is a real function of $x \in U$ and $t \geq 0$; $c \neq 0$, $\alpha > 0$ are strong damping coefficients; \circ denotes the Stratonovich sense of the stochastic term; $g \in L^2(U)$ is a given external force; $f \in C^1(\mathbb{R}, \mathbb{R})$, f' are uniformly bounded and there exist $c_0, c_1, c_2 \geq 0$ such that

$$|f'(u)| \leq c_0, \quad |f(u)| \leq c_1, \quad \forall u \in \mathbb{R}, \tag{1.4}$$

$$|f'(u) - f'(v)| \leq c_2 |u - v|, \quad \forall u, v \in \mathbb{R}, \tag{1.5}$$

where $|\cdot|$ denotes the absolute value of number in \mathbb{R} . $W(t)$ is a one-dimensional two-sided real-valued Wiener process on probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where

$$\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\},$$

the Borel σ -algebra \mathcal{F} on Ω is generated by the compact open topology, and \mathbb{P} is the corresponding Wiener measure on \mathcal{F} . We identify $\omega(t)$ with $W(t)$, i.e., $W(t) = \omega(t), t \in \mathbb{R}$. When $f(u) = \sin u$ and $\alpha = 0$, Equation (1.1) can be regarded as a stochastic perturbed model of a continuous Josephson junction [1], which is stochastic damped sine-Gordon equation [2].

A large amount of studies have been carried out toward the dynamics of a variety of systems related to Equation (1.1). For example, the asymptotical behavior of solutions for deterministic and stochastic wave equations has been studied by many authors, see, e.g. [3]-[27] and the references therein.

In this paper we study the existence of a global random attractor for stochastic strongly damped wave equations with multiplicative noise $c u \circ \frac{dW}{dt}$. The coefficient c of the noise term needs to be suitable small,

which is different from that in stochastic strongly damped wave equations with additive noise, this is because the multiplicative noise depends on the state variable u but the additive noise term is independent of u .

This paper is organized as follows. In the next section, we recall some basic concepts and properties for general random dynamical systems. In Section 3, we provide some basic settings about Equation (1.1) and show that it generates a random dynamical system in proper function space. Section 4 is devoted to proving the existence of a unique random attractor of the random dynamical system.

2. Preliminaries

In this section, we collect some basic knowledge about general random dynamical systems (see [28] [29] for details).

Let $(X, \|\cdot\|_X)$ be a separable Hilbert space with Borel σ -algebra $\mathcal{B}(X)$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space as in Section 1. Define $(\theta_t)_{t \in \mathbb{R}}$ on Ω via

$$\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad t \in \mathbb{R},$$

then $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ is an ergodic metric dynamical system [28] [29].

In the following, a property holds for \mathbb{P} -a.e. $\omega \in \Omega$ means that there is $\Omega_0 \in \Omega$ with $\mathbb{P}(\Omega_0) = 1$ and $\theta_t \Omega_0 = \Omega_0$ for $t \in \mathbb{R}$.

Definition 2.1 A continuous random dynamical system on X over $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ is a mapping

$$\varphi : \mathbb{R}^+ \times \Omega \times X \rightarrow X, \quad (t, \omega, u) \mapsto \varphi(t, \omega, u)$$

which is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$ -measurable and satisfies, for \mathbb{P} -a.e. $\omega \in \Omega$,

- 1) $\varphi(0, \omega, \cdot)$ is the identity on X ;
- 2) $\varphi(t + s, \omega, \cdot) = \varphi(t, \theta_s \omega, \varphi(s, \omega, \cdot))$ for all $s, t \geq 0$;
- 3) $\varphi(t, \omega, \cdot) : X \rightarrow X$ is continuous for all $t \geq 0$.

Definition 2.2 (See [29]).

- 1) A set-valued mapping $\{D(\omega)\} : \Omega \rightarrow 2^X \setminus \{\emptyset\}, \omega \mapsto D(\omega)$ is said to be a random set if the mapping $\omega \mapsto d(u, D(\omega))$ is measurable for any $u \in X$. If $D(\omega)$ is also closed (compact) for each $\omega \in \Omega$, $\{D(\omega)\}$

is called a random closed (compact) set. A random set $\{D(\omega)\}$ is said to be bounded if there exist $u_0 \in X$ and a random variable $R(\omega) > 0$ such that

$$D(\omega) \subset \{u \in X : \|u - u_0\|_X \leq R(\omega)\} \quad \text{for all } \omega \in \Omega.$$

2) A random set $\{D(\omega)\}$ is called tempered provided for \mathbb{P} -a.e. $\omega \in \Omega$,

$$\lim_{t \rightarrow \infty} e^{-\beta t} \sup\{\|b\|_X : b \in D(\theta_{-t}\omega)\} = 0 \quad \text{for all } \beta > 0.$$

3) A random set $\{B(\omega)\}$ is said to be a random absorbing set if for any tempered random set $\{D(\omega)\}$, and \mathbb{P} -a.e. $\omega \in \Omega$, there exists $t_0(\omega)$ such that

$$\varphi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)) \subset B(\omega) \quad \text{for all } t \geq t_0(\omega).$$

4) A random set $\{B_1(\omega)\}$ is said to be a random attracting set if for any tempered random set $\{D(\omega)\}$, and \mathbb{P} -a.e. $\omega \in \Omega$, we have

$$\lim_{t \rightarrow \infty} d_H(\varphi(t, \theta_{-t}\omega, D(\theta_{-t}\omega), B_1(\omega))) = 0,$$

where d_H is the Hausdorff semi-distance given by $d_H(G, F) = \sup_{u \in G} \inf_{v \in F} \|u - v\|_X$ for any $G, F \subset X$.

