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$\mathcal{F}$-Multiautomata on Join Spaces Induced by Differential Operators

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Abstract

In this paper, we introduce the notion of fuzzy multiautomata and we investigate the hyperstructures induced by the linear second-order differential operators which can be used for construction of fuzzy multiautomata serving as a theoretical background for modeling of processes.

Keywords

Fuzzy Systems, Differential Operators, Hyperalgebraic Structures, Multiautomata

1. Introduction

Hyperstructure theory was born in 1934 when Marty defined hypergroups as a generalization of groups. This theory has been studied in the following decades and nowadays by many mathematicians. The hypergroup theory both extends some well-known group results and introduces new topics, thus leading to a wide variety of applications, as well as to a broadening of the investigation fields. There are applications of algebraic hyperstructures to the following subjects: geometry, hypergraphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, combinatorics, codes, artificial intelligence, and probabilistic. A comprehensive review of the theory of hyperstructures appears in [1]-[3].

Further, since the beginning of the first decade of this century relationships between ordinary linear differential operators and the hypergroup theory have been studied [4]-[8].

Zadeh [9] introduced the theory of fuzzy sets and, soon after, Wee [10] introduced the concept of fuzzy automata. Automata have a long history both in theory and application and are the prime examples of general computational systems over discrete spaces. Fuzzy automata not only provide a systematic approach for handling

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uncertainty in such systems, but also can be used in continuous spaces [11]. In this paper, we introduce $\mathcal{F}$-multiautomaton, without output function, where the transition function or next state function satisfies so called Fuzzy Generalized Mixed Condition (FGMC).These $\mathcal{F}$-multiautomata are systems that can be used for the transmission of information of certain type. Then we construct $\mathcal{F}$-multiautomata of commutative hypergroups and join spaces created from second order linear differential operators.

2. Preliminaries

Let J be an open interval of real numbers, and $(C(J),.)$ be the group of all continuous functions from J to interval $(0,1]$. In what follows we denote $L(p,q)y = y'' + p(x)y' + q(x)y;\ p,q \in C(J)$ that named differential operators of second order. And define $LA_2(J) = \{L(p,q);\ p,q \in C(J)\}$. Recall some basic notions of the hypergroup theory. A hypergroupoid is a pair $(H,\bullet)$, where $H \neq \phi$ and $\bullet : H \times H \to \mathcal{P}^*(H)$ is a binary hyperoperation on H. (Here $\mathcal{P}^*(H)$ denotes the system of all nonempty subsets of (H)). If $a \bullet (b \bullet c) = (a \bullet b) \bullet c$ holds for all $a,b,c \in H$ then $(H,\bullet)$ is called a semihypergroup. If moreover, the reproduction axiom ($a \bullet H = H = H \bullet a$, for any element $a \in H$) is satisfied, then the pair $(H,\bullet)$ is called a hypergroup. Join spaces are playing an important role in theories of various mathematical structures and their applications. The concept of a join space has been introduced by Prenowitz [12] and used by him and afterwards together with James Jantoisciak to reconstruct several branches of geometry. In order to define a join space, we need the following notation: If a,b,x are elements of a hypergroupoid $(H,*)$ then we denote $a/b = \{x \in H \mid a \in x * b\}$ and A/B we intend the set $\bigcup_{a \in A, b \in B} a/b$.

Definition 2.1 [12] [13] A commutative hypergroup $(H,*)$ is called a join space (or commutative transposition hypergroup) if the following condition holds for all elements a,b,c,d of H:

$$\frac{a}{b} \cap \frac{c}{d} \neq \varnothing \Rightarrow a * d \cap b * c \neq \varnothing$$

By a quasi-ordered (semi)group we mean a triple $(G,.,\leq)$, where $(G,.)$ is a (semi) group and binary relation $\leq$ is a quasi ordering (*i.e.* is reflexive and transitive) on the set G such that, for any triple $x,y,z \in G$ with the property $x \leq y$ also $x \cdot z \leq y \cdot z$ and $z \cdot x \leq z \cdot y$ hold.

The following lemma is called Ends-Lemma that is proved on [14] [15].

