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Flag-Transitive 6-(v, k, 2) Designs

Xiaolian Liao[1], Shangzhao Li[2], Guohua Chen[1]

[1]Department of Mathematics, Hunan University of Humanities Science and Technology, Loudi, China
[2]Department of Mathematics, Changshu Institute of Technology, Changshu, China
Email: hnldlxl2005@126.com

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Abstract

The automorphism group of a flag-transitive 6–(v, k, 2) design is a 3-homogeneous permutation group. Therefore, using the classification theorem of 3–homogeneous permutation groups, the classification of flag-transitive 6-(v, k,2) designs can be discussed. In this paper, by analyzing the combination quantity relation of 6–(v, k, 2) design and the characteristics of 3-homogeneous permutation groups, it is proved that: there are no 6–(v, k, 2) designs D admitting a flag transitive group G ≤ Aut (D) of automorphisms.

Keywords

Flag-Transitive, Combinatorial Design, Permutation Group, Affine Group, 3-Homogeneous Permutation Groups

1. Introduction

For positive integers $t \le k \le v$ and λ, we define a $t-(v,k,\lambda)$ design to be a finite incidence structure $D=(X,\mathrm{B},I)$, where X denotes a set of points, $|X|=v$ and B a set of blocks, $|\mathrm{B}|=b$, with the properties that each block $B\in\mathrm{B}$ is incident with *k* points, and each *t*-subset of X is incident with λ blocks. A flag of D is an incident point-block pair, that is $x\in X$ and $B\in\mathrm{B}$ such that $(x,B)\in I$. We consider automorphisms of D as pairs of permutations on *X* and B which preserve incidence, and call a group $G\le \mathrm{Aut}(D)$ of automorphisms of D flag-transitive (respectively block-transitive, point t-transitive, point t-homogeneous), if G acts transitively on the flags (respectively transitively on the blocks, t-transitively on the points, *t*-homogeneous on the points) of D. It is a different problem in Combinatorial Maths how to construct a design with given parameters. In this paper, we shall take use of the automorphism groups of designs to find some new designs.

In recent years, the classification of flag-transitive Steiner 2-designs has been completed by W. M. Kantor (See [1]), F. Buekenhout, A. De-landtsheer, J. Doyen, P. B. Kleidman, M. W. Liebeck, J. Sax (See [2]); for flag-

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transitive Steiner *t*-designs $(2<t\le 6)$, Michael Huber has done the classification (See [3]-[7]). But only a few people have discussed the case of flag-transitive t-designs where $t>3$ and $\lambda>1$.

In this paper, we may study a kind of flag-transitive designs with $\lambda=2$. We may consider this problem by making use of the classification of the finite 3-homogeneous permutation groups to study flag-transitive $6-(v,k,2)$ designs. Our main result is:

Theorem: There are no non-trivial $6-(v,k,2)$ designs D admitting a flag transitive group $G\le \mathrm{Aut}(D)$ of automorphisms.

2. Preliminary Results

Lemma 2.1. (Huber M [4]) Let $D=(X,B,I)$ be a $t-$design with $t\ge 3$ .If $G\le \mathrm{Aut}(D)$ acts flag-transitively on D, then G also acts point 2-transitively on D.

Lemma 2.2. (Cameron and Praeger [8]). Let $D=(X,B,I)$ be a $t-(v,k,\lambda)$ design with $\lambda\ge 2$. Then the following holds:

(1) If $G\le \mathrm{Aut}(D)$ acts block-transitively on D, then G also acts point $\lfloor t/2\rfloor$-homogeneously on D;

(2) If $G\le \mathrm{Aut}(D)$ acts flag-transitively on D, then G also acts point $\lfloor (t+1)/2\rfloor$-homogeneously on D.

Lemma 2.3. (Huber M [9]) Let $D=(X,B,I)$ be a $t-(v,k,\lambda)$ design. If $G\le \mathrm{Aut}(D)$ acts flag-transitively on D, then , for any $x\in X$, the division property $r \,\|\, |G_x|$ holds.

