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The Rise of Solitons in Sine-Gordon Field Theory: From Jacobi Amplitude to Gudermannian Function

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Abstract

We show how the famous soliton solution of the classical sine-Gordon field theory in (1 + 1)-dimensions may be obtained as a particular case of a solution expressed in terms of the Jacobi amplitude, which is the inverse function of the incomplete elliptic integral of the first kind.

Keywords

Solitons, Sine-Gordon Field Theory, Elliptic Integrals, Jacobi Amplitude

1. Introduction

The sine-Gordon field theory and the associated massive Thirring model [1] are some of the best studied quantum field theories. In view of its connections to other important physical models, some of which in principle admit actual realizations in nature [2] [3], a huge mass of important exact results have been obtained for this fascinating integrable system [4]-[7]. However, no less fascinating are the remarkable mathematical and physical properties of its soliton (or "solitary wave") solutions which have contributed, along the last decades, to turning the physics of solitons into a very active research topic.

In this work we present a simple and yet appealing step-by-step derivation of a more general solution for the classical sine-Gordon field theory in (1 + 1)-dimensions in terms of a special kind of elliptic function, namely the Jacobi amplitude, which has the famous sine-Gordon soliton solution as a particular case. Despite the fact that the connection between solitons and Jacobi elliptic functions has already been explored in [8], we believe that this work comes to shed more light on this interesting subject, helping to fill in a gap existing in the corres-

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ponding specialized literature.

2. An Alternative Pathway to Solitons in Sine-Gordon Field Theory

2.1. The Jacobi Amplitude Function

We start by considering the following theory describing a real scalar field in (1 + 1)-dimensions $\left(\phi \equiv \phi(x,t)\right)$,

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi), \tag{1}$$

where the potential term is given by

$$V(\phi) = 2\alpha\cos(\beta\phi) + 2\gamma, \tag{2}$$

with α, β and γ being real parameters.

The above Lagrangian gives rise, through the Euler-Lagrange equation, $\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = \frac{\partial\mathcal{L}}{\partial\phi}$, to the following field equation

$$\partial_\mu\partial^\mu\phi \equiv \Box\phi \equiv \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\phi = -\frac{\partial V(\phi)}{\partial\phi}. \tag{3}$$

Notice that since Equation (3) is invariant under Lorentz transformations $\left(x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu\right)$ [9], its solutions may be obtained through the solutions of the corresponding equation for the static case $\left(\phi \equiv \phi(x)\right)$ by a simple Lorentz boost, namely $x - x_0 \to \left(x - x_0 - vt\right)\Big/\sqrt{1-\left(v^2/c^2\right)}$, for arbitrary v $\left(|v| < c \approx 3\times10^8\ \mathrm{m/s}\right)$ [10] [11]. Thus, in what follows, we will focus on the solutions of the equation

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} = \frac{\mathrm{d}V}{\mathrm{d}\phi}. \tag{4}$$

Indeed, by multiplying the above equation by $\mathrm{d}\phi/\mathrm{d}x$ we obtain

$$\frac{\mathrm{d}\phi}{\mathrm{d}x}\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} = \frac{\mathrm{d}\phi}{\mathrm{d}x}\frac{\mathrm{d}V}{\mathrm{d}\phi} \Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)^2\right] = \frac{\mathrm{d}V}{\mathrm{d}x}, \tag{5}$$

which, after an integration with respect to x and some algebra, may be rewritten as

$$\mathrm{d}x' = \pm\frac{\mathrm{d}\phi'}{\sqrt{2V(\phi')}}. \tag{6}$$

By integrating both sides of the above equation, from $x' = x_0$ to $x' = x$ ($\phi' = \phi(x_0)$ to $\phi' = \phi(x)$), we get

$$x - x_0 = \pm\int_{\phi(x_0)}^{\phi(x)}\frac{\mathrm{d}\phi'}{\sqrt{2V(\phi')}}. \tag{7}$$

In order to compute the above integral, we must firstly notice that the potential, shown in Equation (2), may be rewritten as

$$V(\phi') = 2(\alpha+\gamma)\left[1 - \frac{2\alpha}{\alpha+\gamma}\sin^2\left(\frac{\beta\phi'}{2}\right)\right]. \tag{8}$$