Lemma 2.2 Let $(G,\cdot,\leq)$ be a quasi-ordered semigroup. Define a hyperoperation

$$* : G \times G \to p^*(G) \quad \text{by } a * b = [a \cdot b)_{\leq} = \{x \in G; a \cdot b \leq x\}$$

For all pairs of elements $a,b \in G$. Then $(G,*)$ is a semihypergroup which is commutative if the semigroup $(G,\cdot)$ is commutative. If moreover, $(G,\cdot)$ is a group, then $(G,*)$ is a transposition hypergroup. Therefore, if $(G,\cdot)$ is a commutative group, then $(G,*)$ is a join space.

Proposition 2.3 For any pair of differential operators $L(p_1,q_1), L(p_2,q_2) \in LA_2(J)$ define a binary operation as below*:*

$$L(p_1,q_1) \cdot L(p_2,q_2) = L(p_1 \cdot p_2, q_1 \cdot q_2)$$

and define a quasi-ordered relation as following:

$$L(p_1,q_1) \leq L(p_2,q_2) \text{ if } p_1(x) = p_2(x), q_1(x) \leq q_2(x), \text{for all } x \in J.$$

Then $(LA_2(J),.,\leq)$ is a commutative ordered group with the unit element $L(1,1)$. □

Now we apply the simple construction of a hypergroup from Lemma 2.2 into this considered concrete case of differential operators:

For arbitrary pair of operators $L(p_1,q_1), L(p_2,q_2) \in LA_2(J)$ we put:

$$\begin{aligned} L(p_1,q_1) * L(p_2,q_2) &= \{L(p,q) \in LA_2(J) \mid L(p_1,q_1) \cdot L(p_2,q_2) \leq L(p,q)\} \\ &= \{L(p_1 \cdot p_2, \varphi) \mid q_1 \cdot q_2 \leq \varphi; \varphi \in C(J)\} \end{aligned}$$

Then we obtain the following Corollary from Lemma 2. 2 immediately:

Corollary 2.4 For each $L(p_1,q_1), L(p_2,q_2) \in LA_2(J)$, if

$$L(p_1,q_1)*L(p_2,q_2)=\{L(p_1\cdot p_2,\varphi)|q_1\cdot q_2\le\varphi,\varphi\in C(J)\}$$

Then $(LA_2(J),*)$ is a commutative hypergroup and a join space.

Definition 2.5 [16] Let X be a non-empty set, $(H,*)$ be a (semi) hypergroup and $\delta:X\times H\to X$ be a mapping such that, for all $x\in X$, and $s,t\in H$:

$$\delta(\delta(x,t),s)\in\delta(x,t*s),\quad \text{where } \delta(x,t*s)=\{\delta(x,u);u\in t*s\} \tag{2.1}$$

Then (X,H,δ) is called a discrete transformation (semi)hypergroup or an action of the (semi)hypergroup *H* on the set *X*. The mapping δ is usually said to be simply an action.

Remark 2.6 The condition (2.1) used above is called *Generalized Mixed Associativity Condition*, shortly **GMAC**.

Definition 2.7 [6] [7] (Quasi)multiautomaton without output is a triad $M=(H,S,\delta)$, where $(H,*)$ is a (semi)hypergroup, *S* is a non-empty set, and $\delta:H\times S\to S$ is a transition map satisfying **GMAC** condition. The set *S* is called the state set of the (quasi)multiautomaton *M*, the structure $(H,*)$ is called a input (semi)-hypergroup of the (quasi)multiautomaton *M* and δ is called a transition function. Elements of the set *S* are called states and the elements of the set *H* are called input symbols.

3. $\mathcal{F}$-Multi Automata

Definition 3.1 A *fuzzy transformation (semi)hypergroup* (or a fuzzy action) of (semi)hypergroup *H* on *S* is a triple (S,H,μ), where S is a non-empty set, $(H,*)$ is a (semi)hypergroup, and μ is a fuzzy subset of $S\times H\times S$ such that, for all $u,v\in H$ and $p,q\in S$:

$$\vee\{\mu(q,u,r)\wedge\mu(r,v,p)|r\in S\}\in\mu(q,u*v,p)\ \text{ where } \mu(q,u*v,p)=\{\mu(q,x,p)|x\in u*v\} \tag{3.2}$$

Remark 3.2 The condition (3.2) used above is called *Fuzzy Generalized Mixed Condition*, shortly **FGMC**.

Definition 3.3 $\mathcal{F}$ *-(quasi) multiautomaton* without outputs is a triad $\mathcal{F}=(H,S,\mu)$, where $(H,*)$ is a (semi)hyper-group, S is a non-empty set and $\mu:S\times H\times S\to[0,1]$ is a fuzzy transition map satisfying **FGMC** condition.