Lemma 2.4. Let $D=(X,B,I)$ be a $t-(v,k,\lambda)$ design. Then the following holds:

(1) $bk=vr$;

(2) $\binom{v}{t}\lambda=b\binom{k}{t}$;

(3) For $1\le s<t$, a $t-(v,k,\lambda)$ design is also an $s-(v,k,\lambda_s)$ design, where $\lambda_s=\lambda\dfrac{\binom{v-s}{t-s}}{\binom{k-s}{t-s}}$.

(4) In particular, if $t=6$, then

$$r(k-1)(k-2)(k-3)(k-4)(k-5)=\lambda(v-1)(v-2)(v-3)(v-4)(v-5).$$

Lemma 2.5. (Beth T [10]) If $D=(X,B,I)$ is a non-trivial $t-(v,k,\lambda)$ design, then $v>k+t$

Lemma 2.6. (Wei J L [11]) If $D=(X,B,I)$ is a $t-(v,k,\lambda)$ design, then

$$\lambda(v-t+1)\ge (k-t+1)(k-t+2), t>2.$$

In this case, when $t=6$, we deduce from Lemma 2.6 the following upper bound for the positive integer k.

Corollary 2.7. Let $D=(X,B,I)$ be a non-trivial $6-(v,k,2)$ design, then

$$k\le \left\lfloor \sqrt{2v-\frac{39}{4}}+\frac{9}{2}\right\rfloor.$$

Proof: By Lemma 2.6, when $t=6,\lambda=2$, we have $2(v-5)\ge (k-5)(k-4)$, then

$$k\le \left\lfloor \sqrt{2v-\frac{39}{4}}+\frac{9}{2}\right\rfloor.$$

Remark 2.8. Let $D=(X,B,I)$ be a non-trivial $t-(v,k,\lambda)$ design with $t\ge 6$. If $G\le \mathrm{Aut}(D)$ acts flag-transitively on D, then by Lemma 2.2 (1), G acts point 3-homogeneously and in particular point 2-transitively on D. Applying Lemma 2.4 (2) yields the equation

$$b=\frac{\lambda\binom{v}{t}}{\binom{k}{t}}=\frac{v(v-1)|G_{xy}|}{|G_{B1}|}$$

where x and y are two distinct points in X and B_1 is a block in B. If $x\in B_1$, then

$$2\binom{v-2}{4}=(k-1)\binom{k-2}{4}\frac{|G_{xy}|}{|G_{xB1}|}.$$

Corollary 2.9 Let $D=(X,B,I)$ be a $t-(v,k,\lambda)$ design, then

$$\lambda\binom{v-s}{t-s}\equiv 0\left(\bmod\binom{k-s}{t-s}\right).$$

For each positive integers, $s\le t$.

Let G be a finite 3-homogeneous permutation group on a set X with $|X|\ge 4$. Then G is either of

(A) Affine Type:

G contains a regular normal subgroup T which is elementary Abelian of order $v=2^d$.If we identify G with a group of affine transformations

$$x\mapsto x^{\varepsilon}+\mu$$

Of $V=V(d,2)$, where $\varepsilon\in G_0$ and $\mu\in V$, then particularly one of the following occurs:

(1) $G\cong AGL(1,8)$, $A\Gamma L(1,8)$ or $A\Gamma L(1,32)$;

(2) $G\cong SL(d,2), d\ge 2$;

(3) $G\cong A_7, v=2^4$;

or

(B) Almost Simple Type: G contains a simple normal subgroup N, and $N\le G\le \mathrm{Aut}(D)$. In particular, one of the following holds, where N and $v=|X|$ are given as follows:

(1) $A_v, v\ge 5$;

(2) $PLS(2,q), v=q+1, q>3$;

(3) $M_v, v=11,12,22,23,24$;

(4) $M_{11}, v=12$.

3. Proof of the Main Theorem

Let $D=(X,B,I)$ be a non-trivial $6-(v,k,2)$ design, $G\le \mathrm{Aut}(D)$ acts flag-transitively on D, by lemma 2.2, G is a finite 3-homogeneous permutation group. For D is a non-trivial $6-(v,k,2)$ design, then $k>6$. We will prove by contradiction that $G\le\mathrm{Aut}(D)$ cannot act flag-transitively on any non-trivial $6-(v,k,2)$ design.