5) A random compact set $\{A(\omega)\}$ is said to be a random attractor if it is a random attracting set and $\varphi(t, \omega, A(\omega)) = A(\theta_t \omega)$ for \mathbb{P} -a.e. $\omega \in \Omega$ and all $t \geq 0$.

Theorem 2.3 (See [29]). *Let φ be a continuous random dynamical system on X over $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$. If there is a tempered random compact absorbing set $\{B(\omega)\}$ of φ , then $\{A(\omega)\}$ is a compact random attractor of φ , where*

$$A(\omega) = \overline{\bigcap_{t > 0} \bigcup_{\tau \geq t} \varphi(\tau, \theta_{-\tau}\omega, B(\theta_{-\tau}\omega))}, \quad \omega \in \Omega.$$

Moreover, $\{A(\omega)\}$ is the unique random attractor of φ .

3. Stochastic Strongly Damped Wave Equation

In this section, we outline the basic setting of (1.1)-(1.2) and show that it generates a random dynamical system.

Define an unbounded operator

$$A : D(A) \equiv H_0^1(U) \cap H^2(U) \rightarrow L^2(U), \quad u \mapsto -\Delta u.$$

Clearly, A is a self-adjoint, positive linear operator with the eigenvalues $\{\lambda_i\}_{i \in \mathbb{N}}$:

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_i \leq \dots, \lambda_i \rightarrow +\infty (i \rightarrow +\infty).$$

It is well known that $-A$ generates an analytic semigroup of bounded linear operators $\{e^{-At}\}_{t \geq 0}$ on $L^2(U)$.

Let $E = H_0^1(U) \times L^2(U)$, endowed with the usual norm

$$\|Y\|_{H_0^1 \times L^2} = \left(\|\nabla u\|^2 + \|v\|^2 \right)^{\frac{1}{2}} \quad \text{for } Y = (u, v)^T \in E, \tag{3.1}$$

where $\|\cdot\|$ denotes the usual norm in $L^2(U)$ and T stands for the transposition.

It is convenient to reduce (1.1) to an evolution equation of the first order in time

$$\begin{cases} \frac{du}{dt} = v, \\ \frac{dv}{dt} = -\alpha Av - Au - v - f(u) + g + cu \circ \frac{dW}{dt}, \\ u(x, 0) = u_0(x), \quad v(x, 0) = u_1(x), \quad x \in U. \end{cases} \tag{3.2}$$

For our purpose, it is convenient to convert the problems (1.1)-(1.2) into a deterministic system with a random parameter, and then show that it generates a random dynamical system.

We now introduce an Ornstein-Uhlenbeck process given by the Brownian motion. Put

$$z(\theta_t \omega) = -\int_{-\infty}^0 e^s (\theta_t \omega)(s) ds, \quad t \in \mathbb{R}, \quad (3.3)$$

which is called Ornstein-Uhlenbeck process and solves the Itô equation

$$dz + zdt = dW(t). \quad (3.4)$$

From [30] [31], it is known that the random variable $z(\omega)$ is tempered, and there is a θ_t -invariant set $\Omega_0 \subset \Omega$ of full \mathbb{P} measure such that $z(\theta_t \omega)$ is continuous in t for every $\omega \in \Omega_0$.

Lemma 3.1 (See [7]). *For the Ornstein-Uhlenbeck process $z(\theta_t \omega)$ in Equation (3.3), we have the following results*

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t |z(\theta_s \omega)| ds = \mathbf{E} \left[|z(\theta_s \omega)| \right] = \frac{1}{\sqrt{\pi}}, \quad (3.5)$$

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t |z(\theta_s \omega)|^2 ds = \mathbf{E} \left[|z(\theta_s \omega)|^2 \right] = \frac{1}{2}, \quad (3.6)$$

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t |z(\theta_s \omega)|^4 ds = \mathbf{E} \left[|z(\theta_s \omega)|^4 \right] = \frac{3}{4}. \quad (3.7)$$