Set *S* is called the state set and the hyperstructure $(H,*)$ is called the input (semi)hypergroup of the $\mathcal{F}$-(quasi)multiautomaton $\mathcal{F}$ and μ is called fuzzy transition function. Elements of the set S are called states and the elements of the set H are called input symbols.

Definition 3.4 $\mathcal{F}$-(quasi)multiautomaton $\mathcal{F}=(H,S,\mu)$ is said to be *abelian* (or commutative) if

$$\mu(s,x*y,t)=\mu(s,y*x,t),\ \text{ for all } s,x,y,t\in S\times H\times H\times S$$

Example 3.5 Suppose that $H=\{a,b\}$, $S=\{q_1,q_2,q_3\}$. Let hyperoperation $*$ on *H* and fuzzy transition function $\delta:S\times H\times S\to[0,1]$ are defined as follows:

*	a	B
a	{a}	{a,b}
b	{a,b}	{b}

$$\delta(q_1,a,q_1)=\frac{1}{3}\quad\delta(q_2,a,q_2)=\frac{1}{3}\quad\delta(q_1,a,q_2)=\frac{1}{3}\quad\delta(q_2,a,q_3)=\frac{1}{3}$$

$$\delta(q_1,b,q_2)=\frac{2}{3}\quad\delta(q_2,b,q_2)=\frac{2}{3}\quad\delta(q_2,a,q_1)=\frac{1}{3}\quad\delta(q_1,a,q_3)=\frac{1}{3}$$

And for all other ordered triples (q,h,p) we define $\delta(q,h,p)=0$. Then (h,S,δ) is a commutative $\mathcal{F}$-multiautomaton (**Figure 1**).

4. $\mathcal{F}$-Multi Automata on Join Spaces Induced by Differential Operators

Proposition 4.1: Let $\mathcal{F}_1=((C(J),\odot),LA_2(J),\mu_1)$ where, for all $f,g\in C(J)$:

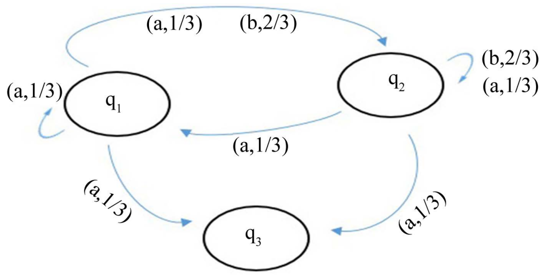

Figure 1. The $\mathcal{F}$-multiautomaton of Example 3.5.

$$f \odot g = [f \cdot g)_{\leq} = \{h \in C(J) | f(x) \cdot g(x) \leq h(x), \forall x \in J\}$$

And define:

$$\mu_1 : LA_2(J) \times C(J) \times LA_2(J) \to [0,1]$$

$$\mu_1(L(p_1,q_1), f, L(p_2,q_2)) = \vee(q_1 \cdot f \cdot q_2)$$

where : $\vee(q_1 \cdot f \cdot q_2) = \bigvee_{\forall x,y,z \in J} (q_1(x) \cdot f(y) \cdot r(z))$

Then $\mathcal{F}_1$ is a commutative $\mathcal{F}$-multiautomaton.

Proof: By Lemma 2.2 the hypergroupoid $(C(J), \odot)$ is a join space. Now, we prove this structure is satisfying FGMC property. Let

$$\vee\{\mu_1(L(p_1,q_1),u,L(t,r)) \wedge \mu_1(L(t,r),v,L(p_2,q_2)) | L(t,r) \in LA_2(J)\} = \imath$$

and

$\mu_1(L(p_1,q_1),u \odot v, L(p_2,q_2)) = \mathcal{A}$, for all $u,v \in C(J)$ and $L(p_1,q_1), L(p_2,q_2) \in LA_2(J)$.