3.1. Groups of Automorphisms of Affine Type

Case (1): $G\cong AGL(1,8)$, $A\Gamma L(1,8)$ or $A\Gamma L(1,32)$;

If $v=8$, then Lemma 2.5 yields $k<v-t=2$, a contradiction to $k>6$. For $v=32$, Corollary 2.7 implies $k\le 12$. Thus $k=7,8,9,10,11,12$. By Lemma 2.4 we have

$$r(k-1)(k-2)(k-3)(k-4)(k-5)=2\times 31\times 30\times 29\times 28\times 27$$

for each values of k, we have

$$r=31\times 29\times 7\times 9, 31\times 29\times 18, \frac{31\times 29\times 27}{4}, 31\times 29\times 3, \frac{31\times 29\times 3}{2}, \frac{31\times 29\times 9,}{11}$$

but r is a positive integer, thus $r=31\times 29\times 7\times 9, 31\times 29\times 18, 31\times 29\times 3$. On the other hand, we have $|G_x|=5(v-1)=5\times 31$, those are contradicting to Lemma 2.3.

Case (2): $G\cong SL(d,2)$, $d\ge 2$.

Here $v=2^d>k>6$. For $d=3$, we have $v=8$, already ruled out in Case (1). So we may assume that $d>3$. Any six distinct points being non-coplanar in $AG(d,2)$, they generate an affine subspace of dimension at least 3. Let ε be the 3-dimensional vector subspace spanned by the first three basis vectors e_1,e_2,e_3 of the vector space $V=V(d,2)$. Then the point-wise stabilizer of ε in $SL(d,2)$ (and therefore also in G) acts point-transitively on $V\setminus\varepsilon$. Let B_1 and B' be the two blocks which are incident with the 6-subset $\{0,e_1,e_2,e_3,e_1+e_2,e_2+e_3\}$, If the block $B_1\cup B'$ contains some point α of $V\setminus\varepsilon$, then $B_1\cup B'$ contains all points of $V\setminus\varepsilon$, and so $2k-12\ge v-8=2^d-8$, this yields $k>2^{d-1}+2>2^{d-1}+1$, a contradiction to Lemma 2.6. Hence $B_1\subseteq\varepsilon$ and $k\le 8$. On the other hand, for D is a flag-transitive 6-design admitting $G\le \mathrm{Aut}(D)$, we deduce from [[12], prop.3.6 (b)] the necessary condition that 2^d-3 must divide $\binom{k}{4}$, and hence it follows for each respective value of k that $d=3$, contradicting our assumption.

Case (3): $G\cong A_7, v=2^4$

For $v=2^4$, we have $k\le 9$, by Corollary 2.7. By Lemma 2.4 and Lemma 2.3, we have $k\ne 7,8,9$.

3.2. Groups of Automorphisms of Almost Simple Type

Case (1): $A_v, v\ge 5$

Since D is non-trivial with $k>6$, we may assume that $v\ge 8$. Then A_v is 6-transitive on X, and hence G is k-transitive, this yields D containing all of the k-subset of X. So D is a trivial design, a contradiction.

Case (2): $PLS(2,q)$, $v=q+1, q=p^e>3$;

Here $N=PLS(2,q), v=q+1, q=p^e\ge 3$ and $p>3$, so $\mathrm{Aut}(N)=P\Gamma L(2,q)$, $|G|=(q+1)q\frac{q-1}{d}a$ with $d=(2,q-1)$ and $a|de$. We may again assume that $v=q+1\ge 8$.

We will first assume that $N=G$. Then, by Remark 2.8, we obtain

$$4(q-2)(q-3)(q-4)\left|PSL(2,q)_{xB}\right|=(k-1)(k-2)(k-3)(k-4)(k-5). \tag{1}$$

In view of Lemma 2.6, we have

$$2(q-4)\ge (k-4)(k-5) \tag{2}$$

It follows from Equation (1) that

$$2(q-2)(q-3)\left|PSL(2,q)_{xB}\right|\le (k-1)(k-2)(k-3) \tag{3}$$

If we assume that $k\ge 21$, then obviously

$$2(k-1)(k-2)(k-3)<\left[(k-4)(k-5)\right]^2$$

and hence

$$(q-2)(q-3)\left|PSL(2,q)_{xB}\right|<2(q-4)^2$$

In view of inequality (2), clearly, this is only possible when $\left|PSL(2,q)_{xB}\right|=1$. In particular, q has not to be even. But then the right-hand side of Equation (1) is always divisible by 16 but never the left-hand side, a contradiction. If $k<21$, then the few remaining possibilities for k can easily be ruled out by hand using Equation (1), Inequality (2), and Corollary 2.9.