To show that problem (3.2) generates a random dynamical system, we let

$$\phi_1 = u, \quad \phi_2 = v + \varepsilon u - cu z(\theta_t \omega),$$

which ε is a given positive number, then problems (1.1)-(1.2) can be rewritten as the equivalent system with random coefficients but without multiplicative noise on E ,

$$\begin{cases} \frac{d\phi_1}{dt} = \phi_2 - \varepsilon \phi_1 + c\phi_1 z(\theta_t \omega), \\ \frac{d\phi_2}{dt} = (\varepsilon - 1 - \alpha A)\phi_2 - (\varepsilon(\varepsilon - 1 - \alpha A) + A)\phi_1 - f(\phi_1) + g \\ \quad - \alpha c A \phi_1 z(\theta_t \omega) - c(\phi_2 - 2\varepsilon \phi_1 + c\phi_1 z(\theta_t \omega))z(\theta_t \omega), \\ \phi_1(x, 0) = u_0(x), \quad \phi_2(x, 0) = u_1(x) + \varepsilon u_0(x) - cu_0(x)z(\omega), \quad x \in U, \end{cases} \quad (3.8)$$

which has the following vector form

$$\begin{cases} \frac{d\phi}{dt} = M_\varepsilon \phi + F_\varepsilon(\theta_t \omega, \phi) \\ \phi_0(x, 0) = (u_0(x), u_1(x) + \varepsilon u_0(x) - cu_0(x)z(\omega))^T, \quad x \in U, \end{cases} \quad (3.9)$$

where

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad M_\varepsilon = \begin{pmatrix} -\varepsilon I & I \\ \varepsilon(1-\varepsilon)I + (\varepsilon\alpha - 1)A & (\varepsilon - 1)I - \alpha A \end{pmatrix},$$

$$F_\varepsilon(\theta_t \omega, \phi) = \begin{pmatrix} c\phi_1 z(\theta_t \omega) \\ -f(\phi_1) + g - \alpha c A \phi_1 z(\theta_t \omega) - c(\phi_2 - 2\varepsilon \phi_1 + c\phi_1 z(\theta_t \omega))z(\theta_t \omega) \end{pmatrix}.$$

We will consider Equation (3.8) or (3.9) for $\omega \in \Omega_0$ and write Ω_0 as Ω from now on.

By the classical theory concerning the existence and uniqueness of the solutions [17] [32], one may show that under conditions (1.4)-(1.5), for each $\omega \in \Omega$, problem (3.9) has a unique solution $\phi(t, \omega, \phi_0)$ which is continuous with respect to ϕ_0 in E for all $t \geq 0$. Then the solution mapping

$$\Phi(t, \omega) : \phi_0 \mapsto \phi(t, \omega, \phi_0), \quad \text{for } \phi_0 \in E, t \geq 0 \text{ and for all } \omega \in \Omega. \quad (3.10)$$

generates a continuous random dynamical system over $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ on E .

Introduce the homeomorphism $T(\varepsilon, \theta_t \omega)(u, v)^\top = (u, v + \varepsilon u - c u z(\theta_t \omega))^\top$, whose inverse homeomorphism is $T^{-1}(\varepsilon, \theta_t \omega)(u, v)^\top = (u, v - \varepsilon u + c u z(\theta_t \omega))^\top$, $(u, v)^\top \in E$. Then the transformation

$$\Psi(t, \omega) = T(\varepsilon, \theta_t \omega) \Phi(t, \omega) T^{-1}(\varepsilon, \theta_t \omega), \quad (3.11)$$

also generates a continuous random dynamical system associated with the problem (3.2) on E .

Note that the two random dynamical systems Ψ and Φ are equivalent. By transformation (3.11), it is easy to see that Ψ has a random attractor provided Φ possesses a random attractor. Thus, we only need to consider the random dynamical system Φ .

4. Random Attractor

In this section, we study the existence of a random attractor. Throughout this section we assume that $\mathcal{S}(E)$ is the collection of all tempered random subsets of E and

$$|c| < \frac{2\sigma\sqrt{\pi\lambda_1\mu}}{\varepsilon + \sqrt{\lambda_1\mu} + \sqrt{(\varepsilon + \sqrt{\lambda_1\mu})^2 + 2\sigma\pi(\alpha\lambda_1 + \sqrt{\lambda_1\mu})}}. \quad (4.1)$$

For our purpose, we introduce a new norm $\|\cdot\|_E$ by

$$\|Y\|_E = \left(\mu \left\| A^{\frac{1}{2}} u \right\|^2 + \|v\|^2 \right)^{\frac{1}{2}}, \quad (4.2)$$

for $Y = (u, v)^\top \in E$, where $A^{\frac{1}{2}} = \nabla$ and μ is chosen such that $\mu = 1 - \alpha\varepsilon \in \left(\frac{1}{2}, 1\right)$ in which $\varepsilon \in (0, 1)$ is a small positive number. It is easy to check that $\|\cdot\|_E$ is equivalent to the usual norm $\|\cdot\|_{H_0^1 \times L^2}$ on E in (3.1).

For $Y_i = (u_i, v_i)^\top \in E$, $i = 1, 2$, let

$$\langle Y_1, Y_2 \rangle_E = \mu \left\langle A^{\frac{1}{2}} u_1, A^{\frac{1}{2}} u_2 \right\rangle + \langle v_1, v_2 \rangle, \quad (4.3)$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on $L^2(U)$. By the Poincaré inequality

$$\left\| A^{\frac{1}{2}} u \right\|^2 \geq \lambda_1 \|u\|^2, \quad \forall u \in H_0^1(U), \quad (4.4)$$

Equation (4.3) is then positive definite.

