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Positive Solutions for Singular Boundary Value Problems of Coupled Systems of Nonlinear Differential Equations

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Abstract

We establish the existence of positive solutions for singular boundary value problems of coupled systems

$$
\begin{cases}
-\left(p_1(t)u' \right)' - q_1(t)u = f_1(t,v) + e_1(t), \\
-\left(p_2(t)v' \right)' - q_2(t)v = f_2(t,u) + e_2(t), \\
\alpha_1 u(0) - \beta_1 u'(0) = 0, \quad \gamma_1 u(1) + \delta_1 u'(1) = 0, \\
\alpha_2 v(0) - \beta_2 v'(0) = 0, \quad \gamma_2 v(1) + \delta_2 v'(1) = 0.
\end{cases}
$$

The proof relies on Schauder's fixed point theorem. Some recent results in the literature are generalized and improved.

Keywords

Positive Solutions, Second-Order Boundary Value Problems, Coupled Systems, Schauder's Fixed Point Theorem

1. Introduction

In this paper, we consider the existence of positive solutions for coupled singular system of second order ordinary differential equations

$$\begin{cases} -\left(p_1(t)u'\right)' - q_1(t)u = f_1(t,v) + e_1(t), \\ -\left(p_2(t)v'\right)' - q_2(t)v = f_2(t,u) + e_2(t), \\ \alpha_1 u(0) - \beta_1 u'(0) = 0, \quad \gamma_1 u(1) + \delta_1 u'(1) = 0, \\ \alpha_2 v(0) - \beta_2 v'(0) = 0, \quad \gamma_2 v(1) + \delta_2 v'(1) = 0. \end{cases} \tag{1.1}$$

Throughout this paper, we always suppose that

(S_1) $p_i(t) \in C^1([0,1], R)$, $p_i(t) > 0$, $q_i(t) \in C([0,1], R)$, $q_i(t) \le 0$, $e_i(t) \in C([0,1], R)$ $\alpha_i, \beta_i, \gamma_i, \delta_i \ge 0$,

and $\beta_i \gamma_i + \alpha_i \gamma_i + \alpha_i \delta_i > 0$ $(i = 1, 2)$. $f_1, f_2 \in C([0,1] \times (0, +\infty), (0, +\infty))$, and may be singular near the zero.

In recent years, singular boundary value problems to second ordinary differential equations have been studied extensively (see [1]-[3]). Some classical tools have been used in the literature to study the positive solutions for second order singular boundary value problems of a coupled system of differential equations. These classical methods include some fixed point theorems in cones for completely continuous operators and Schauder fixed point theorem, for example, see [4]-[6] and literatures therein. Motivated by the recent work on coupled systems of second-order differential equations, we consider the existence of singular boundary value problem. By means of the Schauder fixed point theorem, we study the existence of positive solutions of coupled system (1.1).

2. Preliminary

We consider the scalar equation

$$-\left(p(t)u'\right)' - q(t)u = e(t). \tag{2.1}$$

with boundary conditions

$$\alpha u(0) - \beta u'(0) = 0, \quad \gamma u(1) + \delta u'(1) = 0, \tag{2.2}$$

Suppose that u is a positive solution of (2.1) and (2.2). Then

$$u(t) = \int_0^1 G(t,s) e(s) \, ds.$$

where $G(t,s)$ can be written by

$$G(t,s) := \frac{1}{\omega} \begin{cases} m(t)n(s), & (t,s) \in Q_1, \\ m(s)n(t), & (t,s) \in Q_2. \end{cases}$$

here $I = [0,1]$, $Q = I \times I$ and $Q_1 = \{(t,s) \in Q \mid 0 \le t \le s \le 1\}$, $Q_2 = \{(t,s) \in Q \mid 0 \le s \le t \le 1\}$.

