

Homotopy Analysis Method for Equations of the Type $\nabla^2 u = b(x, y)$ and $\nabla^2 u = b(x, y, u)$

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Abstract

In this paper, the homotopy analysis method (HAM) is presented to solve some of engineering problems. The homotopy analysis method is applied in obtaining exact solutions for equations of the type $\nabla^2 u = b(x, y)$ and $\nabla^2 u = b(x, y, u)$ on an elliptical domain. Exact solutions are presented for several examples involving to demonstrate the applicability and efficiency of HAM.

Keywords

Homotopy Analysis Method, Engineering Problems, Exact Solutions

1. Introduction

The homotopy analysis method is developed in 1992 by Liao [1]-[8]. It is an analytical approach to get the series solution of linear and nonlinear partial differential equations. The difference with the other perturbation methods is that this method is independent of small/large physical parameters. It also provides a simple way to ensure the convergence of series solution [9]. This method has been successfully applied to solve many linear and nonlinear partial differential equations in various fields of science and engineering by many authors [1]-[16]. The homotopy analysis method is useful and efficient for obtaining both analytical and numerical approximations of linear or nonlinear differential equations. In this study, we will concentrate on exact solutions for equations type of $\nabla^2 u = b(x, y)$ and $\nabla^2 u = b(x, y, u)$ frequently used in applied and engineering mathematics.

2. The Engineering Equations on an Elliptical Domain

We refer to the problem given by Partridge and Brebbia [17]. Consider the following engineering equations

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = b(x, y) \quad (1)$$

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$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = b(x, y, u) \tag{2}$$

where $b(x, y)$, that is, considered to be a known function of position and $b(x, y, u)$ will be considered as a known function of the potential.

In all applications, the domain bounded by the ellipse given in **Figure 1** will be used. The boundary condition is the Dirichlet condition with $u = 0$ on the boundary.

The equation of the ellipse is

$$\frac{x^2}{4} + y^2 = 1 \tag{3}$$

3. Homotopy Analysis Method

We apply the HAM to equations of the type $\nabla^2 u = b(x, y)$ and $\nabla^2 u = b(x, y, u)$ with Dirichlet boundary condition. We consider the following differential equation

$$N[u(x, y)] = 0 \tag{4}$$

where N is a linear operator, x and y are independent variables, $u(x, y)$ is an unknown function. In HAM, the zeroth-orderde formation equation is constructed as

$$(1-p)\mathcal{L}[\phi(x, y; p) - u_0(x, y)] = p\hbar N[\phi(x, y; p)] \tag{5}$$

where

$$\phi(x, y; p) = u_0(x, y) + \sum_{m=1}^{\infty} u_m(x, y) p^m \tag{6}$$

\mathcal{L} is an auxiliary parameter, $u_0(x, y)$ is an initial guess, $\hbar \neq 0$ is an auxiliary parameter and $p \in [0, 1]$ is the embedding parameter. Applying the homotopy-derivative [4]

$$u_m(x, y) = \frac{1}{m!} \left. \frac{\partial^m \phi(x, y; p)}{\partial p^m} \right|_{p=0} \tag{7}$$

To both sides of Equation (5), we get the following m th-order deformation equation

$$\mathcal{L}[u_m(x, y) - \chi_m u_{m-1}(x, y)] = \hbar \mathfrak{R}_{m-1} \tag{8}$$

where

$$\mathfrak{R}_{m-1} = \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} N[\phi(x, y; p)]}{\partial p^{m-1}} \right|_{p=0} \tag{9}$$

And

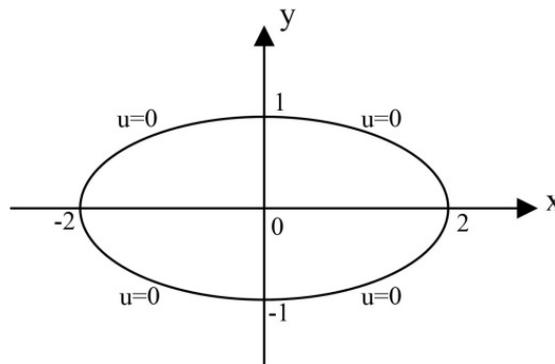


Figure 1. Elliptical domain with Dirichlet boundary condition.