Then

$$\imath = \bigvee_{r \in C(J)} ((\vee(q_1 \cdot u \cdot r)) \wedge (\vee(r \cdot v \cdot q_2))) = (\vee(q_1 \cdot u)) \wedge (\vee(v \cdot q_2))$$

$$\mathcal{A} = \{\vee(q_1 \cdot t \cdot q_2) | t(x) \geq u(x) \cdot v(x)\}$$

Clearly $\imath \in \mathcal{A}$ (since we can take $t(x) = \dfrac{u(x)}{q_2(x)}$ or $t(x) = \dfrac{v(x)}{q_1(x)}$ for each $x \in J$). Then **FGMC** property holds. Hence $\mathcal{F}_1$ is a $\mathcal{F}$-multiautomaton. In addition, since $f \odot g = g \odot f$, for all $f, g \in C(J)$ then $\mathcal{F}_1$ is commutative. □

Proposition 4.2: Let $\mathcal{F}_2 = ((C(J), \odot), LA_2(J), \mu_2)$ where hyperoperation $\odot$ was defined in proposition 4.1.

And define:

$$\mu_2 : LA_2(J) \times C(J) \times LA_2(J) \to [0,1]$$

$$\mu_2(L(p_1,q_1), f, L(p_2,q_2)) = \vee(q_1 \wedge f \wedge q_2)$$

where $\vee(q_1 \wedge f \wedge q_2) = \bigvee_{\forall x,y,z \in J} (q_1(x) \wedge f(y) \wedge q_2(z))$

Then $\mathcal{F}_2$ is a commutative $\mathcal{F}$-multiautomaton.

Proof: By Lemma 2.2 the hypergroupoid $(C(J), \odot)$ is a join space. Now, we prove this structure is satisfying FGMC property. Let

$$\vee\{\mu_2(L(p_1,q_1),u,L(t,r)) \wedge \mu_2(L(t,r),v,L(p_2,q_2)) | L(t,r) \in LA_2(J)\} = \jmath$$

and

$$\mu_2\left(L\left(p_1,q_1\right),u\odot v,L\left(p_2,q_2\right)\right)=\mathcal{B}$$

for all, $u,v\in C\left(J\right)$ and $L\left(p_1,q_1\right),L\left(p_2,q_2\right)\in LA_2\left(J\right)$.

Then

$$\jmath=\bigvee_{r\in C(J)}\left(\left(\vee\left(q_1\wedge u\wedge r\right)\right)\wedge\left(\vee\left(r\wedge v\wedge q_2\right)\right)\right)=\vee\left(q_1\wedge u\wedge v\wedge q_2\right)$$

$$\mathcal{B}=\left\{\vee\left(q_1\wedge t\wedge q_2\right)\middle|u\left(x\right)\cdot v\left(x\right)\le t\left(x\right),x\in J\right\}$$

Since $u(x)\cdot v\left(x\right)\le u\left(x\right)\wedge v\left(x\right)$, for all $x\in J$ then $\jmath\in\mathcal{B}$. Hence **FGMC** property holds. Therefore $\mathcal{F}_2$ is a $\mathcal{F}$-multiautomaton. In addition, It is clear that $\mathcal{F}_2$ is commutative.

Proposition 4.3: Let $\mathcal{F}_3=\left(\left(LA_2\left(J\right),*\right),C\left(J\right),\mu_3\right)$ where, for all $L\left(p_1,q_1\right),L\left(p_2,q_2\right)\in LA_2\left(J\right)$:

$$L\left(p_1,q_1\right)*L\left(p_2,q_2\right)=\left\{L\left(p_1.p_2,\varphi\right)\middle|q_1\left(x\right)\cdot q_2\left(x\right)\le\varphi\left(x\right),\varphi\in C\left(J\right)\right\}$$

And define:

$$\mu_3:C\left(J\right)\times LA_2\left(J\right)\times C\left(J\right)\to\left[0,1\right]$$

$$\mu_3\left(f,L\left(p,q\right),g\right)=\vee\left(f\wedge q\wedge g\right)$$

where $\vee\left(f\wedge q\wedge g\right)=\bigvee_{\forall x,y,z\in J}\left(f\left(x\right)\wedge q\left(y\right)\wedge g\left(z\right)\right)$

Then $\mathcal{F}_3$ is a commutative $\mathcal{F}$-multiautomaton.