Lemma 2.1. Suppose that (S_1) holds, then the Green's function $G(t,s)$, defined by (2.3) possesses the following properties:
1): $m(t) \in C^2(I, R)$ is increasing and $m(t) > 0$, $x \in (0,1]$.
2): $m(t) \in C^2(I, R)$ is decreasing and $n(t) > 0$, $x \in [0,1)$.
3): $(Lm)(t) \equiv 0$, $m(0) = \beta$, $m'(0) = \alpha$.
4): $(Ln)(t) \equiv 0$, $n(1) = \delta$, $n'(1) = -\gamma$.
5): ω is a positive constant. Moreover, $p(t)\left(m'(t)n(t) - m(t)n'(t)\right) \equiv \omega$.
6): $G(t,s)$ is continuous and symmetrical over Q.
7): $G(t,s)$ has continuously partial derivative over Q_1, Q_2.
8): For each fixed $s \in I$, $G(t,s)$ satisfies $LG(t,s) = 0$ for $s \ne t$, $t \in I$. Moreover, $R_1(G) = R_2(G) = 0$ for $s \in (0,1)$.
9): G_t' has discontinuous point of the first kind at $t = s$ and

$$G_t'(s+0,s) - G_t'(s-0,s) = -\frac{1}{p(s)}, \quad s \in (0,1).$$

We define the function $\gamma_i(t):[0,1] \to R$ by

$$\gamma_i(t) = \int_0^1 G_i(t,s) e_i(s) \, ds, \qquad i = 1, 2,$$

which is the unique solution of

$$\begin{cases} -\left(p_i(t) u'(t)\right)' - q_i(t) u(t) = e_i(t), \, 0 < t < 1, \; i = 1, 2 \\ \alpha_i u(0) - \beta_i u'(0) = 0, \\ \gamma_i u(1) + \delta_i u'(1) = 0. \end{cases}$$

Following from Lemma 2.1 and (S_1), it is easy to see that

$$G_i(t,s) > 0, \text{ for all } (t,s) \in [0,1] \times [0,1], \; i = 1, 2$$

Let us fix some notation to be used in the following: For a given function $h \in C[0,1]$, we denote the essential supremum and infimum by h^* and h_*. if they exist. Let, $\gamma_{i^*} = \min_t \gamma_i(t)$, $\gamma_i^* = \max_t \gamma_i(t)$.

3. Main Results

1) $\gamma_{1^*} \geq 0$, $\gamma_{2^*} \geq 0$.

Theorem 3.1. We assume that there exists $b_i \geq 0$, $\hat{b}_i \geq 0$, and $0 < \alpha_i < 1$ such that

$$(H_1) \qquad \frac{\hat{b}_i(t)}{u^{\alpha_i}} \leq f_i(t,u) \leq \frac{b_i(t)}{u^{\alpha_i}}, \text{ for all } u > 0, \; a.e. \; t \in (0,1), \; i = 1, 2$$

If $\gamma_{1^*} \geq 0$, $\gamma_{2^*} \geq 0$, then there exists a positive solution of (1.1).

Proof A positive solution of (1.1) is just a fixed point of the completely continuous map $A(u,v) = (A_1 u, A_2 v): C[0,1] \times C[0,1] \to C[0,1] \times C[0,1]$ defined as

$$(A_1 u)(t) := \int_0^1 G_1(t,s) \left[f_1(s, v(s)) + e_1(s) \right] ds = \int_0^1 G_1(t,s) f_1(s, v(s)) \, ds + \gamma_1(t);$$

$$(A_2 v)(t) := \int_0^1 G_2(t,s) \left[f_2(s, u(s)) + e_2(s) \right] ds = \int_0^1 G_2(t,s) f_2(s, u(s)) \, ds + \gamma_2(t);$$

By a direct application of Schauder's fixed point theorem, the proof is finished if we prove that A maps the closed convex set defined as

$$K = \left\{ (u,v) \in C[0,1] \times C[0,1] : r_1 \leq u(t) \leq R_1, r_2 \leq v(t) \leq R_2, \text{ for all } t \in [0,1] \right\}$$

into itself, where $R_1 > r_1 > 0$, $R_2 > r_2 > 0$ are positive constants to be fixed properly. For convenience, we introduce the following notations

$$\beta_i(t) = \int_0^1 G_i(t,s) b_i(s) \, ds, \quad \hat{\beta}_i(t) = \int_0^1 G_i(t,s) \hat{b}_i(s) \, ds, \; i = 1, 2.$$

Given $(u,v) \in K$, by the nonnegative sign of G_i and f_i, $i = 1, 2$ we have

$$(A_1 u)(t) = \int_0^1 G_1(t,s) f_1(s, v(s)) \, ds + \gamma_1(t) \geq \int_0^1 G_1(t,s) \frac{\hat{b}_1(s)}{v^{\alpha_1}(s)} \, ds \geq \int_0^1 G_1(t,s) \frac{\hat{b}_1(s)}{R_2^{\alpha_1}} \, ds \geq \hat{\beta}_{1^*} \frac{1}{R_2^{\alpha_1}}$$