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \tag{10}$$

Note that $u_m(x, y)$ for $m \geq 1$ can be obtained by solving the linear Equation (8) with linear boundary conditions that come from original problem. If the power series Equation (6) of $\phi(x, y; p)$ converges at $p = 1$, then we get the following series solution:

$$u(x, y) = u_0(x, y) + \sum_{m=1}^{\infty} u_m(x, y) \tag{11}$$

4. Applications

We apply Homotopy Analysis Method to equations of the type $\nabla^2 u = b(x, y)$ and $\nabla^2 u = b(x, y, u)$, as follows:

Equation (1) suggests that we define an equation of linear operator as

$$N[\phi(x, y; p)] = \frac{\partial^2 \phi(x, y; p)}{\partial x^2} + \frac{\partial^2 \phi(x, y; p)}{\partial y^2} + b(x, y) \tag{12}$$

And Equation (2) suggests that we define an equation of linear operator as

$$N[\phi(x, y; p)] = \frac{\partial^2 \phi(x, y; p)}{\partial x^2} + \frac{\partial^2 \phi(x, y; p)}{\partial y^2} + b(x, y, \phi(x, y; p)) \tag{13}$$

Using the above definitions, the zeroth-orderde formation equation is constructed as

$$(1 - p)\mathcal{L}[\phi(x, y; p) - u_0(x, y)] = p\hbar N[\phi(x, y; p)]$$

Applying the homotopy-derivative to the zeroth-orderde formation equation, we obtain the following m th-orderde formation equations

$$\mathcal{L}[u_m(x, y) - \chi_m u_{m-1}(x, y)] = \hbar \mathfrak{R}_{m-1}$$

Since $m \geq 1$, $\chi_m = 1$, now the solution of the m th-orderde formation equation becomes

$$u_m(x, y) = u_{m-1}(x, y) + \hbar \mathcal{L}^{-1}[\mathfrak{R}_{m-1}] \tag{14}$$

where

$$\mathfrak{R}_{m-1} = \frac{\partial^2 u_{m-1}(x, y)}{\partial x^2} + \frac{\partial^2 u_{m-1}(x, y)}{\partial y^2} + b(x, y) \tag{15}$$

$$\mathfrak{R}_{m-1} = \frac{\partial^2 u_{m-1}(x, y)}{\partial x^2} + \frac{\partial^2 u_{m-1}(x, y)}{\partial y^2} + b(x, y, u_{m-1}(x, y)) \tag{16}$$

Example 1. Consider the equation of the type $\nabla^2 u = b$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -2 \tag{17}$$

With initial guess

$$u_0(x, y) = -\frac{4}{5}y^2 + \frac{4}{5} \tag{18}$$

using HAM, were cursively obtain

$$u_1(x, y) = \hbar \int_0^x \int_0^x \frac{\partial^2 u_0(x, y)}{\partial x^2} dx dx + \hbar \int_0^x \int_0^x \frac{\partial^2 u_0(x, y)}{\partial y^2} dx dx + \hbar \int_0^x \int_0^x 2 dx dx \tag{19}$$

$$u_1(x, y) = \hbar \frac{1}{5}x^2 \tag{20}$$

$$u_2(x, y) = u_1(x, y) + \hbar \int_0^x \int_0^x \frac{\partial^2 u_1(x, y)}{\partial x^2} dx dx + \hbar \int_0^x \int_0^x \frac{\partial^2 u_1(x, y)}{\partial y^2} dx dx \quad (21)$$

$$u_2(x, y) = (\hbar + \hbar^2) \frac{1}{5} x^2 \quad (22)$$

$$u_3(x, y) = u_2(x, y) + \hbar \int_0^x \int_0^x \frac{\partial^2 u_2(x, y)}{\partial x^2} dx dx + \hbar \int_0^x \int_0^x \frac{\partial^2 u_2(x, y)}{\partial y^2} dx dx \quad (23)$$

$$u_3(x, y) = (\hbar + 2\hbar^2 + \hbar^3) \frac{1}{5} x^2 \quad (24)$$

⋮

$$u(x, y) = -\frac{4}{5} y^2 + \frac{4}{5} + \hbar \frac{1}{5} x^2 + (\hbar + \hbar^2) \frac{1}{5} x^2 + (\hbar + 2\hbar^2 + \hbar^3) \frac{1}{5} x^2 + \dots \quad (25)$$