Proof: According to Corollary 2.4 $\left(LA_2\left(J\right),*\right)$ is a join space. Now we check the FGMC property for this structure. Let

$$\vee\left\{\mu_3\left(f,L\left(p_1,q_1\right),r\right)\wedge\mu_3\left(r,L\left(p_2,q_2\right),g\right)\middle|r\in C\left(J\right)\right\}=\ell$$

And

$$\mu_3\left(f,L\left(p_1,q_1\right)*L\left(p_2,q_2\right),g\right)=\mathcal{C}\text{, for all } f,g\in C\left(J\right) \text{ and } L\left(p_1,q_1\right),L\left(p_2,q_2\right)\in LA_2\left(J\right).$$

Then

$$\ell=\bigvee_{r\in C(J)}\left(\left(\vee\left(f\wedge q_1\wedge r\right)\right)\wedge\left(\vee\left(r\wedge q_2\wedge g\right)\right)\right)=\vee\left(f\wedge q_1\wedge q_2\wedge g\right)$$

$$\begin{aligned}\mathcal{C}&=\left\{\mu_3\left(f,L\left(p_1\cdot p_2,\varphi\right),g\right)\middle|q_1\left(x\right)\cdot q_2\left(x\right)\le\varphi\left(x\right),x\in J\right\}\\&=\left\{\vee\left(f\wedge\varphi\wedge g\right)\middle|q_1\left(x\right)\cdot q_2\left(x\right)\le\varphi\left(x\right),x\in J\right\}\end{aligned}$$

Since $q_1\left(x\right)\wedge q_2\left(x\right)\ge q_1\left(x\right)\cdot q_2\left(x\right)$, for all $x\in J$ then $\ell\in\mathcal{C}$. Hence $\mathcal{F}_3$ is a $\mathcal{F}$-multiautomaton. It is clear that $\mathcal{F}_3$ is commutative. □

Proposition 4.4: Let $\mathcal{F}_4=\left(\left(LA_2\left(J\right),*\right),C\left(J\right),\mu_4\right)$, where hyperoperation $*$ was defined in proposition 3.4.

And define:

$$\mu_4:C\left(J\right)\times LA_2(J)\times C\left(J\right)\to\left[0,1\right]$$

$$\mu_4\left(f,L\left(p,q\right),g\right)=\vee\left(f\cdot q\cdot g\right)$$

where: $\vee\left(f\cdot q\cdot g\right)=\bigvee_{\forall x,y,z\in J}\left(f\left(x\right)\cdot q\left(y\right)\cdot g\left(z\right)\right)$

Then $\mathcal{F}_4$ is a commutative $\mathcal{F}$-multiautomaton.

Proof: According to Corollary 2.4 $\left(LA_2\left(J\right),*\right)$ is a join space. Now, we prove this structure is satisfying FGMC property. Let

$$\vee\left\{\mu_4\left(f, L\left(p_1, q_1\right), r\right) \wedge \mu_4\left(r, L\left(p_2, q_2\right), g\right) \middle| r \in C(J)\right\} = \mathcal{m}$$

$$\mu_4\left(f, L\left(p_1, q_1\right) * L\left(p_2, q_2\right), g\right) = \mathcal{M}$$

for all $f, g \in C(J)$ and $L(p_1, q_1), L(p_2, q_2) \in LA_2(J)$.

Then

$$\mathcal{m} = \bigvee_{r \in C(J)} \left(\left(\vee\left(f \cdot q_1 \cdot r\right)\right) \wedge \left(\vee\left(r \cdot q_2 \cdot g\right)\right)\right) = \left(\vee\left(f \cdot q_1\right)\right) \wedge \left(\vee\left(q_2 \cdot g\right)\right)$$

$$\mathcal{M} = \left\{\vee\left(f \cdot t \cdot g\right) \middle| q_1(x) \cdot q_2(x) \leq t(x)\right\}$$

Since $\frac{q_1(x)}{g(x)} \geq q_1(x) \cdot q_2(x)$ and $\frac{q_2(x)}{f(x)} \geq q_1(x) \cdot q_2(x)$, for all $x \in J$ then $\mathcal{m} \in \mathcal{M}$. Hence $\mathcal{F}_4$ is a $\mathcal{F}$-multiautomaton. It is clear that $\mathcal{F}_4$ is commutative.

5. Conclusion

In this research, we introduced $\mathcal{F}$-multistructures which can be used for construction of $\mathcal{F}$-multiautomata serving as a theoretical background for modeling of processes. Then we obtain some $\mathcal{F}$-multiautomata of linear second-order differential operators. In future work, we can introduce $\mathcal{F}$-multiautomaton with output and concrete interpretations of these structures can be studied.

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