Now, we present a property of the operator M_ε in E that plays an important role in this article.

Lemma 4.1 *Let $E_1 = D(A) \times H_0^1(U)$. There exists a small positive constant $\sigma \in (0, \varepsilon)$ such that for any $Y = (u, v)^\top \in E_1$,*

$$\langle M_\varepsilon Y, Y \rangle_E \leq -\sigma \|Y\|_E^2 - \frac{\alpha}{2} \left\| A^{\frac{1}{2}} v \right\|^2 - \frac{1}{2} \|v\|^2. \quad (4.5)$$

The proof of **Lemma 4.1** is similar to that of Lemma 1 in [24]. We hence omit it here.

Lemma 4.2 *Assume that $g \in H^1(U)$, conditions (1.4), (1.5) and (4.1) hold. Then, there exists a random ball $\{B_0(\omega)\} \in \mathcal{S}(E)$ centered at 0 with random radius $\rho(\omega) > 0$ such that for any $\{\tilde{B}(\omega)\} \in \mathcal{S}(E)$, there is a $T_{\tilde{B}}(\omega, \rho) > 0$ such that for any $\phi_0(\theta_{-t}\omega) \in \tilde{B}(\theta_{-t}\omega)$ satisfies for \mathbb{P} -a.e. $\omega \in \Omega$ and all $t \geq T_{\tilde{B}}(\omega, \rho)$,*

$$\left\| \phi(t, \theta_{-t}\omega, \phi_0(\theta_{-t}\omega)) \right\|_E \leq \rho(\omega). \quad (4.6)$$

Proof. Take the inner product $\langle \cdot, \cdot \rangle_E$ of problem (3.9) with ϕ . By the Cauchy-Schwartz inequality and the Young inequality, we find that

$$\begin{aligned} \mu \left\langle cA^{\frac{1}{2}}\phi_1 z(\theta, \omega), A^{\frac{1}{2}}\phi_1 \right\rangle &\leq |\mu cz(\theta, \omega)| \cdot \left\| A^{\frac{1}{2}}\phi_1 \right\|^2, \\ \langle \alpha cA\phi_1 z(\theta, \omega), \phi_2 \rangle &\leq |\alpha cz(\theta, \omega)| \cdot \left\| A^{\frac{1}{2}}\phi_1 \right\| \cdot \left\| A^{\frac{1}{2}}\phi_2 \right\| \leq \frac{\alpha}{2} |cz(\theta, \omega)|^2 \cdot \left\| A^{\frac{1}{2}}\phi_1 \right\|^2 + \frac{\alpha}{2} \left\| A^{\frac{1}{2}}\phi_2 \right\|^2, \\ \langle f(\phi_1), \phi_2 \rangle &\leq c_1^2 |U| \cdot \|\phi_2\| \leq c_1^4 |U|^2 + \frac{1}{4} \|\phi_2\|^2, \\ \langle g, \phi_2 \rangle &\leq \|g\| \cdot \|\phi_2\| \leq \|g\|^2 + \frac{1}{4} \|\phi_2\|^2, \\ \langle c\phi_2 z(\theta, \omega), \phi_2 \rangle &\leq |cz(\theta, \omega)| \cdot \|\phi_2\|^2, \end{aligned}$$

where $|U|$ is the volume of the set U .

By using the Poincaré inequality (4.4), we have that

$$\begin{aligned} \langle 2c\varepsilon\phi_1 z(\theta, \omega), \phi_2 \rangle &\leq 2|c\varepsilon z(\theta, \omega)| \cdot \|\phi_1\| \cdot \|\phi_2\| \\ &\leq \frac{2|c\varepsilon z(\theta, \omega)|}{\sqrt{\lambda_1}} \left\| A^{\frac{1}{2}}\phi_1 \right\| \cdot \|\phi_2\| \leq \frac{|c\varepsilon z(\theta, \omega)|}{\sqrt{\lambda_1 \mu}} \|\phi\|_E^2, \\ \langle c^2\phi_1 z^2(\theta, \omega), \phi_2 \rangle &\leq |cz(\theta, \omega)|^2 \cdot \|\phi_1\| \cdot \|\phi_2\| \\ &\leq \frac{|cz(\theta, \omega)|^2}{\sqrt{\lambda_1}} \left\| A^{\frac{1}{2}}\phi_1 \right\| \cdot \|\phi_2\| \leq \frac{|cz(\theta, \omega)|^2}{2\sqrt{\lambda_1 \mu}} \cdot \|\phi\|_E^2. \end{aligned}$$

By all the above inequalities and **Lemma 4.1**, we have

$$\frac{d}{dt} \|\phi\|_E^2 + 2\sigma \|\phi\|_E^2 \leq \left(2 \left(1 + \frac{\varepsilon}{\sqrt{\lambda_1 \mu}} \right) |cz(\theta, \omega)| + \left(\frac{\alpha}{\mu} + \frac{1}{\sqrt{\lambda_1 \mu}} \right) |cz(\theta, \omega)|^2 \right) \|\phi\|_E^2 + 2\|g\|^2 + 2c_1^4 |U|^2. \quad (4.7)$$