When $\hbar = -1$, we obtain the exact solution as follows:

$$u(x, y) = -\frac{1}{5} x^2 - \frac{4}{5} y^2 + \frac{4}{5} \quad (26)$$

$$u(x, y) = -\frac{4}{5} \left(\frac{x^2}{4} + y^2 - 1 \right) \quad (27)$$

Example 2. Consider the equation of the type $\nabla^2 u = b(x, y)$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -x \quad (28)$$

with initial guess

$$u_0(x, y) = -\frac{x^3}{14} + \frac{2x}{7} \quad (29)$$

using HAM, were cursively obtain

$$u_1(x, y) = \hbar \int_0^y \int_0^y \frac{\partial^2 u_0(x, y)}{\partial x^2} dy dy + \hbar \int_0^y \int_0^y \frac{\partial^2 u_0(x, y)}{\partial y^2} dy dy + \hbar \int_0^y \int_0^y x dy dy \quad (30)$$

$$u_1(x, y) = \hbar \frac{2x}{7} y^2 \quad (31)$$

$$u_2(x, y) = u_1(x, y) + \hbar \int_0^y \int_0^y \frac{\partial^2 u_1(x, y)}{\partial x^2} dy dy + \hbar \int_0^y \int_0^y \frac{\partial^2 u_1(x, y)}{\partial y^2} dy dy \quad (32)$$

$$u_2(x, y) = (\hbar + \hbar^2) \frac{2x}{7} y^2 \quad (33)$$

$$u_3(x, y) = u_2(x, y) + \hbar \int_0^y \int_0^y \frac{\partial^2 u_2(x, y)}{\partial x^2} dy dy + \hbar \int_0^y \int_0^y \frac{\partial^2 u_2(x, y)}{\partial y^2} dy dy \quad (34)$$

$$u_3(x, y) = (\hbar + 2\hbar^2 + \hbar^3) \frac{2x}{7} y^2 \quad (35)$$

⋮

$$u(x, y) = -\frac{x^3}{14} + \frac{2x}{7} + \hbar \frac{2x}{7} y^2 + (\hbar + \hbar^2) \frac{2x}{7} y^2 + (\hbar + 2\hbar^2 + \hbar^3) \frac{2x}{7} y^2 + \dots \quad (36)$$

When $\hbar = -1$, we obtain the exact solution as follows:

$$u(x, y) = -\frac{x^3}{14} + \frac{2x}{7} - \frac{2x}{7}y^2 \tag{37}$$

$$u(x, y) = -\frac{2x}{7} \left(\frac{x^2}{4} + y^2 - 1 \right) \tag{38}$$

Example 3. Consider the equation of the type $\nabla^2 u = b(x, y, u)$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -u \tag{39}$$

with initial guess

$$u_0(x, y) = x \tag{40}$$

using HAM, were cursively obtain

$$u_1(x, y) = \hbar \int_0^x \int_0^x \frac{\partial^2 u_0(x, y)}{\partial x^2} dx dx + \hbar \int_0^x \int_0^x \frac{\partial^2 u_0(x, y)}{\partial y^2} dx dx + \hbar \int_0^x \int_0^x u_0(x, y) dx dx \tag{41}$$

$$u_1(x, y) = \hbar \frac{x^3}{3!} \tag{42}$$

$$u_2(x, y) = u_1(x, y) + \hbar \int_0^x \int_0^x \frac{\partial^2 u_1(x, y)}{\partial x^2} dx dx + \hbar \int_0^x \int_0^x \frac{\partial^2 u_1(x, y)}{\partial y^2} dx dx + \hbar \int_0^x \int_0^x u_1(x, y) dx dx \tag{43}$$