Note for every $(u,v) \in K$

$$(A_1 u)(t) = \int_0^1 G_1(t,s) f_1(s, v(s)) \, ds + \gamma_1(t) \leq \int_0^1 G_1(t,s) \frac{b_1(s)}{v^{\alpha_1}(s)} \, ds + \gamma_1^* \leq \int_0^1 G_1(t,s) \frac{b_1(s)}{r_2^{\alpha_1}} \, ds + \gamma_1^* \leq \beta_1^* \frac{1}{r_2^{\alpha_1}} + \gamma_1^*$$

Similarly, by the same strategy, we have

$$(A_2 v)(t) = \int_0^1 G_2(t,s) f_2(s, u(s)) \, ds + \gamma_2(t) \geq \int_0^1 G_2(t,s) \frac{\hat{b}_2(s)}{u^{\alpha_2}(s)} \, ds \geq \int_0^1 G_2(t,s) \frac{\hat{b}_2(s)}{R_1^{\alpha_2}} \, ds \geq \hat{\beta}_{2^*} \frac{1}{R_1^{\alpha_2}}$$

$$(A_2 v)(t) = \int_0^1 G_2(t,s) f_2(s, u(s)) ds + \gamma_2(t) \leq \int_0^1 G_2(t,s) \frac{b_2(s)}{u^{\alpha_2}(s)} ds + \gamma_2^* \leq \int_0^1 G_2(t,s) \frac{b_2(s)}{r_1^{\alpha_2}} ds + \gamma_2^* \leq \beta_2^* \frac{1}{r_1^{\alpha_2}} + \gamma_2^*$$

Thus $(A_1 u, A_2 v) \in K$ if r_1, r_2, R_1, R_2 are chosen so that

$$\hat{\beta}_{1*} \cdot \frac{1}{R_2^{\alpha_1}} \geq \gamma_1, \quad \beta_1^* \cdot \frac{1}{r_2^{\alpha_1}} + \gamma_1^* \leq R_1$$

$$\hat{\beta}_{2*} \cdot \frac{1}{R_1^{\alpha_2}} \geq \gamma_2, \quad \beta_2^* \cdot \frac{1}{r_1^{\alpha_2}} + \gamma_2^* \leq R_1.$$

Note that $\hat{\beta}_{i*} > 0$, $\beta_{i*} > 0$ and taking $R = R_1 = R_2$, $r = r_1 = r_2$, $r = \frac{1}{R}$, it is sufficient to find $R > 1$ such that

$$\hat{\beta}_{1*} \cdot R^{1-\alpha_1} \geq 1, \quad \beta_1^* \cdot R^{\alpha_1} + \gamma_1^* \leq R$$

$$\hat{\beta}_{2*} \cdot R^{1-\alpha_2} \geq 1, \quad \beta_2^* \cdot R^{\alpha_2} + \gamma_2^* \leq R.$$

and these inequalities hold for R big enough because $\alpha_i < 1$.

2) $\gamma_1^* \leq 0$, $\gamma_2^* \leq 0$.

The aim of this section is to show that the presence of a weak singular nonlinearity makes it possible to find positive solutions if $\gamma_1^* \leq 0$, $\gamma_2^* \leq 0$.

Theorem 3.2. We assume that there exists $b_i \geq 0$, $\hat{b}_i \geq 0$, and $0 < \alpha_i < 1$ such that (H_1) is satisfied. If $\gamma_1^* \leq 0$, $\gamma_2^* \leq 0$ and

$$r_{1*} \geq \left[\alpha_1 \alpha_2 \cdot \frac{\hat{\beta}_{1*}}{\left(\beta_2^*\right)^{\alpha_1}} \right]^{\frac{1}{1-\alpha_1 \alpha_2}} \left(1 - \frac{1}{\alpha_1 \alpha_2}\right),$$

$$r_{2*} \geq \left[\alpha_1 \alpha_2 \cdot \frac{\hat{\beta}_{2*}}{\left(\beta_1^*\right)^{\alpha_2}} \right]^{\frac{1}{1-\alpha_1 \alpha_2}} \left(1 - \frac{1}{\alpha_1 \alpha_2}\right).$$

(3.1)

then there exists a positive solution of (1.1).