By the Gronwall lemma, we have that, for all $t \geq 0$,

$$\begin{aligned} \|\phi(t, \omega, \phi_0(\omega))\|_E^2 &\leq e^{\int_0^t \left(-2\sigma + 2 \left(1 + \frac{\varepsilon}{\sqrt{\lambda_1 \mu}} \right) |cz(\theta_s, \omega)| + \left(\frac{\alpha}{\mu} + \frac{1}{\sqrt{\lambda_1 \mu}} \right) |cz(\theta_s, \omega)|^2 \right) ds} \|\phi_0(\omega)\|_E^2 \\ &\quad + 2 \int_0^t \left(\|g\|^2 + c_1^4 |U|^2 \right) e^{\int_s^t \left(-2\sigma + 2 \left(1 + \frac{\varepsilon}{\sqrt{\lambda_1 \mu}} \right) |cz(\theta_\tau, \omega)| + \left(\frac{\alpha}{\mu} + \frac{1}{\sqrt{\lambda_1 \mu}} \right) |cz(\theta_\tau, \omega)|^2 \right) d\tau} ds. \end{aligned} \quad (4.8)$$

By replacing ω by $\theta_{-t}\omega$, we get from problem (4.8) that,

$$\begin{aligned} \|\phi(t, \theta_{-t}\omega, \phi_0(\theta_{-t}\omega))\|_E^2 &\leq e^{\int_0^t \left(-2\sigma + 2 \left(1 + \frac{\varepsilon}{\sqrt{\lambda_1 \mu}} \right) |cz(\theta_{s-t}, \omega)| + \left(\frac{\alpha}{\mu} + \frac{1}{\sqrt{\lambda_1 \mu}} \right) |cz(\theta_{s-t}, \omega)|^2 \right) ds} \|\phi_0(\theta_{-t}\omega)\|_E^2 \\ &\quad + 2 \int_0^t \left(\|g\|^2 + c_1^4 |U|^2 \right) e^{\int_s^t \left(-2\sigma + 2 \left(1 + \frac{\varepsilon}{\sqrt{\lambda_1 \mu}} \right) |cz(\theta_{\tau-t}, \omega)| + \left(\frac{\alpha}{\mu} + \frac{1}{\sqrt{\lambda_1 \mu}} \right) |cz(\theta_{\tau-t}, \omega)|^2 \right) d\tau} ds \\ &\leq e^{\int_{-t}^0 \left(-2\sigma + 2 \left(1 + \frac{\varepsilon}{\sqrt{\lambda_1 \mu}} \right) |cz(\theta_s, \omega)| + \left(\frac{\alpha}{\mu} + \frac{1}{\sqrt{\lambda_1 \mu}} \right) |cz(\theta_s, \omega)|^2 \right) ds} \|\phi_0(\theta_{-t}\omega)\|_E^2 \\ &\quad + 2 \int_{-\infty}^0 \left(\|g\|^2 + c_1^4 |U|^2 \right) e^{\int_s^0 \left(-2\sigma + 2 \left(1 + \frac{\varepsilon}{\sqrt{\lambda_1 \mu}} \right) |cz(\theta_\tau, \omega)| + \left(\frac{\alpha}{\mu} + \frac{1}{\sqrt{\lambda_1 \mu}} \right) |cz(\theta_\tau, \omega)|^2 \right) d\tau} ds. \end{aligned} \quad (4.9)$$

By inequality (4.1), it is easy to see that

$$\frac{|c|}{\sqrt{\pi}} \left(1 + \frac{\varepsilon}{\sqrt{\lambda_1 \mu}} \right) + \frac{c^2}{4} \left(\frac{\alpha}{\mu} + \frac{1}{\sqrt{\lambda_1 \mu}} \right) < \sigma. \quad (4.10)$$

It then follows from inequality (4.10), **Lemma 3.1**, $\phi_0(\theta_{-t}\omega) \in \tilde{B}(\omega)$ and $\{\tilde{B}(\omega)\} \in \mathcal{S}(E)$ that

$$e^{\int_{-t}^0 \left(-2\sigma + 2 \left(1 + \frac{\varepsilon}{\sqrt{\lambda_1 \mu}} \right) |cz(\theta_s \omega)| + \left(\frac{\alpha}{\mu} + \frac{1}{\sqrt{\lambda_1 \mu}} \right) |cz(\theta_s \omega)|^2 \right) ds} \left\| \phi_0(\theta_{-t}\omega) \right\|_E^2 \rightarrow 0, \text{ as } t \rightarrow \infty. \quad (4.11)$$

By **Lemma 3.1**, inequality (4.10) and $g \in H^1(U)$, we have

$$\int_{-\infty}^0 \left(\|g\|^2 + c_1^4 |U|^2 \right) e^{\int_s^0 \left(-2\sigma + 2 \left(1 + \frac{\varepsilon}{\sqrt{\lambda_1 \mu}} \right) |cz(\theta_r \omega)| + \left(\frac{\alpha}{\mu} + \frac{1}{\sqrt{\lambda_1 \mu}} \right) |cz(\theta_r \omega)|^2 \right) dr} ds \leq +\infty. \quad (4.12)$$