$$u_2(x, y) = (\hbar + \hbar^2) \frac{x^3}{3!} + \hbar^2 \frac{x^5}{5!} \tag{44}$$

$$u_3(x, y) = u_2(x, y) + \hbar \int_0^x \int_0^x \frac{\partial^2 u_2(x, y)}{\partial x^2} dx dx + \hbar \int_0^x \int_0^x \frac{\partial^2 u_2(x, y)}{\partial y^2} dx dx + \hbar \int_0^x \int_0^x u_2(x, y) dx dx \tag{45}$$

$$u_3(x, y) = (\hbar + 3\hbar^2 + 2\hbar^3) \frac{x^3}{3!} + (\hbar^2 + \hbar^3) \frac{x^5}{5!} + \hbar^3 \frac{x^7}{7!} \tag{46}$$

⋮

$$u(x, y) = x + \hbar \frac{x^3}{3!} + (\hbar + \hbar^2) \frac{x^3}{3!} + \hbar^2 \frac{x^5}{5!} + (\hbar + 3\hbar^2 + 2\hbar^3) \frac{x^3}{3!} + (\hbar^2 + \hbar^3) \frac{x^5}{5!} + \hbar^3 \frac{x^7}{7!} + \dots \tag{47}$$

For $\hbar = -1$, we obtained the closed form series solution as

$$u(x, y) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \tag{48}$$

$$u(x, y) = \sin x \tag{49}$$

which is the exact solution.

Example 4. Consider the equation of the type $\nabla^2 u = b(x, y, u)$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\frac{\partial u}{\partial x} \tag{50}$$

with initial guess

$$u_0(x, y) = 1 - x \tag{51}$$

using HAM, were cursively obtain

$$u_1(x, y) = \hbar \int_0^x \int_0^x \frac{\partial^2 u_0(x, y)}{\partial x^2} dx dx + \hbar \int_0^x \int_0^x \frac{\partial^2 u_0(x, y)}{\partial y^2} dx dx + \hbar \int_0^x \int_0^x \frac{\partial u_0(x, y)}{\partial x} dx dx \quad (52)$$

$$u_1(x, y) = -\hbar \frac{x^2}{2!} \quad (53)$$

$$u_2(x, y) = u_1(x, y) + \hbar \int_0^x \int_0^x \frac{\partial^2 u_1(x, y)}{\partial x^2} dx dx + \hbar \int_0^x \int_0^x \frac{\partial^2 u_1(x, y)}{\partial y^2} dx dx + \hbar \int_0^x \int_0^x \frac{\partial u_1(x, y)}{\partial x} dx dx \quad (54)$$

$$u_2(x, y) = -(\hbar + \hbar^2) \frac{x^2}{2!} - \hbar^2 \frac{x^3}{3!} \quad (55)$$

$$u_3(x, y) = u_2(x, y) + \hbar \int_0^x \int_0^x \frac{\partial^2 u_2(x, y)}{\partial x^2} dx dx + \hbar \int_0^x \int_0^x \frac{\partial^2 u_2(x, y)}{\partial y^2} dx dx + \hbar \int_0^x \int_0^x \frac{\partial u_2(x, y)}{\partial x} dx dx \quad (56)$$

$$u_3(x, y) = -(\hbar + 2\hbar^2 + \hbar^3) \frac{x^2}{2!} - 2(\hbar^2 + \hbar^3) \frac{x^3}{3!} - \hbar^3 \frac{x^4}{4!} \quad (57)$$

⋮

$$u(x, y) = 1 - x - \hbar \frac{x^2}{2!} - (\hbar + \hbar^2) \frac{x^2}{2!} - \hbar^2 \frac{x^3}{3!} - (\hbar + 2\hbar^2 + \hbar^3) \frac{x^2}{2!} - 2(\hbar^2 + \hbar^3) \frac{x^3}{3!} - \hbar^3 \frac{x^4}{4!} - \dots \quad (58)$$

For $\hbar = -1$, we obtained the closed form series solution as

$$u(x, y) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots \quad (59)$$

$$u(x, y) = e^{-x} \quad (60)$$

which is the exact solution.