Proof In this case, to prove that $A: K \to K$ it is sufficient to find $0 < r_1 < R_1$, $0 < r_2 < R_2$ such that

$$\frac{\hat{\beta}_{1*}}{R_2^{\alpha_1}} + \gamma_{1*} \geq r_1, \quad \frac{\beta_1^*}{r_2^{\alpha_1}} \leq R_1 \tag{3.2}$$

$$\frac{\hat{\beta}_{2*}}{R_1^{\alpha_2}} + \gamma_{2*} \geq r_2, \quad \frac{\beta_2^*}{r_1^{\alpha_2}} \leq R_2 \tag{3.3}$$

If we fix $R_1 = \frac{\beta_1^*}{r_2^{\alpha_1}}$, $R_2 = \frac{\beta_2^*}{r_1^{\alpha_2}}$, then the first inequality of (3.3) holds if r_2 satisfies

$$\hat{\beta}_{2*} \left(\beta_1^*\right)^{-\alpha_2} r_2^{\alpha_1 \alpha_2} + \gamma_{2*} \geq r_2$$

or equivalently

$$\gamma_{2*} \geq g(r_2) := r_2 - \frac{\hat{\beta}_{2*}}{\left(\beta_1^*\right)^{\alpha_2}} r_2^{\alpha_1 \alpha_2}$$

The function $g(r_2)$ possesses a minimum at

$$r_{20} := \left[\alpha_1 \alpha_2 \cdot \frac{\hat{\beta}_{2*}}{\left(\beta_1^*\right)^{\alpha_2}} \right]^{\frac{1}{1-\alpha_1 \alpha_2}}$$

Taking $r_2 = r_{20}$, then (3.3) holds if

$$\gamma_{2^*} \geq g(r_{20}) = \left[\alpha_1 \alpha_2 \cdot \frac{\hat{\beta}_{2^*}}{\left(\beta_1^*\right)^{\alpha_2}}\right]^{\frac{1}{1-\alpha_1\alpha_2}} \left(1 - \frac{1}{\alpha_1 \alpha_2}\right)$$

Similarly,

$$\gamma_{1^*} \geq h(r_1) := r_1 - \frac{\hat{\beta}_{1^*}}{\left(\beta_2^*\right)^{\alpha_1}} r_1^{\alpha_1\alpha_2}$$

$h(r_1)$ possesses a minimum at

$$r_{10} := \left[\alpha_1 \alpha_2 \cdot \frac{\hat{\beta}_{1^*}}{\left(\beta_2^*\right)^{\alpha_1}}\right]^{\frac{1}{1-\alpha_1\alpha_2}}$$

$$\gamma_{1^*} \geq \left[\alpha_1 \alpha_2 \cdot \frac{\hat{\beta}_{1^*}}{\left(\beta_2^*\right)^{\alpha_1}}\right]^{\frac{1}{1-\alpha_1\alpha_2}} \left(1 - \frac{1}{\alpha_1 \alpha_2}\right)$$

Taking $r_1 = r_{10}$, $r_2 = r_{20}$, then the first inequalities in (3.2) and (3.3) hold if $\gamma_{1^*} \geq h(r_1)$ and $\gamma_{2^*} \geq g(r_2)$, which are just condition (3.1). The second inequalities hold directly from the choice of R_1 and R_2, so it remains to prove that $R_1 = \frac{\beta_1^*}{r_{20}^{\alpha_1}} > r_{10}$, $R_2 = \frac{\beta_2^*}{r_{10}^{\alpha_2}} > r_{20}$ This is easily verified through elementary computations:

$$R_1 = \frac{\beta_1^*}{r_{20}^{\alpha_1}} = \frac{\beta_1^*}{\left\{\left[\alpha_1 \alpha_2 \cdot \frac{\hat{\beta}_{2^*}}{\left(\beta_1^*\right)^{\alpha_2}}\right]^{\frac{1}{1-\alpha_1\alpha_2}}\right\}^{\alpha_1}} = \frac{\beta_1^*}{\left[\alpha_1 \alpha_2 \cdot \frac{\hat{\beta}_{2^*}}{\left(\beta_1^*\right)^{\alpha_2}}\right]^{\frac{\alpha_1}{1-\alpha_1\alpha_2}}} = \frac{\left(\beta_1^*\right)^{1+\frac{\alpha_1\alpha_2}{1-\alpha_1\alpha_2}}}{\left(\alpha_1 \alpha_2 \cdot \hat{\beta}_{2^*}\right)^{\frac{\alpha_1}{1-\alpha_1\alpha_2}}} = \frac{\left(\beta_1^*\right)^{\frac{1}{1-\alpha_1\alpha_2}}}{\left[\left(\alpha_1 \alpha_2 \cdot \hat{\beta}_{2^*}\right)^{\alpha_1}\right]^{\frac{1}{1-\alpha_1\alpha_2}}}$$