We choose

$$\rho^2(\omega) = 4 \int_{-\infty}^0 \left(\|g\|^2 + c_1^4 |U|^2 \right) e^{\int_s^0 \left(-2\sigma + 2 \left(1 + \frac{\varepsilon}{\sqrt{\lambda_1 \mu}} \right) |cz(\theta_r \omega)| + \left(\frac{\alpha}{\mu} + \frac{1}{\sqrt{\lambda_1 \mu}} \right) |cz(\theta_r \omega)|^2 \right) dr} ds. \quad (4.13)$$

Then, for any tempered random set $\{\tilde{B}(\omega)\} \in \mathcal{S}(E)$, there exists a $T_{\tilde{B}}(\omega, \rho) > 0$ such that for any $\phi_0(\theta_{-t}\omega) \in \tilde{B}(\omega)$, satisfies for \mathbb{P} -a.e. $\omega \in \Omega$, $\forall t \geq T_{\tilde{B}}(\omega, \rho)$,

$$\left\| \phi(t, \theta_{-t}\omega, \phi_0(\theta_{-t}\omega)) \right\|_E^2 \leq \rho^2(\omega). \quad (4.14)$$

So, the proof is completed. \square

We now construct a random compact attracting set for RDS Φ . For this purpose, we decompose the solution ϕ of Equation (3.9) with the initial value $\phi_0(\omega) = (u_0, v_0 + \varepsilon u_0 - cu_0 z(\omega))^T$ into two parts

$\phi = \phi^a + \phi^b = (u^a, v^a + \varepsilon u^a)^T + (u^b, v^b + \varepsilon u^b - cu z(\theta_t \omega))^T$, satisfy, respectively

$$\frac{d\phi^a}{dt} = M_\varepsilon \phi^a, \quad \phi_0^a = (u_0, v_0 + \varepsilon u_0)^T, \quad (4.15)$$

$$\frac{d\phi^b}{dt} = M_\varepsilon \phi^b + F_\varepsilon(\theta_t \omega, \phi), \quad \phi_0^b = (0, -cu_0 z(\omega))^T. \quad (4.16)$$

Lemma 4.3 Assume that $g \in H^1(U)$, conditions (1.4), (1.5) and (4.1) hold. Then, for any $\{\tilde{B}(\omega)\} \in \mathcal{S}(E)$, and $\phi_0(\theta_{-t}\omega) \in \tilde{B}(\omega)$, we have for \mathbb{P} -a.e. $\omega \in \Omega$,

$$\left\| \phi^a(t, \theta_{-t}\omega, \phi_0^a(\theta_{-t}\omega)) \right\|_E \rightarrow 0, \text{ as } t \rightarrow \infty, \quad (4.17)$$

and there exist a tempered random variable $\rho_1(\omega) > 0$ and $T_{\tilde{B}}(\omega, \rho_1) > 0$ such that for \mathbb{P} -a.e. $\omega \in \Omega$ and all $t \geq T_{\tilde{B}}(\omega, \rho_1)$,

$$\left\| A^{\frac{1}{2}} \phi^b(t, \theta_{-t}\omega, \phi_0^b(\theta_{-t}\omega)) \right\|_E \leq \rho_1(\omega), \quad (4.18)$$

where ϕ^a and ϕ^b satisfy Equations (4.15), (4.16).

Proof. We first take the inner product $\langle \cdot, \cdot \rangle_E$ of Equation (4.15) with ϕ^a . By **Lemma 4.1**, we obtain

$$\left\| \phi^a(t, \theta_{-t}\omega, \phi_0^a(\theta_{-t}\omega)) \right\|_E^2 \leq e^{-2\sigma t} \left\| \phi_0^a(\theta_{-t}\omega) \right\|_E^2. \quad (4.19)$$

Then by $\phi_0(\theta_{-t}\omega) \in \tilde{B}(\omega)$ and $\{\tilde{B}(\omega)\} \in \mathcal{S}(E)$, we have

$$\left\| \phi^a(t, \theta_{-t}\omega, \phi_0^a(\theta_{-t}\omega)) \right\|_E \rightarrow 0, \text{ as } t \rightarrow \infty. \quad (4.20)$$

Thus, the first assertion is valid.

Next, we take the inner product $\langle \cdot, \cdot \rangle_E$ of Equation (4.16) with $A\phi^b$. From the positivity of the operator A , we easily obtain

$$\langle M_\varepsilon \phi^b, A\phi^b \rangle_E \leq -\sigma \left\| A^{\frac{1}{2}} \phi^b \right\|_E^2 - \frac{\alpha}{2} \|A\phi_2^b\|^2 - \frac{1}{2} \left\| A^{\frac{1}{2}} \phi_2^b \right\|_E^2. \quad (4.21)$$