Thus, by making the change of variables $\phi' \to \theta' = \frac{\beta}{2}\phi'$, defining $k^2 = \frac{2\alpha}{\alpha+\gamma}$ and choosing x_0 such that $\phi(x_0) = 0 \Rightarrow \theta_0 = 0$, we are left with

$$x - x_0 = \pm\frac{k}{\beta\sqrt{2\alpha}}\int_0^\theta\frac{\mathrm{d}\theta'}{\sqrt{1-k^2\sin^2\theta'}}. \tag{9}$$

The integral appearing in Equation (9) is called an incomplete elliptic integral of the first kind, $F(\theta,k)$, whereas k is called the elliptic modulus or eccentricity. The upper limit, θ, of this integral may be written in terms of the *Jacobi amplitude* (the inverse function of the incomplete elliptic integral of the first kind) as [12] [13]

$$\theta = \pm F^{-1}\left(\frac{\beta\sqrt{2\alpha}}{k}(x-x_0),k\right) \equiv \pm\mathrm{am}\left(\frac{\beta\sqrt{2\alpha}}{k}(x-x_0),k\right). \tag{10}$$

Notice that, from the above definition, we have $F\left(\mathrm{am}(x,k),k\right)=x$.

The solution of Equation (4) may be, finally, written as

$$\phi(x) = \pm\frac{2}{\beta}\mathrm{am}\left(\frac{\beta\sqrt{2\alpha}}{k}(x-x_0),k\right). \tag{11}$$

Hence, from the above equation, we may notice that

$$\phi(x_0) = \pm\frac{2}{\beta}\mathrm{am}(0,k) = 0, \tag{12}$$

as it should.

2.2. The Case $k=1$: The Gudermannian Function and the Soliton Solution of Sine-Gordon Equation

From the definition $k^2 = 2\alpha/(\alpha+\gamma)$ we may obviously see that when $\gamma=\alpha$ we have $k=1$. Hence, the solution for Equation (4) with the potential given by

$$V(\phi) = 2\alpha\left[1+\cos(\beta\phi)\right], \tag{13}$$

may be obtained as a special case of the solution presented in Equation (11). Indeed, since

$$\mathrm{am}(x,1) = \mathrm{gd}\,x \equiv 2\arctan\left(\mathrm{e}^x\right) - \pi/2,$$

where $\mathrm{gd}\,x$ is called the *Gudermannian function* (a special function which relates the circular functions to the hyperbolic ones without using complex numbers, named after Christoph Gudermann (1798-1852)), we are left with

$$\begin{aligned}\phi(x) &= \pm\frac{2}{\beta}\mathrm{am}\left(\beta\sqrt{2\alpha}(x-x_0),1\right) = \pm\frac{2}{\beta}\mathrm{gd}\left(\beta\sqrt{2\alpha}(x-x_0)\right)\\ &\equiv \pm\frac{4}{\beta}\arctan\left[\exp\left(\beta\sqrt{2\alpha}(x-x_0)\right)\right] \mp \frac{\pi}{\beta}.\end{aligned} \tag{14}$$

Last but not least, we must notice that by substituting the Equation (14) into Equation (3) and making the change (Lorentz boost) $x-x_0 \to (x-x_0-vt)\Big/\sqrt{1-(v^2/c^2)}$, we obtain the famous sine-Gordon field equation, namely

$$\Box\phi_S + 2\alpha\beta\sin\beta\phi_S = 0, \tag{15}$$

where $\phi_S \equiv \phi_S(x,t)$ is the no less famous soliton/anti-soliton solution [10] [11], given by

$$\phi_S(x,t) = \pm\frac{4}{\beta}\arctan\left[\exp\left(\beta\sqrt{2\alpha}\frac{x-x_0-vt}{\sqrt{1-(v^2/c^2)}}\right)\right]. \tag{16}$$

This result allows us to characterize the Lorentz boosted, and shifted by π/β, version of the solution in terms of the Jacobi amplitude shown in Equation (11), namely

$$\phi(x,t) = \pm\frac{2}{\beta}\mathrm{am}\left(\frac{\beta\sqrt{2\alpha}}{k}\frac{x-x_0-vt}{\sqrt{1-(v^2/c^2)}},k\right) \pm \frac{\pi}{\beta}, \tag{17}$$

as a generalization of the sine-Gordon soliton/anti-soliton solution for $k \neq 1$.