$$= \left[\frac{\beta_1^*}{\left(\alpha_1 \alpha_2 \cdot \hat{\beta}_{2^*}\right)^{\alpha_1}}\right]^{\frac{1}{1-\alpha_1\alpha_2}} = \left[\frac{1}{\left(\alpha_1 \alpha_2\right)^{\alpha_1}} \cdot \frac{\beta_1^*}{\left(\hat{\beta}_{2^*}\right)^{\alpha_1}}\right]^{\frac{1}{1-\alpha_1\alpha_2}} > \left[\alpha_1 \alpha_2 \cdot \frac{\hat{\beta}_{1^*}}{\left(\beta_2^*\right)^{\alpha_1}}\right]^{\frac{1}{1-\alpha_1\alpha_2}} = r_{10},$$

since $\hat{\beta}_{i^*} \leq \beta_i^*$, $i = 1,2$ Similarly, we have $R_2 > r_{20}$.

3) $\gamma_{1^*} \geq 0$, $\gamma_2^* \leq 0$ $\left(\gamma_1^* \leq 0, \gamma_{2^*} \geq 0\right)$

Theorem 3.3. Assume that (H_1) is satisfied. If $\gamma_{1^*} \geq 0$, $\gamma_2^* \leq 0$ and

$$\gamma_{2^*} \geq r_{21} - \hat{\beta}_{2^*} \cdot \frac{r_{21}^{\alpha_1\alpha_2}}{\left(\beta_1^* + \gamma_1^* r_{21}^{\alpha_1}\right)^{\alpha_2}} \tag{3.4}$$

where $0 < r_{21} < +\infty$ is a unique positive solution of equation

$$r_2^{1-\alpha_1\alpha_2} \left(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1}\right)^{1+\alpha_2} = \alpha_1 \alpha_2 \beta_1^* \hat{\beta}_{2^*} \tag{3.5}$$

then there exists a positive solution of (1.1).

Proof We follow the same strategy and notation as in the proof of ahead theorem. In this case, to prove that $A : K \to K$, it is sufficient to find $r_1 < R_1$, $r_2 < R_2$ such that

$$\frac{\hat{\beta}_{1^*}}{R_2^{\alpha_1}} \geq r_1, \quad \frac{\beta_2^*}{r_1^{\alpha_2}} \leq R_2 \tag{3.6}$$

$$\frac{\hat{\beta}_{2^*}}{R_1^{\alpha_2}} + \gamma_{2^*} \geq r_2, \quad \frac{\beta_1^*}{r_2^{\alpha_1}} + \gamma_1^* \leq R_1 \tag{3.7}$$

If we fix $R_2 = \dfrac{\beta_2^*}{r_1^{\alpha_2}}$, then the first inequality of (3.6) holds if r_1 satisfies

$$\frac{\hat{\beta}_{1^*}}{\left(\beta_2^*\right)^{\alpha_1}} \cdot r_1^{\alpha_1 \alpha_2} \geq r_1, \tag{3.8}$$

or equivalently

$$0 < r_1 \leq \left[\frac{\hat{\beta}_{1^*}}{\left(\beta_2^*\right)^{\alpha_1}}\right]^{\frac{1}{1-\alpha_1\alpha_2}} \tag{3.9}$$

If we chose $r_1 > 0$ small enough, then (3.9) holds, and R_2 is big enough.

If we fix $R_1 = \dfrac{\beta_1^*}{r_2^{\alpha_1}} + \gamma_1^*$ then the first inequality of (3.7) holds if r_2 satisfies

$$\gamma_{2^*} \geq r_2 - \frac{\hat{\beta}_{2^*}}{R_1^{\alpha_2}} = r_2 - \hat{\beta}_{2^*} \cdot \frac{1}{\left(\dfrac{\beta_1^*}{r_2^{\alpha_1}} + \gamma_1^*\right)^{\alpha_2}} = r_2 - \hat{\beta}_{2^*} \cdot \frac{1}{\left(\dfrac{\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1}}{r_2^{\alpha_1}}\right)^{\alpha_2}} = r_2 - \hat{\beta}_{2^*} \cdot \frac{r_2^{\alpha_1\alpha_2}}{\left(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1}\right)^{\alpha_2}},$$

or equivalently

$$\gamma_{2^*} \geq f(r_2) := r_2 - \hat{\beta}_{2^*} \cdot \frac{r_2^{\alpha_1\alpha_2}}{\left(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1}\right)^{\alpha_2}} \tag{3.10}$$