By the Cauchy-Schwartz inequality and the Young inequality, we find that

$$\begin{aligned} \mu \left\langle cA^{\frac{1}{2}} \phi_1^b z(\theta_t, \omega), A^{\frac{1}{2}} A\phi_1^b \right\rangle &= -\mu \langle cA\phi_1^b z(\theta_t, \omega), A\phi_1^b \rangle \leq |\mu cz(\theta_t, \omega)| \cdot \|A\phi_1^b\|^2, \\ \langle \alpha cA\phi_1^b z(\theta_t, \omega), A\phi_2^b \rangle &\leq \alpha |cz(\theta_t, \omega)| \cdot \|A\phi_1^b\| \cdot \|A\phi_2^b\| \leq \frac{\alpha |cz(\theta_t, \omega)|^2}{2} \|A\phi_1^b\|^2 + \frac{\alpha}{2} \|A\phi_2^b\|^2, \\ \langle f(\phi_1^b), A\phi_2^b \rangle &\leq c_0^2 |U| \cdot \left\| A^{\frac{1}{2}} \phi_2^b \right\| \leq c_0^4 |U|^2 + \frac{1}{4} \left\| A^{\frac{1}{2}} \phi_2^b \right\|_E^2, \\ \langle g, A\phi_2^b \rangle &\leq \left\| A^{\frac{1}{2}} g \right\| \cdot \left\| A^{\frac{1}{2}} \phi_2^b \right\| \leq \left\| A^{\frac{1}{2}} g \right\|^2 + \frac{1}{4} \left\| A^{\frac{1}{2}} \phi_2^b \right\|_E^2, \\ \langle c\phi_2^b z(\theta_t, \omega), A\phi_2^b \rangle &\leq |cz(\theta_t, \omega)| \cdot \left\| A^{\frac{1}{2}} \phi_2^b \right\|_E^2. \end{aligned}$$

By using inequality (4.4), we have that

$$\begin{aligned} \langle 2c\varepsilon\phi_1^b z(\theta_t, \omega), A\phi_2^b \rangle &= \langle 2c\varepsilon A\phi_1^b z(\theta_t, \omega), \phi_2^b \rangle \leq 2\varepsilon |cz(\theta_t, \omega)| \cdot \|A\phi_1^b\| \cdot \|\phi_2^b\| \\ &\leq \frac{2\varepsilon |cz(\theta_t, \omega)|}{\sqrt{\lambda_1}} \cdot \|A\phi_1^b\| \cdot \left\| A^{\frac{1}{2}} \phi_2^b \right\| \\ &\leq \frac{\varepsilon |cz(\theta_t, \omega)|}{\sqrt{\lambda_1 \mu}} \cdot \left\| A^{\frac{1}{2}} \phi^b \right\|_E^2, \\ \langle c^2 \phi_1^b z^2(\theta_t, \omega), A\phi_2^b \rangle &= \langle c^2 A\phi_1^b z^2(\theta_t, \omega), \phi_2^b \rangle \leq |cz(\theta_t, \omega)|^2 \cdot \|A\phi_1^b\| \cdot \|\phi_2^b\| \\ &\leq \frac{|cz(\theta_t, \omega)|^2}{\sqrt{\lambda_1}} \cdot \|A\phi_1^b\| \cdot \left\| A^{\frac{1}{2}} \phi_2^b \right\| \leq \frac{|cz(\theta_t, \omega)|^2}{2\sqrt{\lambda_1 \mu}} \cdot \left\| A^{\frac{1}{2}} \phi^b \right\|_E^2. \end{aligned}$$

Combining all the above inequalities and inequality (4.21), we have

$$\begin{aligned} &\frac{d}{dt} \left\| A^{\frac{1}{2}} \phi^b \right\|_E^2 + 2\sigma \left\| A^{\frac{1}{2}} \phi^b \right\|_E^2 \\ &\leq \left(2 \left(1 + \frac{\varepsilon}{\sqrt{\lambda_1 \mu}} \right) |cz(\theta_t, \omega)| + \left(\frac{\alpha}{\mu} + \frac{1}{\sqrt{\lambda_1 \mu}} \right) |cz(\theta_t, \omega)|^2 \right) \left\| A^{\frac{1}{2}} \phi^b \right\|_E^2 + 2 \left\| A^{\frac{1}{2}} g \right\|^2 + 2c_0^4 |U|^2. \end{aligned} \quad (4.22)$$

Using the Gronwall lemma, for all $t \geq 0$, we get

$$\left\| A^{\frac{1}{2}} \phi^b(t, \omega, \phi_0^b(\omega)) \right\|_E^2 \leq 2 \int_0^t \left(\left\| A^{\frac{1}{2}} g \right\|^2 + c_0^4 |U|^2 \right) e^{\int_s^t \left(-2\sigma + 2 \left(1 + \frac{\varepsilon}{\sqrt{\lambda_1 \mu}} \right) |cz(\theta_\tau, \omega)| + \left(\frac{\alpha}{\mu} + \frac{1}{\sqrt{\lambda_1 \mu}} \right) |cz(\theta_\tau, \omega)|^2 \right) d\tau} ds. \quad (4.23)$$