Now, let us assume that $N<G\le \mathrm{Aut}(N)$. We recall that $q=p^e\ge 7$, and will distinguish in the following the case $p>3, p=2,$ and $p=3$.

First, let $p>3$. We define $G^*=G\cap\left(PSL(2,q):\langle\tau_\alpha\rangle\right)$ with $\tau_\alpha\in Sym\left(GF\left(p^e\right)\right)\cup\{\infty\}\cong S_v$ of order e induced by the Frobenius automorphism $\alpha: GF\left(p^e\right)\to GF\left(p^e\right), x\mapsto x^p$. Then, by Dedekind's law, we can write

$$G^*=PSL(2,q):\left(G^*\cap\langle\tau_\alpha\rangle\right)$$

Defining $P\Sigma L(2,q)=PSL(2,q):\langle\tau_\alpha\rangle$, it can easily be calculated that $P\Sigma L(2,q)_{0,1,\infty}=\langle\tau_\alpha\rangle$, and $\langle\tau_\alpha\rangle$ has precisely $p+1$ distinct fixed points (cf. e.g., [[13] Ch. 6.4, Lemma 2]). As $p>3$, we have therefore that $G^*_{0B_1}\cap\langle\tau_\alpha\rangle\le G^*\cap\langle\tau_\alpha\rangle\le G^*_F$ for a flag $F=\{(0,B_1),(0,B')\}$ fixed with $\langle\tau_\alpha\rangle$ by the definition of $6-(v,k,2)$ designs. On the other hand, every element of $G^*\cap\langle\tau_\alpha\rangle$ either fixes block B_1, or commute block B_1 with block B', thus the index $\left[G^*_{0B_1}\cap\langle\tau_\alpha\rangle : G^*\cap\langle\tau_\alpha\rangle\right]\le 2$. Clearly $PSL(2,q)\cap\left(G^*\cap\langle\tau_\alpha\rangle\right)=1$.

Hence, we have

$$\left|(0,B_1)^{G^*}\right| = \left[G^* : G^*_{0B_1}\right] \le \left[PSL(2,q)\cap\left(G^*\cap\langle\tau_\alpha\rangle\right) : PSL(2,q)_{0B_1}\cap\left(G^*_{0B_1}\cap\langle\tau_\alpha\rangle\right)\right]$$
$$= c\left[PSL(2,q) : PSL(2,q)_{0B_1}\right] = c\left|(0,B_1)^{PSL(2,q)}\right|.$$

where $c=1$ or 2. Thus, if we assume that $G^* \le \mathrm{Aut}(D)$ acts already flag-transitively on D, then we obtain $bk = \left|(0,B_1)^{G^*}\right| \le c\left|(0,B_1)^{PSL(2,q)}\right|$. Then either $bk = \left|(0,B_1)^{PSL(2,q)}\right|$, and $PSL(2,q)$ acts on D flag-transitively, that is the case when $N=G$; or $bk = 2\left|(0,B_1)^{PSL(2,q)}\right|$, and $PSL(2,q)$ has exactly two orbits of equal length on the sets of flags. Then, proceeding similarly to the case $N=G$ for each orbit on the set of the flags, we have that

$$2(q-2)(q-3)(q-4)\left|PSL(2,q)_{0B_1}\right| = (k-1)(k-2)(k-3)(k-4)(k-5) \tag{4}$$

Using again

$$2(q-4) \ge (k-4)(k-5) \tag{5}$$

We obtain

$$2(q-2)(q-3)\left|PSL(2,q)_{0B_1}\right| \le (k-1)(k-2)(k-3) \tag{6}$$

If we assume that $k \ge 21$, then again

$$(k-1)(k-2)(k-3) \le 2\left[(k-4)(k-5)\right]^2 \tag{7}$$

and thus

$$4(q-2)(q-3)\left|PSL(2,q)_{0B_1}\right| \le (q-4)^2$$

but this is impossible. The few remaining possibilities for $k<21$ can again easily be ruled out by hand.