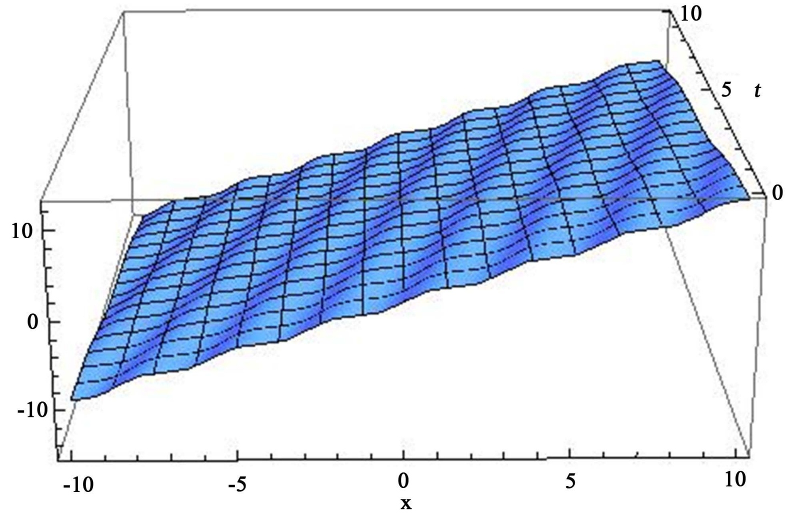

Figure 1. The Jacobi amplitude solution given by Equation (17) with $\alpha = 0.50$, $\beta = 2.00$, $x_0 = 0$, $k = 0.99$ and $v = 0.50c$.

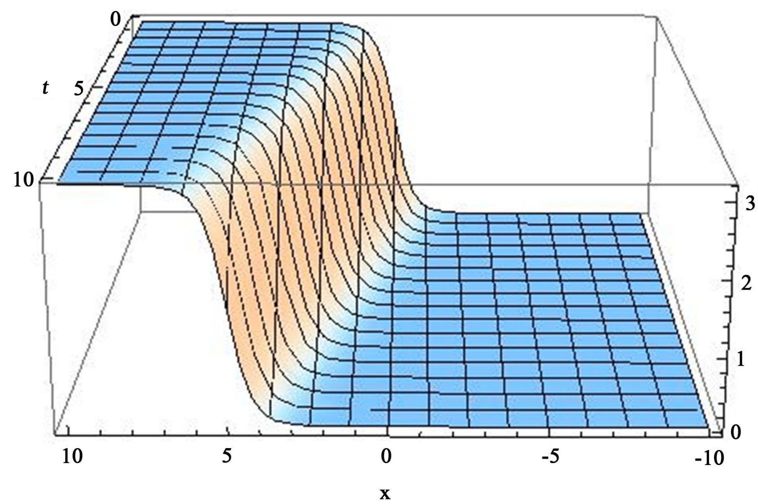

Figure 2. The soliton solution given by Equation (16) with $\alpha = 0.50$, $\beta = 2.00$, $x_0 = 0$, $k = 1.00$ and $v = 0.50c$.

3. Concluding Remarks

We would like to make a few comments about the soliton solution, shown in Equation (16), and its generalized version, shown in Equation (17). Firstly, we may notice by comparing **Figure 1** and **Figure 2** how different are these solutions, where we would like to highlight the doubly periodic behaviour of the Jacobi amplitude solution.

Finally, let us observe that, as remarked in [10], this soliton solution, though arising in a classical field theory, looks very much like a classical particle since its energy density is localized at a point $(x = x_0)$ and its total energy for a static field configuration $(\phi_S \equiv \phi_S(x))$, namely

$$E(\phi_S) = \int_{-\infty}^{\infty} \mathrm{d}x \left[\frac{1}{2} \left(\frac{\mathrm{d}\phi}{\mathrm{d}x} \right)^2 + V(\phi) \right] = \frac{8\sqrt{2\alpha}}{\beta}, \tag{18}$$

is finite, just as we should expect.

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