Replacing ω by $\theta_{-t}\omega$, we get from the above that,

$$\begin{aligned} \left\| A^{\frac{1}{2}} \phi^b(t, \theta_{-t} \omega, \phi_0^b(\theta_{-t} \omega)) \right\|_E^2 &\leq 2 \int_0^t \left(\left\| A^{\frac{1}{2}} g \right\|^2 + c_0^4 |U|^2 \right) e^{\int_s^t \left(-2\sigma + 2 \left(1 + \frac{\varepsilon}{\sqrt{\lambda_1 \mu}} \right) |cz(\theta_{-r} \omega)| + \left(\frac{\alpha}{\mu} + \frac{1}{\sqrt{\lambda_1 \mu}} \right) |cz(\theta_{-r} \omega)|^2 \right) dr} ds \\ &\leq 2 \int_{-\infty}^0 \left(\left\| A^{\frac{1}{2}} g \right\|^2 + c_0^4 |U|^2 \right) e^{\int_s^0 \left(-2\sigma + 2 \left(1 + \frac{\varepsilon}{\sqrt{\lambda_1 \mu}} \right) |cz(\theta_r \omega)| + \left(\frac{\alpha}{\mu} + \frac{1}{\sqrt{\lambda_1 \mu}} \right) |cz(\theta_r \omega)|^2 \right) dr} ds. \end{aligned} \quad (4.24)$$

By **Lemma 3.1**, inequality (4.10) and $g \in H^1(U)$, we have

$$\int_{-\infty}^0 \left(\left\| A^{\frac{1}{2}} g \right\|^2 + c_0^4 |U|^2 \right) e^{\int_s^0 \left(-2\sigma + 2 \left(1 + \frac{\varepsilon}{\sqrt{\lambda_1 \mu}} \right) |cz(\theta_r \omega)| + \left(\frac{\alpha}{\mu} + \frac{1}{\sqrt{\lambda_1 \mu}} \right) |cz(\theta_r \omega)|^2 \right) dr} ds \leq +\infty. \quad (4.25)$$

We can choose

$$\rho_1^2(\omega) = 4 \int_{-\infty}^0 \left(\left\| A^{\frac{1}{2}} g \right\|^2 + c_0^4 |U|^2 \right) e^{\int_s^0 \left(-2\sigma + 2 \left(1 + \frac{\varepsilon}{\sqrt{\lambda_1 \mu}} \right) |cz(\theta_r \omega)| + \left(\frac{\alpha}{\mu} + \frac{1}{\sqrt{\lambda_1 \mu}} \right) |cz(\theta_r \omega)|^2 \right) dr} ds, \quad (4.26)$$

then the second assertion is valid. □

By **Lemma 4.2** and **Lemma 4.3**, for any $\{\tilde{B}(\omega)\} \in \mathcal{D}(E)$, $\phi_0(\theta_{-t} \omega) \in \tilde{B}(\theta_{-t} \omega)$, $t \geq T_{\tilde{B}}(\omega, \rho) + T_{\tilde{B}}(\omega, \rho_1)$, and for some constant $\kappa > 0$, we have for \mathbb{P} -a.e. $\omega \in \Omega$,

$$\left\| \phi^b(t, \theta_{-t} \omega, \phi_0^b(\theta_{-t} \omega)) \right\|_{E_1} \leq \kappa(\rho(\omega) + \rho_1(\omega)), \quad (4.27)$$

where $E_1 = D(A) \times H_0^1(U)$. Let $\{B(\omega)\} \in \mathcal{D}(E)$ be a closed ball of E :

$$B(\omega) = \left\{ \rho(\omega) \in E : \left\| \rho(\omega) \right\|_{E_1} \leq \kappa(\rho(\omega) + \rho_1(\omega)) \right\}. \quad (4.28)$$

Then, by the compact embedding of E_1 into E , $\{B(\omega)\}$ is compact in E .

Note that

$$\phi(t, \theta_{-t} \omega, \phi_0(\theta_{-t} \omega)) = \phi^a(t, \theta_{-t} \omega, \phi_0^a(\theta_{-t} \omega)) + \phi^b(t, \theta_{-t} \omega, \phi_0^b(\theta_{-t} \omega)). \quad (4.29)$$

Then by **Lemma 4.3** and inequality (4.27), we have for \mathbb{P} -a.e. $\omega \in \Omega$,

$$d_H(\phi(t, \theta_{-t} \omega, \tilde{B}(\theta_{-t} \omega)), B(\omega)) \rightarrow 0 \text{ as } t \rightarrow +\infty. \quad (4.30)$$

which implies that $\{B(\omega)\}$ is a random compact attracting set for Φ . It follows from Equations (4.13) and (4.26) that $\{B(\omega)\}$ is tempered. Thus by **Theorem 2.3**, the main result of this section can now be stated as follows.

Theorem 4.4 Assume that $g \in H^1(U)$, conditions (1.4), (1.5) and (4.1) hold. Then, the random dynamical system Φ has a unique compact random attractor $\mathcal{A}(\omega)$ in E , where

$$\mathcal{A}(\omega) = \overline{\bigcap_{t>0} \bigcup_{\tau \geq t} \phi(\tau, \theta_{-\tau} \omega, B(\theta_{-\tau} \omega))}, \quad \omega \in \Omega,$$

in which $\{B(\omega)\}$ is a tempered random compact attracting set for Φ .

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