Example 5. Consider the equation of the type $\nabla^2 u = b(x, y, u)$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} \quad (61)$$

with initial guess

$$u_0(x, y) = e^{-x} + 1 - y \quad (62)$$

using HAM, we recursively obtain

$$u_1(x, y) = \hbar \int_0^y \int_0^y \frac{\partial^2 u_0(x, y)}{\partial x^2} dy dy + \hbar \int_0^y \int_0^y \frac{\partial^2 u_0(x, y)}{\partial y^2} dy dy + \hbar \int_0^y \int_0^y \frac{\partial u_0(x, y)}{\partial x} dy dy + \hbar \int_0^y \int_0^y \frac{\partial u_0(x, y)}{\partial y} dy dy \quad (63)$$

$$u_1(x, y) = -\hbar \frac{y^2}{2!} \quad (64)$$

$$u_2(x, y) = u_1(x, y) + \hbar \int_0^y \int_0^y \frac{\partial^2 u_1(x, y)}{\partial x^2} dy dy + \hbar \int_0^y \int_0^y \frac{\partial^2 u_1(x, y)}{\partial y^2} dy dy + \hbar \int_0^y \int_0^y \frac{\partial u_1(x, y)}{\partial x} dy dy + \hbar \int_0^y \int_0^y \frac{\partial u_1(x, y)}{\partial y} dy dy, \quad (65)$$

$$u_2(x, y) = -(\hbar + \hbar^2) \frac{y^2}{2!} - \hbar^2 \frac{y^3}{3!} \quad (66)$$

$$u_3(x, y) = u_2(x, y) + \hbar \int_0^y \int_0^y \frac{\partial^2 u_2(x, y)}{\partial x^2} dy dy + \hbar \int_0^y \int_0^y \frac{\partial^2 u_2(x, y)}{\partial y^2} dy dy + \hbar \int_0^y \int_0^y \frac{\partial u_2(x, y)}{\partial x} dy dy + \hbar \int_0^y \int_0^y \frac{\partial u_2(x, y)}{\partial y} dy dy, \quad (67)$$

$$u_3(x, y) = -(\hbar + 2\hbar^2 + \hbar^3) \frac{y^2}{2!} - 2(\hbar^2 + \hbar^3) \frac{y^3}{3!} - \hbar^3 \frac{y^4}{4!} \quad (68)$$

$$\vdots$$

$$u(x, y) = e^{-x} + 1 - y - \hbar \frac{y^2}{2!} - (\hbar + \hbar^2) \frac{y^2}{2!} - \hbar^2 \frac{y^3}{3!} - (\hbar + 2\hbar^2 + \hbar^3) \frac{y^2}{2!} - 2(\hbar^2 + \hbar^3) \frac{y^3}{3!} - \hbar^3 \frac{y^4}{4!} - \dots \quad (69)$$

For $\hbar = -1$, we obtained the closed form series solution as

$$u(x, y) = e^{-x} + 1 - y + \frac{y^2}{2!} - \frac{y^3}{3!} + \frac{y^4}{4!} - \dots \quad (70)$$

$$u(x, y) = e^{-x} + e^{-y} \quad (71)$$

which is the exact solution.

5. Conclusion

In this paper, the homotopy analysis method has been applied to solve some of engineering problems defined on an elliptical domain. Exact solutions for equations of the type $\nabla^2 u = b(x, y)$ and $\nabla^2 u = b(x, y, u)$ are obtained using the HAM. Obviously for $\hbar = -1$ the obtained solutions are as the same Reference [17]. There suits show that HAM is very efficient technique in finding the exact solutions for equations of the type $\nabla^2 u = b(x, y)$ and $\nabla^2 u = b(x, y, u)$ having wide applications in engineering mathematics.

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