According to

$$f'(r_2) = 1 - \hat{\beta}_{2^*} \cdot \frac{1}{\left(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1}\right)^{2\alpha_2}}\left[\alpha_1\alpha_2 r_2^{\alpha_1\alpha_2 - 1}\left(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1}\right)^{\alpha_2} - r_2^{\alpha_1\alpha_2}\alpha_2\left(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1}\right)^{\alpha_2 - 1}\alpha_1\gamma_1^* r_2^{\alpha_1 - 1}\right]$$

$$= 1 - \frac{\hat{\beta}_{2^*}\alpha_1\alpha_2 r_2^{\alpha_1\alpha_2 - 1}}{\left(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1}\right)^{\alpha_2}}\left[1 - \frac{r_2^{\alpha_1}\gamma_1^*}{\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1}}\right] = 1 - \alpha_1\alpha_2\beta_1^*\hat{\beta}_{2^*} r_2^{\alpha_1\alpha_2 - 1}\left(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1}\right)^{-1-\alpha_2},$$

we have $f'(0) = -\infty$, $f'(+\infty) = 1$, then there exists r_{21} such that $f'(r_{21}) = 0$, and

$$f''(r_2) = -\left[\alpha_1\alpha_2\beta_1^*\hat{\beta}_{2^*}\left(\alpha_1\alpha_2 - 1\right)r_2^{\alpha_1\alpha_2 - 2}\left(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1}\right)^{-1-\alpha_2}\right.$$

$$\left. + \alpha_1\alpha_2\beta_1^*\hat{\beta}_{2^*} r_2^{\alpha_1\alpha_2 - 1}\left(-1-\alpha_2\right)\left(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1}\right)^{-2-\alpha_2}\gamma_1^*\alpha_1 r_2^{\alpha_1 - 1}\right] > 0.$$

Then the function $f(r_2)$ possesses a minimum at r_{21}, i.e., $f(r_{21}) = \min_{r_2 \in (0,+\infty)} f(r_2)$. Note $f'(r_{21}) = 0$ then we have

$$1 - \alpha_1\alpha_2\beta_1^*\hat{\beta}_{2^*} r_{21}^{\alpha_1\alpha_2 - 1}\left(\beta_1^* + \gamma_1^* \cdot r_{21}^{\alpha_1}\right)^{-1-\alpha_2} = 0$$

or equivalently

$$r_{21}^{1-\alpha_1\alpha_2}\left(\beta_1^* + \gamma_1^* \cdot r_{21}^{\alpha_1}\right)^{1+\alpha_2} = \alpha_1\alpha_2\beta_1^*\hat{\beta}_{2^*}$$

Taking $r_2 = r_{21}$, then the first inequality in (3.7) holds if $\gamma_{2^*} \geq f(r_{21})$, which is just condition (3.4). The second inequalities hold directly by the choice of R_1, and it would remain to prove that $r_{21} < R_2$ and $r_{10} < R_1$. These inequalities hold for R_2 big enough and r_1 small enough.

Remark 1. In theorem 3.3 the right-hand side of condition (3.4) always negative, this is equivalent to proof that $f(r_{21}) < 0$. This is obviously established through the proof of Theorem 3.3.

Similarly, we have the following theorem.

Theorem 3.4. Assume (H_1) is satisfied. If $\gamma_1^* \leq 0$, $\gamma_{2^*} \geq 0$ and

$$\gamma_{1^*} \geq r_{11} - \hat{\beta}_{1^*} \cdot \frac{r_{11}^{\alpha_1\alpha_2}}{\left(\beta_2^* + \gamma_2^* r_{11}^{\alpha_2}\right)^{\alpha_1}},$$

where $0 < r_{11} < +\infty$ is a unique positive solution of the equation

$$r_1^{1-\alpha_1\alpha_2}\left(\beta_2^* + \gamma_2^* r_1^{\alpha_2}\right)^{1+\alpha_1} = \alpha_1\alpha_2\beta_2^*\hat{\beta}_{1^*},$$

then there exists a positive solution of (1.1).

Funding

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