Now, let $p=2$, then, clearly $N = PSL(2,q) = PGL(2,q)$, and we have $\mathrm{Aut}(N) = P\Sigma L(2,q)$. If we assume that $\langle\tau_\alpha\rangle$ is the subgroup of $P\Sigma L(2,q)_{0B_1}$ for a flag $(0,B_1)\in \mathrm{B}$, then we have $G^* = G = P\Sigma L(2,q)$ and as clearly $PSL(2,q)\cap\langle\tau_\alpha\rangle = 1$, we can apply Equation $(*)$. Thus, $PSL(2,q)$ must also be flagtransitive, which has already been considered. Therefore, we assume that $\langle\tau_\alpha\rangle$ is not the subgroup of $P\Sigma L(2,q)_{0B_1}$. Let $s>2$ be a prime divisor of $e = \left|\langle\tau_\alpha\rangle\right|$. As the normal subgroup $H := \left(P\Sigma L(2,q)_{0,1,\infty}\right)^s \le \langle\tau_\alpha\rangle$ of index s has precisely p^s+1 distinct fix points, we have $G\cap H \le G_{0B_1}$ for a flag $F = \{(0,B_1),(0,B')\}$ fixed with $\langle\tau_\alpha\rangle$ by the definition of $6-(v,k,2)$ designs. It can then be deduced that $e = s^u$ for some $u \in \underline{N}$. Since if we assume for $G = P\Sigma L(2,q)$ that there exists a further prime divisor $\bar{s} > 2$ of e with $\bar{s} \ne s$, then $\bar{H} := \left(P\Sigma L(2,q)_{0,1,\infty}\right)^{\bar{s}} \le \langle\tau_\alpha\rangle$ and H are both subgroups of $P\Sigma L(2,q)_{0B_1}$ by the flag-transitivity of $P\Sigma L(2,q)$, and hence $\langle\tau_\alpha\rangle \le P\Sigma L(2,q)_{0B_1}$, a contradiction. Furthermore, as $\langle\tau_\alpha\rangle$ is not the subgroup of $P\Sigma L(2,q)_{0B_1}$. We may, by applying Dedekind's law, assume that

$$G_{0B_1} = PSL(2,q)_{0B_1} : (G\cap H)$$

Thus, by Remark 2.8, we obtain

$$(q-2)(q-3)(q-4)\left|PSL(2,q)_{0B_1}\right|\left|G\cap H\right| = k(k-1)(k-2)(k-3)(k-4)(k-5)\left|G\cap\langle\tau_\alpha\rangle\right|$$

More precisely:

(A) if $G = PSL(2,q):(G\cap H)$,

$$(q-2)(q-3)(q-4)\left|PSL(2,q)_{0B_1}\right| = k(k-1)(k-2)(k-3)(k-4)(k-5)$$

(B) if $G = P\Sigma L(2,q)$,

$$(q-2)(q-3)(q-4)\left|PSL(2,q)_{0B_1}\right| = k(k-1)(k-2)(k-3)(k-4)(k-5)s$$

As far as condition (A) is concerned, we may argue exactly as in the earlier case $N=G$. Thus, only condition (B) remains. If e is a power of 2, then Remark 2.8 gives

$$(q-2)(q-3)(q-4)\left|G_{0B_1}\right| = k(k-1)(k-2)(k-3)(k-4)(k-5)a$$

with $a|e$. In particular, a must divide $\left|G_{0B_1}\right|$, and we may proceed similarly as in the case $N=G$, yielding a contradiction.

The case $p=3$ may be treated as the case $p=2$.

Case (3): $M_v, v=11,12,22,23,24$

By Corollary 2.7, we get $k=7$ for $v=11$ or 12, and $k=7$ or 8 for $v=22,23$ or 24, and the very small number of cases for k can easily be eliminated by hand using Corollary 2.9 and Remark 2.8.

Case (4): $M_{11}, v=12$

As in case (3), for $v=12$, we have $k=7$ in view of Corollary 2.7, a contradiction since no 6-(12, 7, 2) design can exist by Corollary 2.9. This completes the proof of the Main Theorem.

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